Aggregate Implications of Credit Market Imperfections (III)

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Lecture 3: Dynamic Models with Heterogeneous Agents

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- A Model with Savers (Unfinished)
- A World Economy Model with International Capital Flows
- Dynamics of Household Wealth Distribution
  - A Single Dynasty's Problem; Individual Poverty Traps
  - A Model of Interacting Dynasties; Collective Poverty Traps
  - A Model of Emergent Class Society: Symmetry-Breaking

A Model with Savers (as one way of introducing heterogeneous agents in the models of Lecture 2.)

Time: Discrete ($t = 0, 1, 2, \ldots$)

Demography: 2-period lived OG agents

- Two-types of agents with the mass L_j ($j = 1$ or 2) in each generation.
- Different types endowed with different types of the endowment, by one unit, in the first period only, which is supplied inelastically.
- Each agent consumes only in the second. They save everything.
- Only type-j agents have the access to type-j projects characterized by m_j, R_j, λ_j, B_j, and μ_j.

Final Good: produced by CRS Technology: $Y_t = F(K_t, L_1, L_2)$, with the factor rewards, $\rho_t \equiv F_K(K_t, L_1, L_2) \equiv \Pi(K_t)$; $w_{jt} = F_j(K_t, L_1, L_2) \equiv W_j(K_t)$.

Aggregate Saving: $S_t = L_1W_1(K_t) + L_2W_2(K_t)$.

Equilibrium Conditions;

(1) $L_1W_1(K_t) + L_2W_2(K_t) = \sum_j(m_jX_{jt})$.

(2) $k_{t+1} = \sum_j(m_jR_jX_{jt})$.

(3) $$\frac{1}{r_{t+1}} \le Max\left\{\frac{1-W^j(k_t)/m_j}{\lambda_j R_j \Pi(k_{t+1}) + \mu_j B_j}, \frac{1}{R_j \Pi(k_{t+1}) + B_j}\right\} \quad (j = 1, 2)$$

where X_{jt} is the measure of type-j agents investing in period t, and $X_{jt} > 0$ (j = 1, 2,…J) implies the equality in (3).

Consider the special case, where $R_1 = R > R_2 = 0$; $B_1 = 0 < B_2 = B$; $0 < \lambda_1 = \lambda < 1$ and $\mu_2 = 1$.

If R > B, this effectively makes Type-1 "Investors" and Type-2 "Savers" who can only store at the rate equal to B.

Unfinished

A World Economy Model with International Capital Flows

Time: Discrete (t = 0, 1, 2,…)

Demography: 2-period lived OG agents

- Type-j ($j \in J$) agents with mass L_j in each cohort.
- Each type-j agent is endowed with one unit of the endowment, "Type-j Labor", in the first period only, which is supplied inelastically.
- Each agent consumes only in the second. They save everything.

Final Good: J different technologies to produce the final good. Type-j technology produces the final good using Type-j capital and type-j labor.

$$Y_t = \sum_j F_j(K_{jt}, L_j) = \sum_j f_j(k_{jt})L_j,$$

where K_{jt} is **type-j capital,** and $k_{jt} = K_{jt}/L_j$ is **the type-j capital-labor ratio**.

Competitive Factor Prices: $\rho_{jt} \equiv f_j'(k_{jt})$; $w_{jt} = f_j(k_{jt}) - k_{jt}f_j'(k_{1t}) \equiv W_j(k_{jt})$.

Investment Technologies:

Only type-j agents can produce type-j capital with type-j project, characterized by m_j, R_j, and λ_j.

Aggregate Saving: $S_t = \sum_j L_j W_j(k_{jt})$.

Primary Interpretation:

A World Economy Model, where type-j agents are those living in country-j, supply nontradeable labor that work with nontradeable type-j capital, and only they know how to invest in country-j. And the final good is tradeable.

Alternative Interpretation:
Type-j is Industry-j, producing good-j, in a small open economy that takes the world prices of J-tradeable goods given, but does not lend nor borrow with the rest of the world.

Equilibrium Conditions:

S = I condition: $\sum_j W_j(k_{jt})L_j = S_t = I_t = \sum_j m_j X_{jt} L_j$.

Capital Stock Adjustment: $k_{jt+1} = m_j R_j X_{jt}$ (j = 1, 2, …, J)

which can be combined as

(WRC) $\sum_j W_j(k_{jt})L_j = \sum_j k_{jt+1}(L_j/R_j)$.

(PC)+(BC) + (Inada Condition) for each →

(RRE)
$$r_{t+1} = \frac{R_j f_j'(k_{jt+1})}{Max\left\{\dfrac{1-W_j(k_{jt})/m_j}{\lambda_j}, 1\right\}}.$$

Symmetric Case: $m_j = m$; $L_j = 1/J$; $\lambda_j = \lambda$; $R_j = R$; $f_j(\bullet) = f(\bullet)$ for all j.

(WRC): $R\sum_j W_j(k_{jt}) = \sum_j k_{jt+1}$.

(RRE): $$r_{t+1} = \frac{\lambda R f'(k_{jt+1})}{Max\{1 - W(k_{jt})/m, \lambda\}}$$

If $m(1-\lambda) = 0$, $r_{t+1} = Rf'(k_{jt+1}) = Rf'(k_{t+1})$ where $k_{t+1} = (R/J)\sum_j W(k_{jt})$.

→ Convergence Across Countries Complete After One Period!

After One Period,

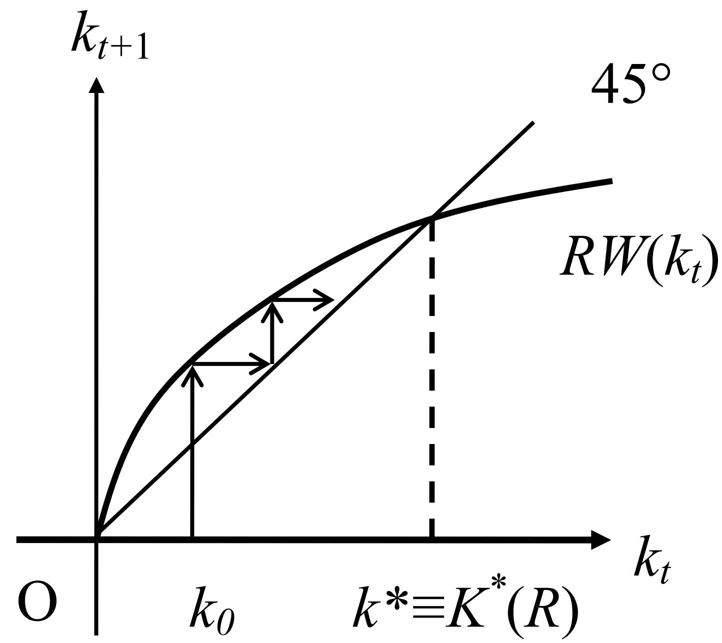
→ $k_{t+1} = RW(k_t)$ for all j.

What if $m(1-\lambda) > 0$?

Suppose J =1 (or the Autarky Case).

(WRC): $k_{t+1} = RW(k_t)$

(RRE): $r_{t+1} = \dfrac{\lambda R f'(k_{t+1})}{Max\{1 - W(k_t)/m, \lambda\}}$



The dynamics is governed by (WRC) regardless of m, and λ.

The model is indeed identical with "A Model with Convergence" in Lecture 2.

Suppose J = 2. Furthermore, suppose that $W(k_{1t})$, $W(k_{2t}) < m(1-\lambda)$.

***Caution**: This assumption is problematic, since it is made on endogenous variables.*

(WRC): $R\{W(k_{1t}) + W(k_{2t})\} = k_{1t+1} + k_{2t+1}$

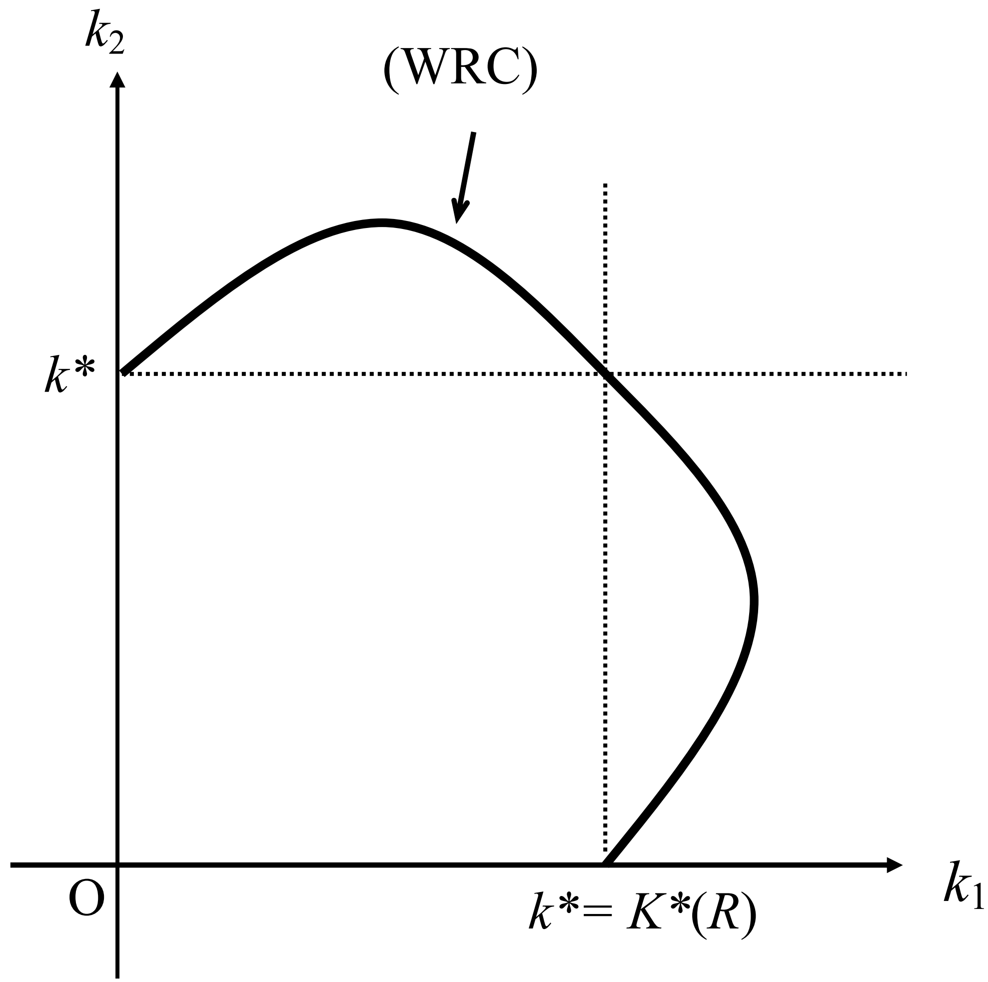
(RRE): $$\frac{f'(k_{1t+1})}{m - W(k_{1t})} = \frac{f'(k_{2t+1})}{m - W(k_{2t})}$$

Steady State Conditions:

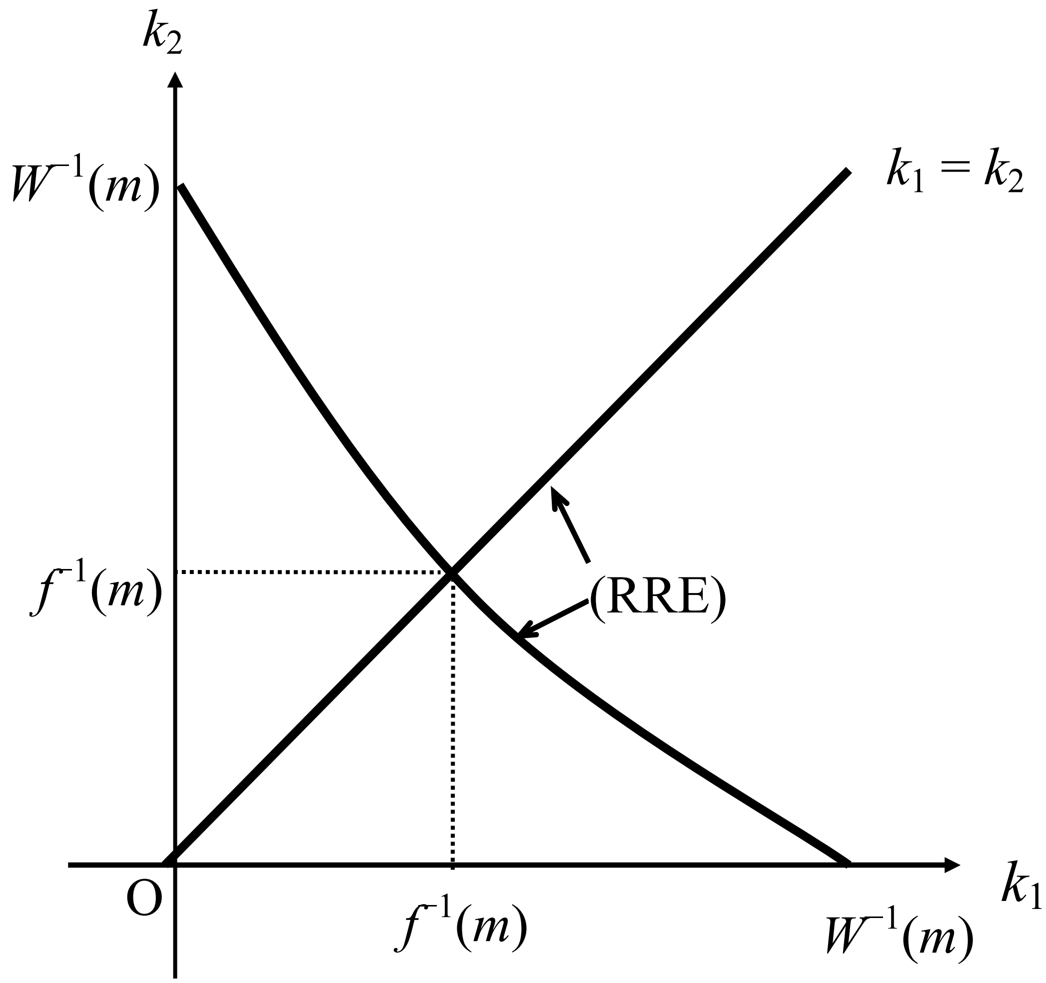
(WRC): $R\{W(k_1) + W(k_2)\} = k_1 + k_2$

(RRE): $$\frac{f'(k_1)}{m - W(k_1)} = \frac{f'(k_2)}{m - W(k_2)}.$$

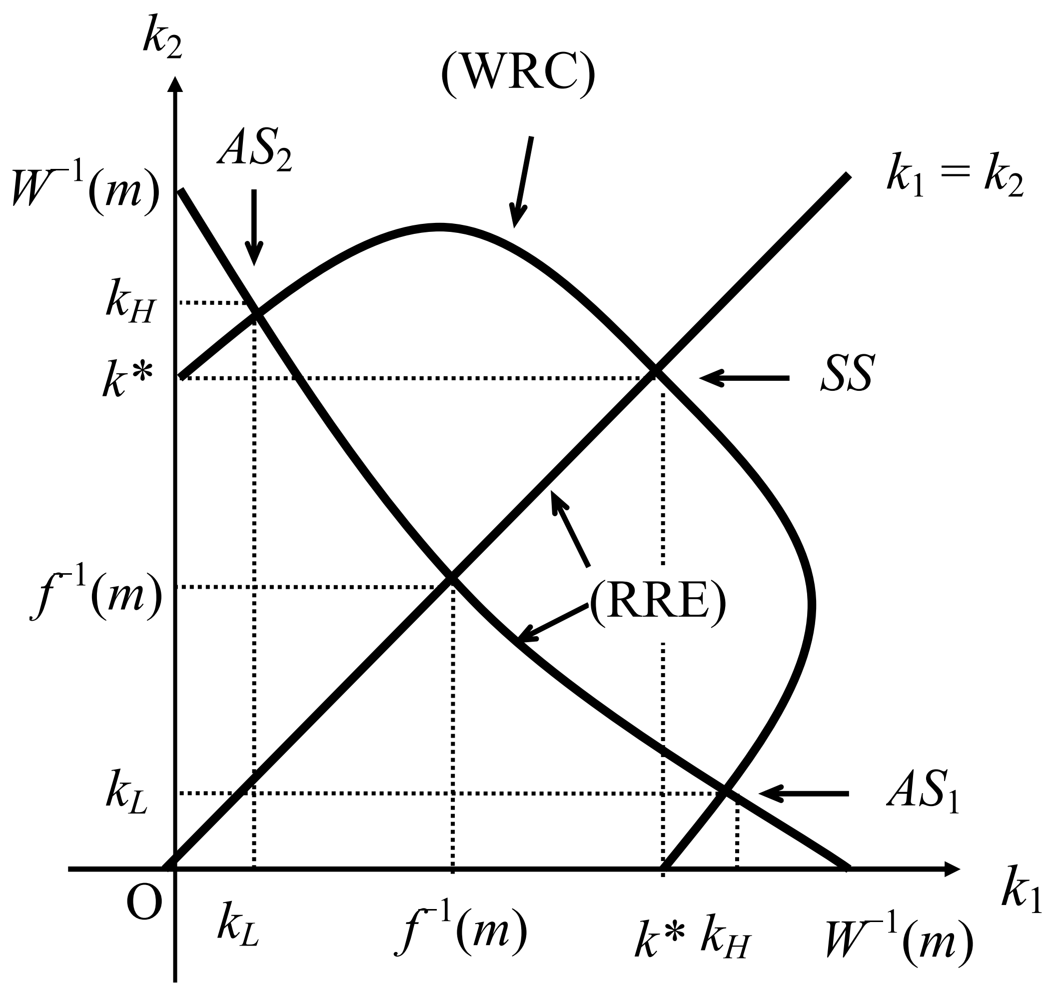
A Graphic Illustration of (WRC): $R\{W(k_1) + W(k_2)\} = k_1 + k_2$



A Graphic Illustration of (RRE): $\dfrac{f'(k_1)}{m - W(k_1)} = \dfrac{f'(k_2)}{m - W(k_2)}$



For an Intermediate Value of R, there are multiple steady states.



When multiple steady states exist,

- A Unique Symmetric Steady State, (SS) = (k^*, k^*), is *Unstable.*
- A Symmetric Pair of Asymmetric Steady States;

  (AS_1) = (k_H, k_L), (AS_2) = (k_L, k_H). *Are they Stable?*

Numerical simulations suggest that (AS_1) and (AS_2) seem stable.

While suggestive, the above analysis has some flaws.

- Hard to examine the stability of Asymmetric Steady States analytically.
- Hard to verify the assumption, $W(k_{1t})$, $W(k_{2t}) < m(1-\lambda)$.
- Hard to characterize the steady states for the entire parameter spaces. → We cannot examine the effects of changing the parameter values.

Let us modify the model in order to get the analytical results.

Suppose J = [0,1] so that each country is small.

(WRC): $$R\int_0^1 W(k_t(j))dj = \int_0^1 k_{t+1}(j)dj$$

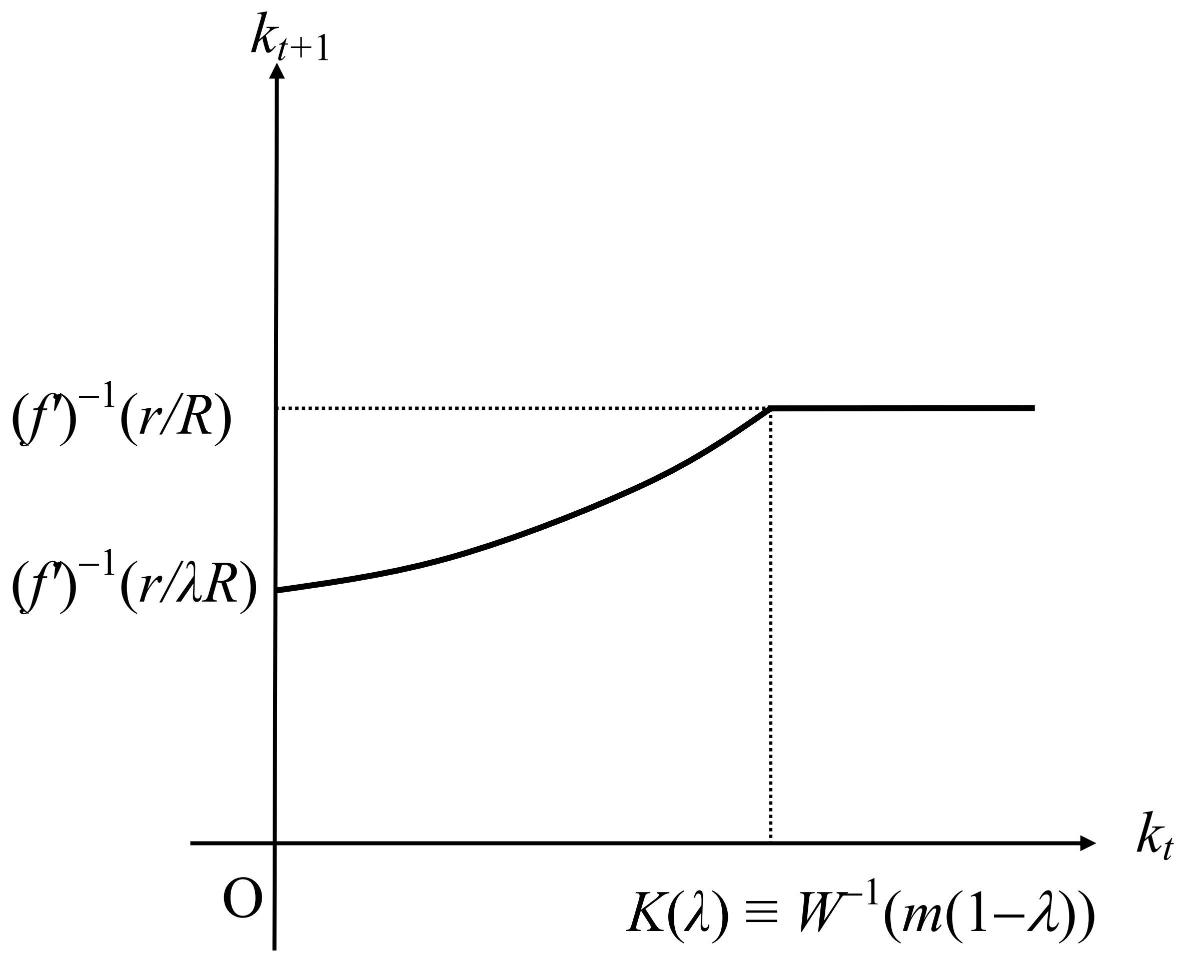
(RRE): $$r_{t+1} = \frac{\lambda R f'(k_{t+1}(j))}{Max\{1 - W(k_t(j))/m, \lambda\}}$$

Consider the dynamics of one (small) country, when the world as the whole is in steady state, where r is constant over time:

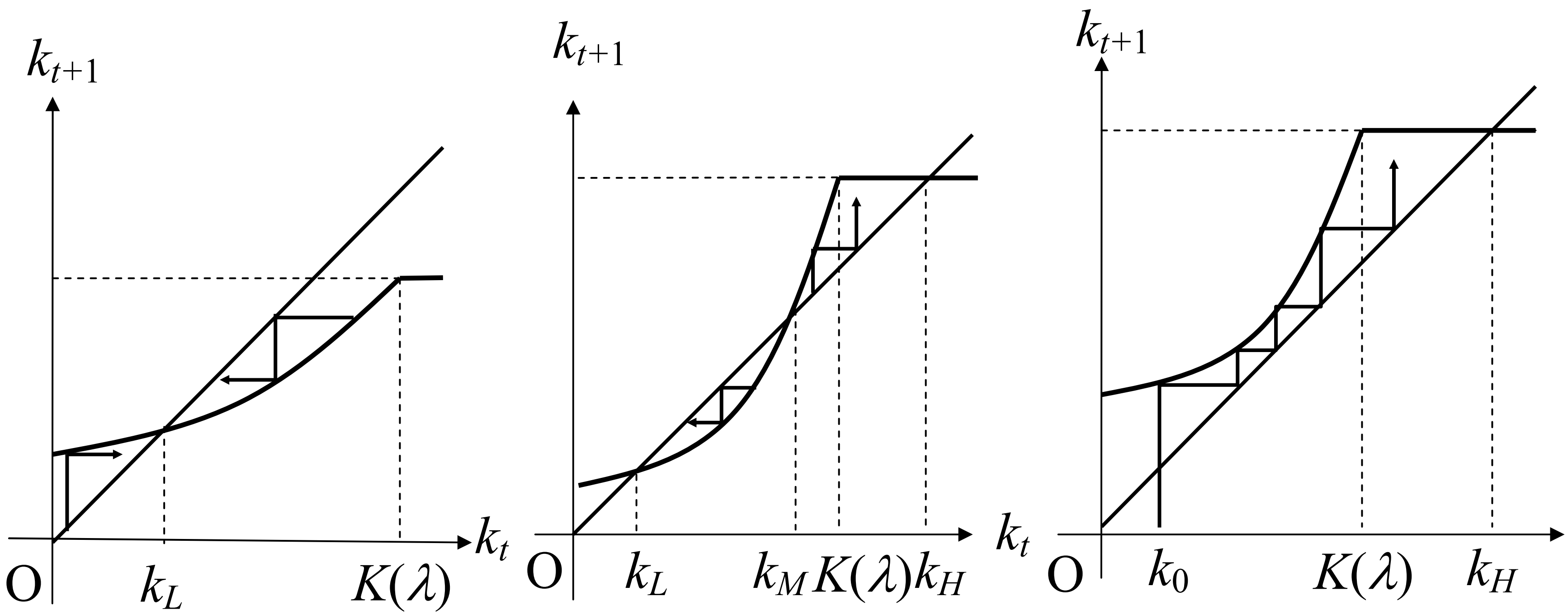
(WRC): $$R\int_0^1 W(k^*(j))dj = \int_0^1 k^*(j)dj$$

(RRE): $$r = \frac{\lambda R f'(k_{t+1}(j))}{Max\{1 - W(k_t(j))/m, \lambda\}}$$

Note that (RRE) is equivalent to the dynamics of the Under-Investment case in the Model with Good and Bad Projects in Lecture 2, without the non-negativity constraint on the Bad (if we set r = B).



Three generic ways in which the graph intersects with 45º line.



A B C

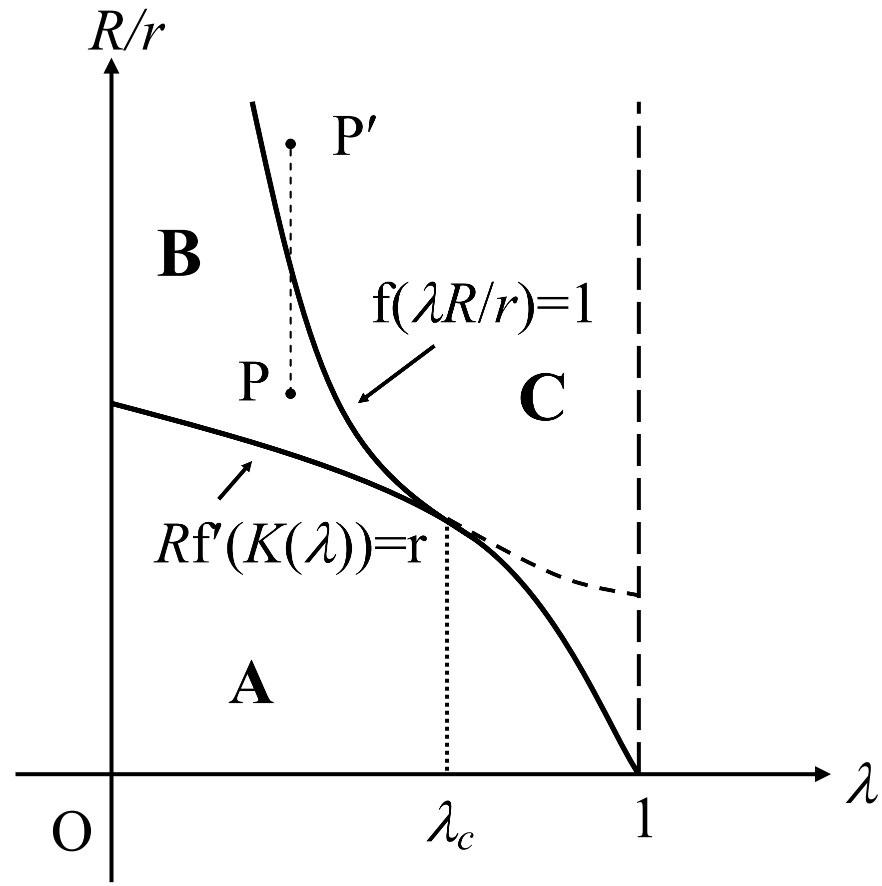
Parameter Configurations (Note: r is taken as a parameter, here.)

When interpreted as a small open economy model with the exogenous r,

- Even a small exogenous decline in r, illustrated by a move from P to P', could help the small economy trapped at the lower steady state, escape from it.

Likewise,

- Even a small exogenous rise in r could dislocate the small open economy from the higher steady state, causing a downward spiral.



The three generic cases imply that, in any *stable* steady state of the world economy consisting of a continuum of small countries, the steady state value of each country, k*(j), could take at most two different values.

If k*(j) = k* for all j, then the steady state is symmetric and

(WRC): $k^* = RW(k^*) \rightarrow k^* = K^*(R)$.

(RRE): $r = \dfrac{\lambda R f'(k^*)}{Max\{1 - W(k^*)/m, \lambda\}} = \dfrac{\lambda R f'(K^*(R))}{Max\{1 - W(K^*(R))/m, \lambda\}}$

In this symmetric steady state,

(PC) is binding, if $K^*(R) \geq K(\lambda)$.

(BC) is binding, if $K^*(R) \leq K(\lambda)$.

This steady state is identical with the steady state for the case of J = 1.

Or, the steady state may be characterized by a two-point distribution, where, for a fraction X of countries, $k^*(j) = k_H$, or for a fraction, 1–X, $k_t{}^*(j) = k_L$.

(WRC): $$X[k_H - RW(k_H)] = (1-X)[RW(k_L) - k_L] > 0$$

(RRE): $$Rf'(k_H) = r = \frac{\lambda Rf'(k_L)}{1-W(k_L)}$$

where 0 < X < 1 is also endogenous.

- ✓(WRC) implies $k_H > K^*(R) > k_L$.
- ✓(RRE) implies $k_H > K(\lambda) > k_L$.
- ✓Endogenous Polarization of the World Economy into the Rich & the Poor.
- ✓Investment Distortion among the Poor is endogenous.
- ✓The Rich's investment is financed by the Poor's saving.

In summary, there are only two possible types of steady states:

I. *Symmetric Steady State*, where $k^*(j) = K^*(R)$ for all j.

II. *Asymmetric Steady States,* where some countries have k_H and others have k_L, where $k_H > K^*(R)$, $K(\lambda) > k_L$.

The two types of steady states may or may not co-exist, depending on the parameter values.

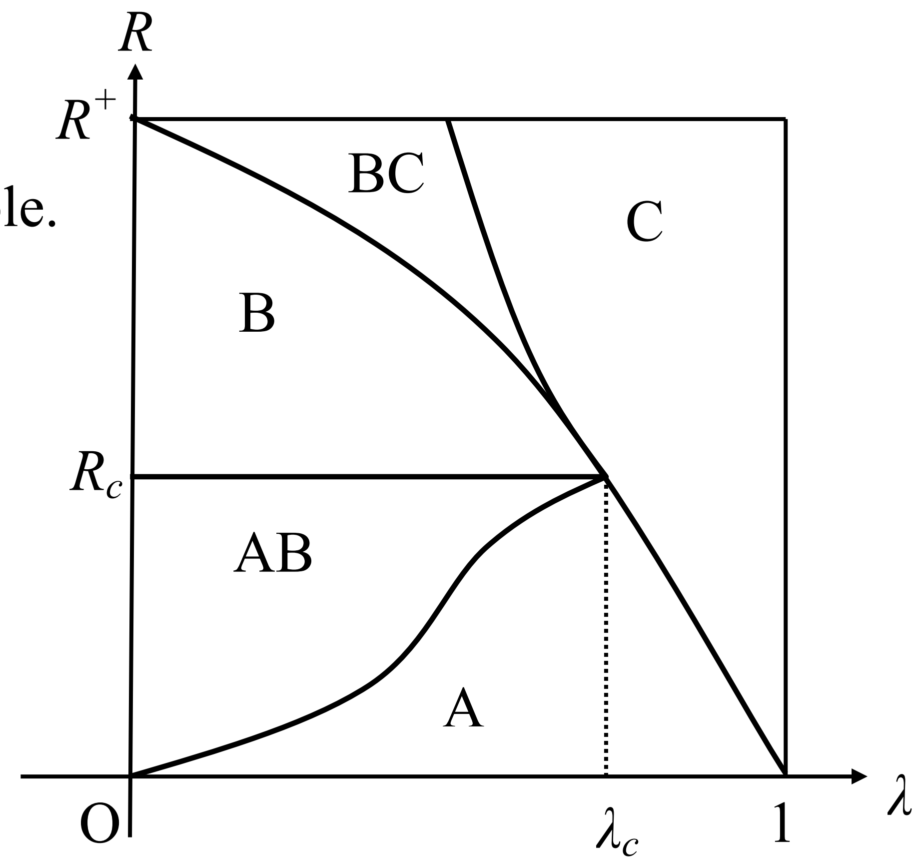
Parameter Configuration:

In A+AB, Symmetric Steady State, where (BC) is binding, exists.
In AB+B+BC, Stable Asymmetric Steady States exist.
In BC+C, Symmetric Steady State, where (PC) is binding, exists.

In Region B,

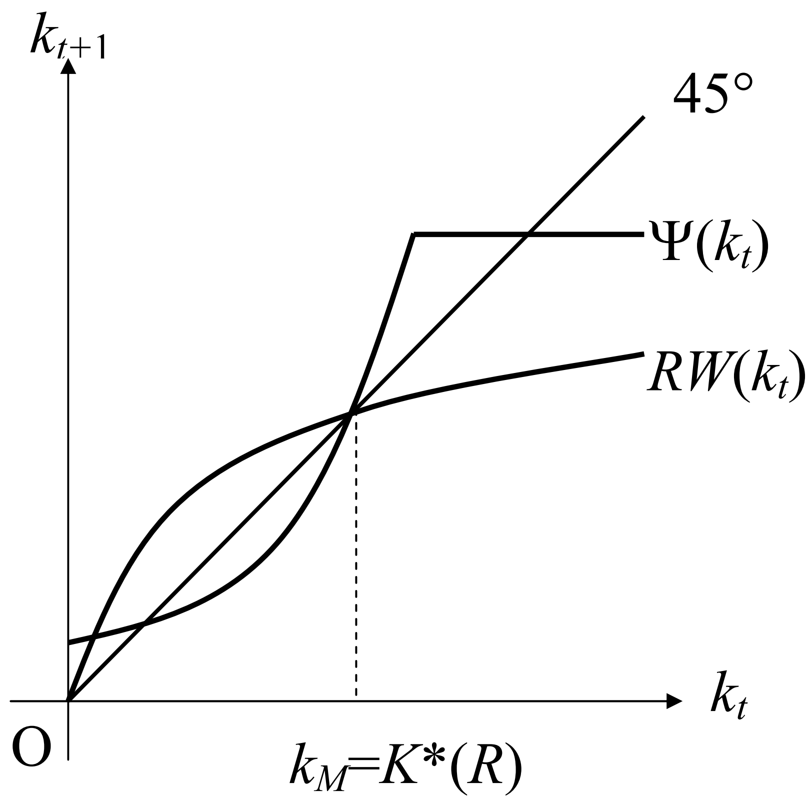
Only Asymmetric Steady States are stable.
Symmetric Steady State is unstable.

→ Symmetry-Breaking!!



Symmetry-Breaking when $K^*(R_c) < K^*(R) < K(\lambda)$.

What would happen if financial markets are fully integrated, when all the countries were located in the autarky steady state, $K^*(R)$?



Intuitions:

Why Symmetry-Breaking caused by Global Financial Integration?

- WITHOUT the international financial market, the domestic market rate adjusts to equate S = I, which offsets any country-specific shock, restoring the symmetry.
- WITH the international financial market, the domestic market rates are all linked. Without offsetting changes in the domestic market rate, positive (negative) country-specific shocks start virtuous (vicious) circles of high (low) wealth/high (low) investment.

Why Asymmetric Stable Steady States?

Diminishing Returns eventually put a break on the spiral process

The model captures the two contrasting views on global financial markets

1. Neoclassical View: *An Equalizing Force*

- Facilitate the Efficient Allocation of the World Saving
- Help the poor countries to grow faster and catch up with the rich

2. Structualist View: *An Unequalizing Force*

- The poor cannot compete with the rich in the global capital market
- Magnifying the gap between the rich and the poor
- Creating the International Economic Order of the Rich and the Poor

Efficiency Implication:

Because of the convexity of technologies (Aggregate Diminishing returns at the country level), the world output is smaller in (stable) asymmetric steady states than in the (unstable) symmetric steady state.

Proof: Maximizing the steady state world output means;

$$\text{Max} \int_0^1 f(k(j))dj \quad \text{s.t.} \int_0^1 k(j)dj \le R\int_0^1 W(k(j))dj$$

Since the feasibility set is convex, the objective is symmetric and strictly-quasi concave, the solution is $k(j) = k^* = K^*(R)$ for all $j \in [0,1]$.

Note: This feature is in contrast to models of endogenous inequality and symmetry-breaking based on IRS and/or Agglomeration Economies.

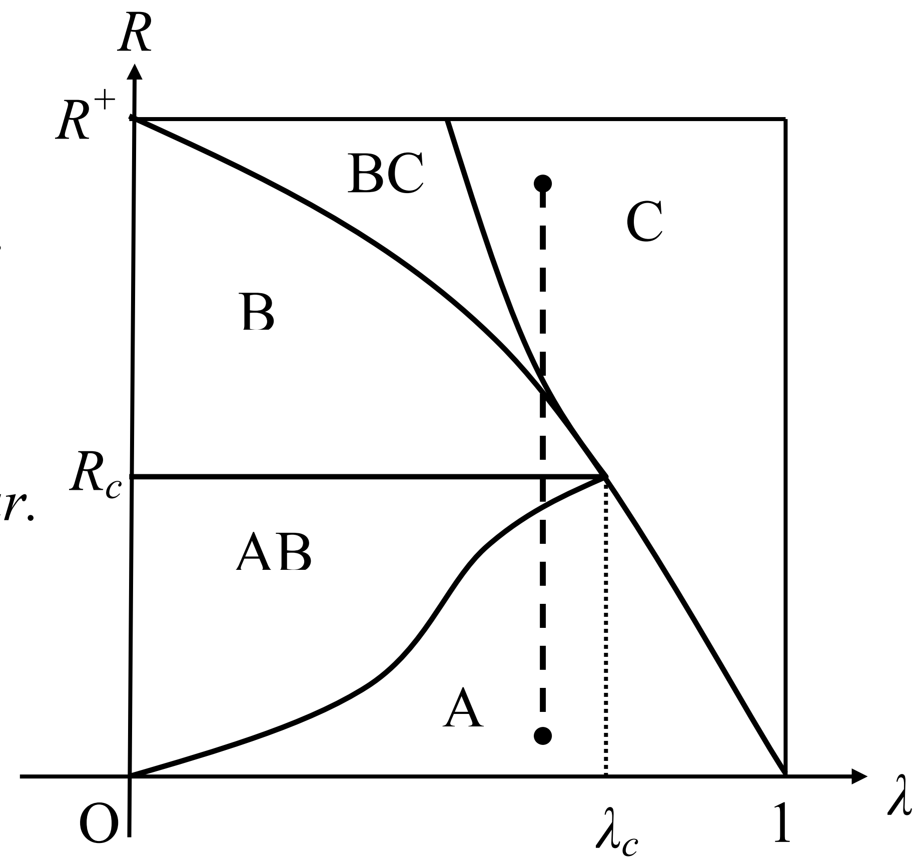
Application: Technical Progress and Inverted U-Curve Patterns of Inequality

Suppose $\lambda < \lambda_c$, and R was initially so small that the World Economy was in Region A applies. Countries are equally poor. Then, gradually, technology starts improving.

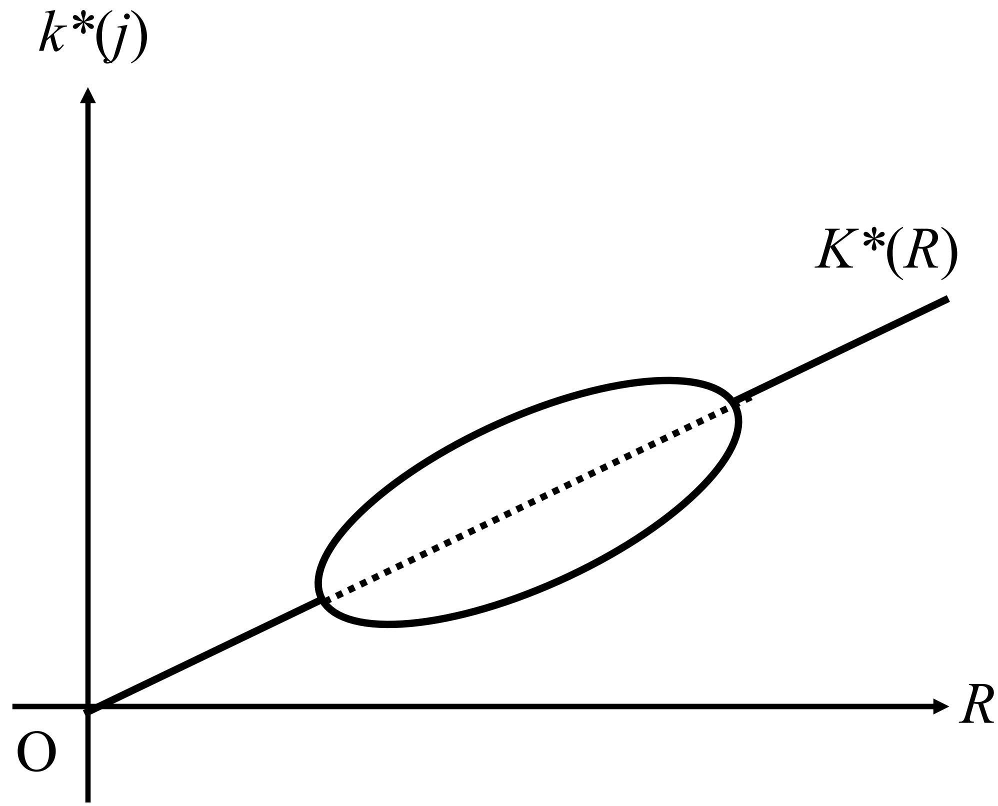
As R becomes greater than R_c, the world economy enters Region B.
Symmetric Steady State becomes unstable.

As R becomes even greater, the world economy enters Region C.
Stable Asymmetric Steady States disappear.

Over Region B, the world economy experiences:
First, divergence; then convergence.



Schematically,



Some Additional Remarks:

The model suggests

- a greater financial integration may cause a polarization of the world economy into the rich and the poor.
- inequality among nations might go up initially, and then go down as technology improves.

The model does not say

- The world economy has become increasingly unequal.
- The inequality of nations should be blamed for the international financial market.

More generally,

Symmetry-Breaking does not mean divergence

- Symmetry-Breaking means endogenous inequality.
- Symmetry-Breaking can be consistent with convergence.
- Symmetry-Breaking means, however, that there is a limit to convergence.

Endogenous Inequality does not mean that exogenous heterogeneity is not important. It suggests that

- a small amount of exogenous heterogeneity can be magnified to generate a huge inequality
- possible endogeneity of observed heterogeneities that are treated as exogenous in the growth accounting, growth calibration literature (e.g., there may be the two-way causality between Per Capita Income ↔ the Investment distortions)

Some Open Questions and Possible Extensions:

- Convergence Speed; even if the steady state continues to be unique, symmetric and stable under globalization, financial integration might affect the speed of convergence. We know that, when $\lambda = 1$, convergence is faster under globalization than under autarky. But, with a smaller λ, convergence might be slower under globalization than under autarky.

- Allow the agents to produce capital abroad (with reduced productivity), which could lead to Two-Way Flow of Financial Capital and FDI.
  - Savers in the South lends to Firms in the North, which invest in the South.
  - FDI can be used to bypass the external capital market in the South.

- Introducing Trade in Inputs, subject to some trade costs, which could lead to positive spillovers in neighboring countries;
  - Regional contagions (East Asian booms and Latin American stagnations)

- Endogenizing Investment Technology could lead to two-way causality between Productivity Difference vs. Institutional difference.

- The above analysis treats λ as exogenously fixed. However,
  - Economic development might change λ endogenously.
  - Globalization might affect λ.

- Interactions Between Inequality Within and Across Countries:

Dynamics of Household Wealth Distribution

A Single Dynasty's Problem; Individual Poverty Traps

Time: Discrete ($t = 0, 1, 2, \ldots$)

Final Good: used for both Consumption and Investment

A Dynasty: Infinite-sequence of one-period lived agents linked by inherence

An Agent (living in period t):

- Receive his wealth, w_t, in the form of bequest at the beginning of the period
- Make investment "choices" to maximize the end-of-the period wealth.
- Earn some additional income, y.
- Consume by c_t and Bequest w_{t+1} at the end of the period

Two Ways of Allocating the Inherence, w_t.

- Run a **non-divisible investment project**, which converts F units of the input at the beginning of period t into ***R*** units in **Final Good** at the end of period t, by **borrowing** $F-w_t$ at the market rate of return equal to ***r*.**
- **Lend** $x_t \le w_t$ units of the input at the beginning of period t for rx_t units of the final good at the end of period t. (Or, **Storage** with the rate of return, r.)

Agent's End-of-Period Wealth:

$U_t = y + R - r(F-w_t) = y + R - rF + rw_t$, if borrow and run the project,
$U_t = y + rw_t$ if lend (or put in storage).

***Profitability Constraint* (PC):** $R \ge rF$

***Borrowing Constraint* (BC):** $\lambda R \ge r(F-w_t) \rightarrow w_t \ge w_c \equiv F - \lambda R/r$.

Let $R > rF$. Then, the agent invests if and only if (BC) holds.

Agent's Consumption and Bequest Decisions:

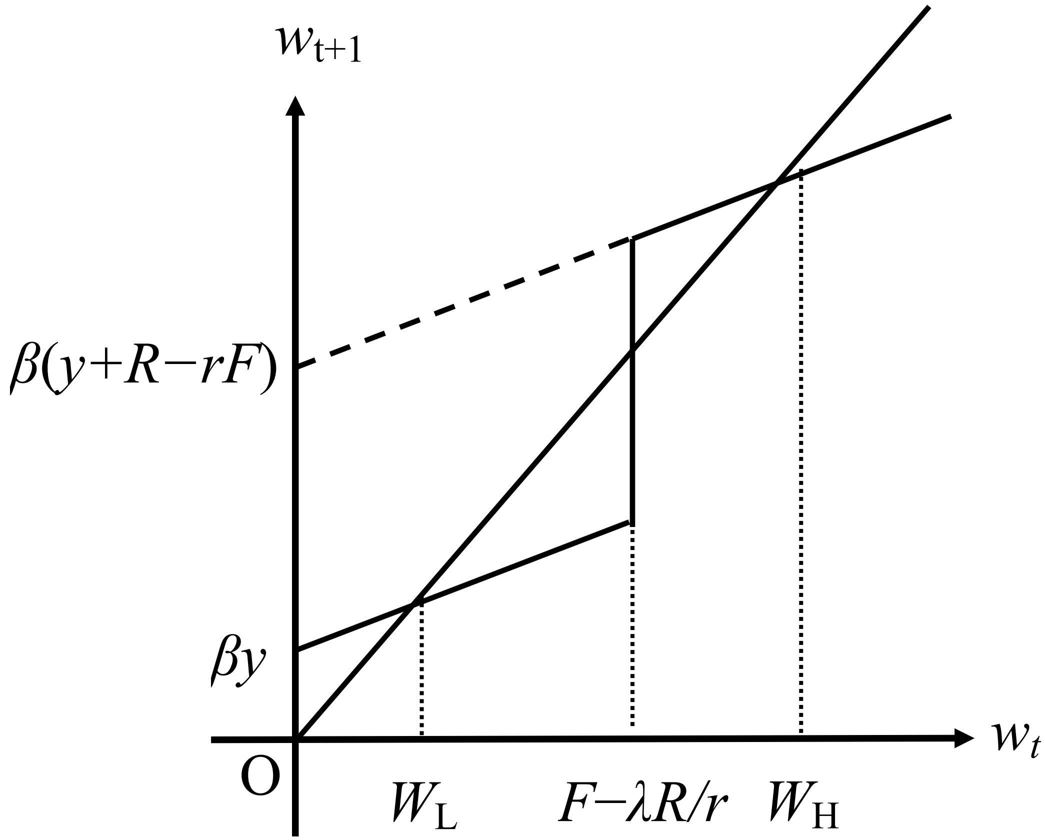
$$Max\left(\frac{c_t}{1-\beta}\right)^{1-\beta}\left(\frac{w_{t+1}}{\beta}\right)^{\beta} \text{ s.t. } c_t + w_{t+1} \le U_t \rightarrow c_t = (1-\beta)U_t,\ w_{t+1} = \beta U_t$$

Dynasty's Wealth Accumulation:

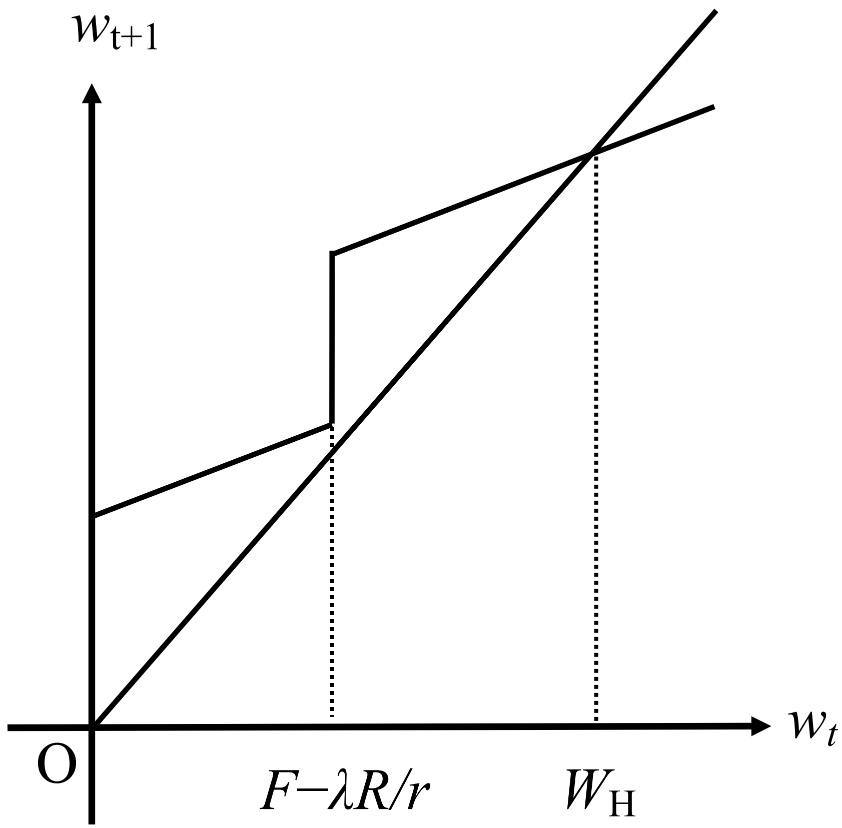
$$w_{t+1} = \beta U_t = \begin{cases} \beta(y + rw_t) & \text{if} \quad w_t < w_c \equiv F - \lambda R/r, \\ \beta[y + rw_t + (R-rF)] & \text{if} \quad w_t \ge w_c \equiv F - \lambda R/r. \end{cases}$$

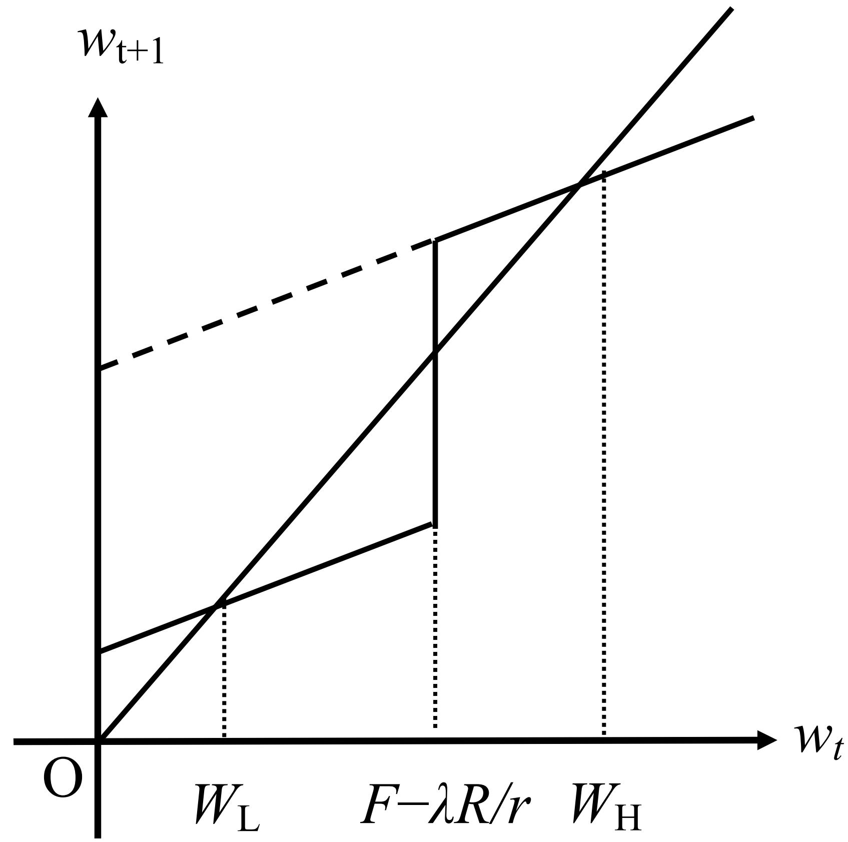
Assume $\beta r < 1$, and define $W_L \equiv \dfrac{\beta y}{1-\beta r}$ and $W_H \equiv \dfrac{\beta(y+R-rF)}{1-\beta r}$.

If $W_L < w_c < W_H$, the dynasty's long run wealth depends on the initial wealth.
→ Individual Poverty Trap



A Tale of Two (Non-Interacting) Families





A Model of Interacting Dynasties; Collective Poverty Traps

Time: Discrete (t = 0, 1, 2,…)

Final Good: used both for Consumption and Investment

A Continuum of Inherently-Identical Infinitely-Lived Dynasties:

- Each is linked by one period lived agent through inherence
- In each period, they differ only in inherence. $w_t \sim G_t(w)$.

An Agent of a Particular Dynasty, living in period t:

- Receives the initial wealth, w_t, in bequest at the beginning of the period
- Make occupational and investment "choices" to maximize the end-of-the period wealth.
- Consume by c_t and bequest w_{t+1} at the end of the period

Occupational and Investment Choices:

- *Worker*: Earns the wage rate, v_t; lends w_t at the gross return r
- *Entrepreneur:* Borrows $F-w_t$ at the gross rate of return, r, and sets up a firm, which requires F units of the final good at the beginning of period. The firm hires labor at the wage rate, v_t, and produces the final good at the end of period, with the technology, $\varphi(n)$; $\varphi'(n) > 0 > \varphi''(n)$; $\varphi(0) = 0$ and $\varphi(\infty) = \infty$.

Labor Employment: $n(v_t) \equiv \text{Argmax}_n\{\varphi(n) - v_t n\}$

Gross Profit: $\pi(v_t) \equiv \text{Max}_n\{\varphi(n) - v_t n\} \equiv \varphi(n(v_t)) - v_t n(v_t)\}$

$$\pi'(v) = -n(v) < 0,\ \pi''(v) = -n'(v) > 0$$

$$\pi(0) = n(0) = \varphi(\infty) = \infty.$$

Agent's End-of-Period Wealth:

$U_t = v_t + rw_t$, by becoming a worker

$U_t = \pi(v_t) + r(w_t-F)$ by becoming an entrepreneur

***Profitability Constraint* (PC):** $\pi(v_t) - v_t \geq rF \leftrightarrow v_t \leq V$, with $\pi(V) - V \equiv rF$.

- $v_t < V$, every agent wants to be an employer.
- $v_t = V$, indifferent.
- $v_t > V$, every agent wants to a worker.

V: the "fair" value of labor

***Borrowing Constraint* (BC):** $\lambda\pi(v_t) \geq r(F-w_t) \rightarrow w_t \geq C(v_t) \equiv \mathrm{Max}\{0, F-\lambda\pi(v_t)/r\}$

- $C'(v) > 0$ and $C''(v) < 0$, if $C(v) > 0$ and $\lambda > 0$.
- $C(v) = 0$ for a small v if $\lambda > 0$.

Alternative Interpretation:

- The worker supplies one unit of labor and earns $v_t = W(k_t) \equiv f(k_t) - k_t f'(k_t)$.
- The entrepreneur supplies R units of capital and earns $\Pi_t \equiv Rf'(k_t)$.

The two interpretations give the same result, if we set $k_t = R/n_t$ and $\varphi(n_t) \equiv F(R, n_t) = f(k_t)n_t$. **Prove it!**

Combining (PC) and (BC):

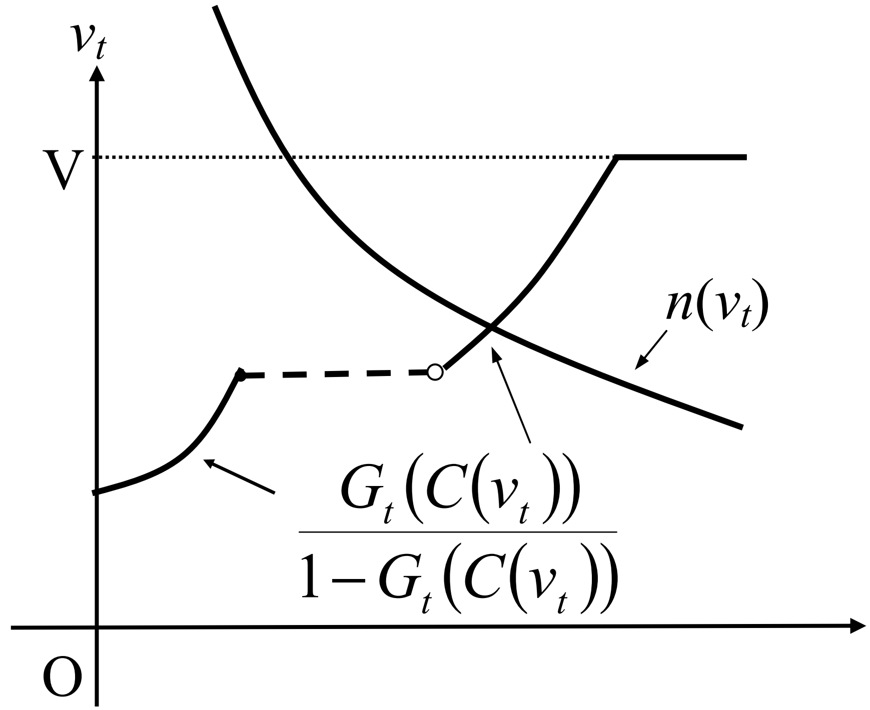
- $v_t > V$, then $v_t > \pi(v_t) - rF$:
  - nobody sets up a firm, no demand for labor.

- $v_t < V$, then $v_t < \pi(v_t) - rF$:
  - The agents with $w_t < C(v_t)$ have no choice but to become workers.
  - The agents with $w_t \geq C(v_t)$ become employers and hire $n(v_t)$ each.

- $v_t = V$, then $v_t = \pi(v_t) - rF$:
  - The agents with $w_t < C(v_t)$ have no choice but to become workers.
  - The agents with $w_t \geq C(v_t)$ are willing to be employers and hire $n(v_t)$ each.

Labor Market Equilibrium (LME): $\dfrac{G_t(C(v_t))}{1-G_t(C(v_t))} \leq n(v_t)$; $v_t \leq V$,

The dotted vertical line indicates a mass point in G_t.

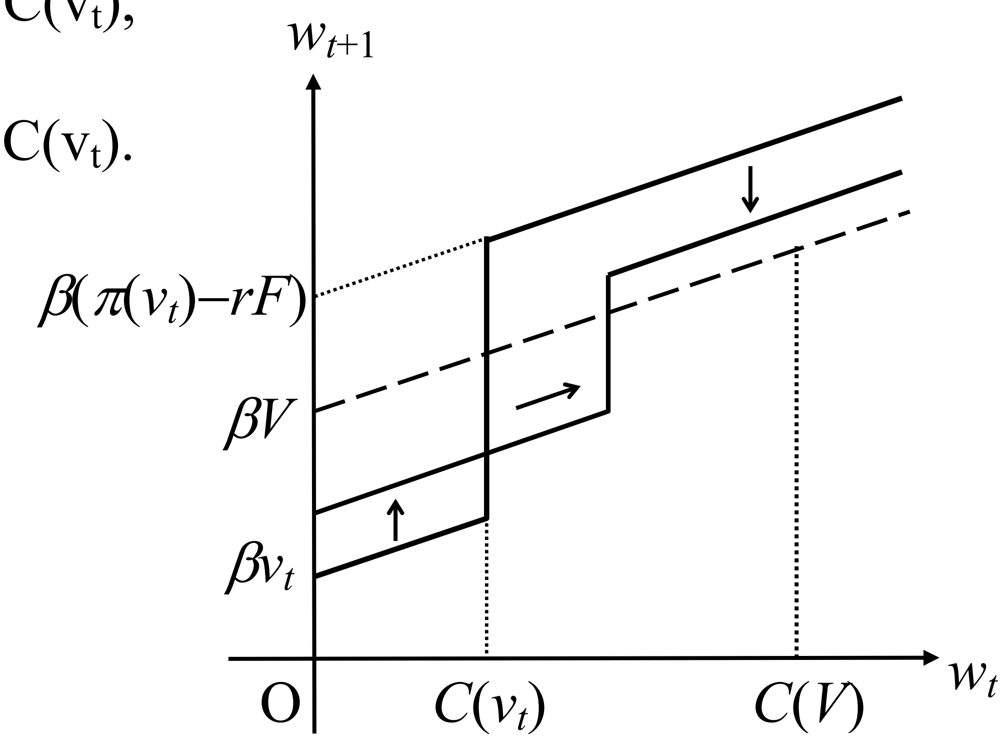


$G_t(\bullet) \Rightarrow v_t, \pi(v_t)$

Wealth Accumulation (WA)

$$w_{t+1} = \begin{cases} \beta(v_t + rw_t) & \text{if } w_t < C(v_t), \\ \beta(\pi(v_t) - rF + rw_t) & \text{if } w_t \geq C(v_t). \end{cases}$$

The arrows indicate
the effects of a higher v_t.

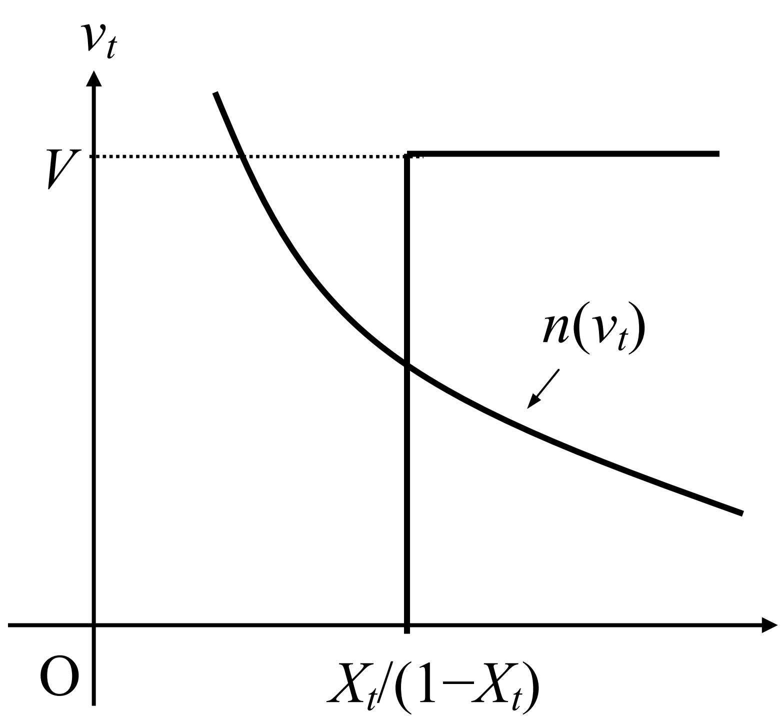


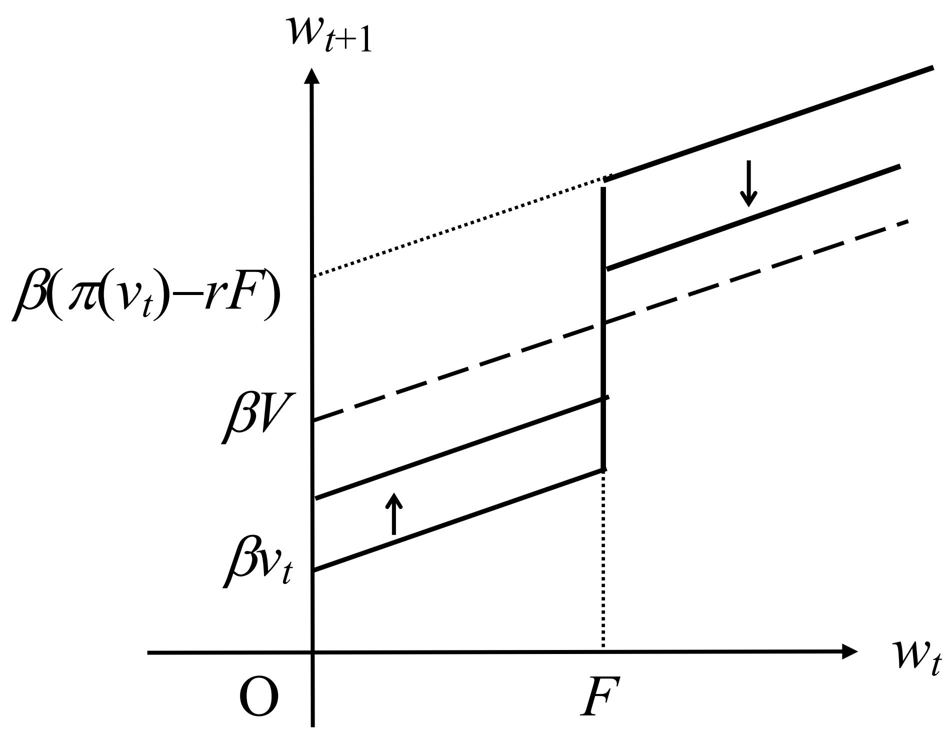
$$\mathbf{G_t(\bullet) \Rightarrow v_t, \pi(v_t) \Rightarrow G_{t+1}(\bullet) \Rightarrow v_{t+1}, \pi(v_{t+1}) \Rightarrow \ldots}$$

Special Case: $\lambda = 0 \rightarrow C(v_t) = F$.

(LME): $\dfrac{X_t}{1-X_t} \leq n(v_t)$; $v_t \leq V$, where $X_t \equiv G_t(F)$

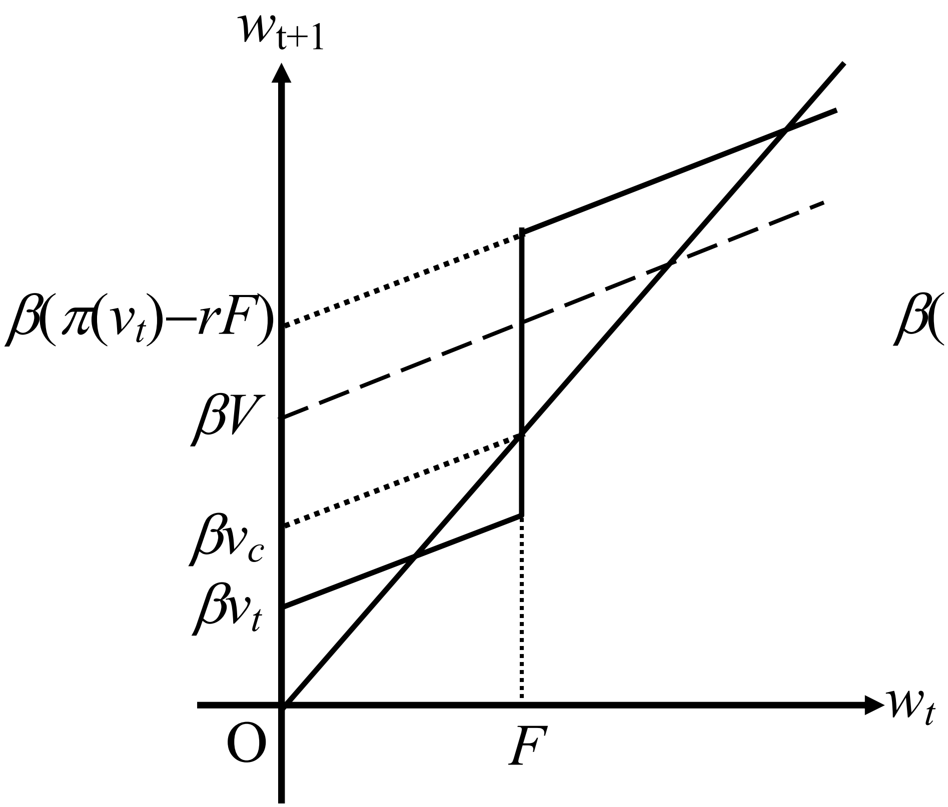
(WA): $w_{t+1} = \begin{cases} \beta(v_t + rw_t) & \text{if } w_t < F, \\ \beta(\pi(v_t) - rF + rw_t) & \text{if } w_t \geq F. \end{cases}$



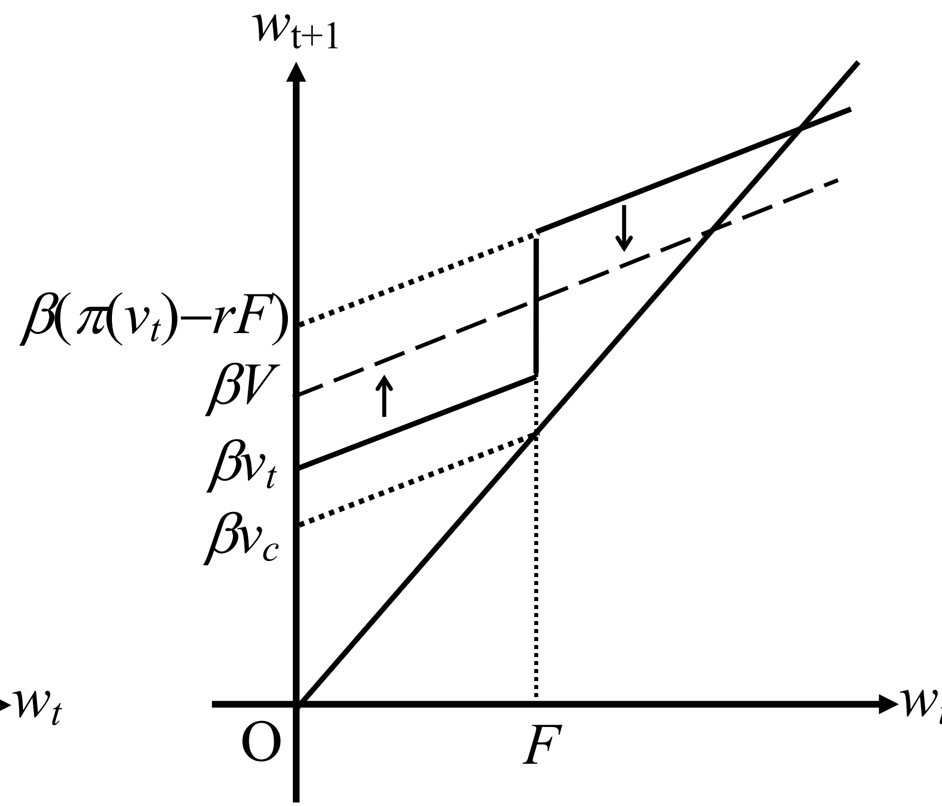


Suppose $v_c \equiv (1-\beta r)F/\beta < V$. Then,

Either $v_t \leq v_c < V < \pi(v_t) - rF$ OR $v_c < v_t \leq V \leq \pi(v_t) - rF$.



$X_t = X_{t+1} = \ldots = X_\infty$;
$v_t = v_{t+1} = \ldots = v_\infty$.



X_t declines until $X_\infty = 0$.
v_t goes up until $v_\infty = V$.

In period 0, the wage rate, v_0, is given by $X_0 = n(v_0)/[1+n(v_0)]$.

- A fraction, $G_0(F) = X_0$, of the agents becomes workers;
- A fraction, $1-X_0$, of the agents becomes entrepreneurs;

If $G_0(F) = X_0 \geq X_c \equiv n(v_c)/[1+n(v_c)]$, this is a steady state.

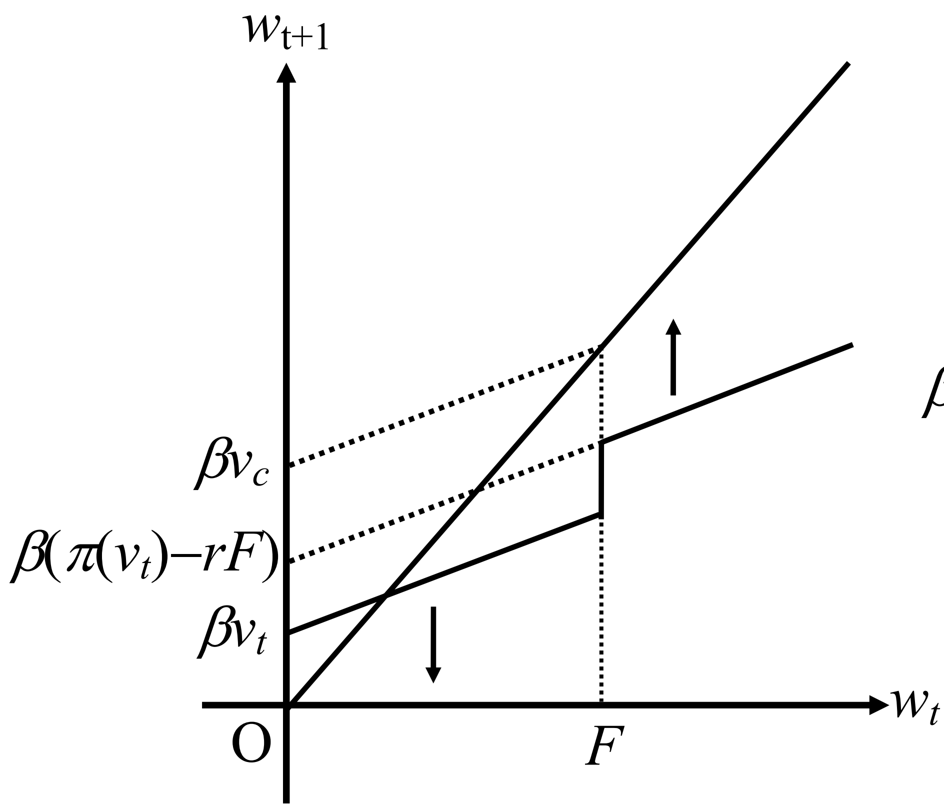
- A fraction, X_0, of the dynasties becomes the proletariat; their wealth converges to $\beta v_0/(1-\beta r)$. They are trapped in poverty.
- A fraction $1-X_0$ of the dynasties becomes the bourgeoisie; their wealth converges to $\beta[\pi(v_0)-rF]/(1-\beta r)$.

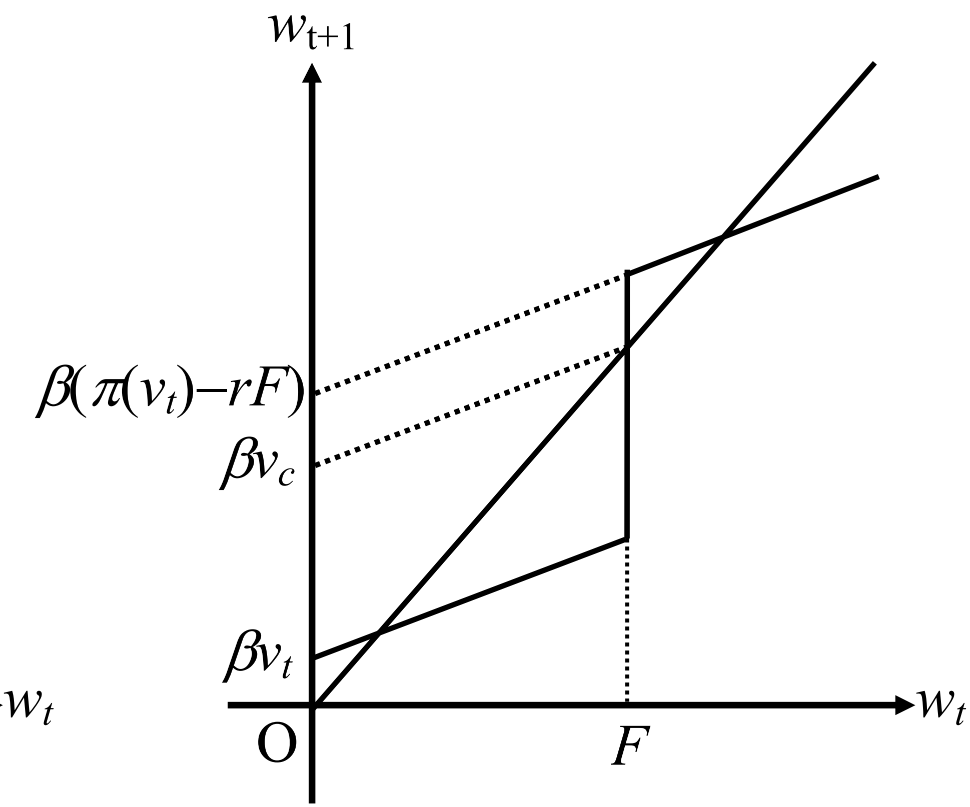
If $G_0(F) = X_0 < X_c \equiv n(v_c)/[1+n(v_c)]$, the wage rate is sufficiently high that some workers leave enough wealth that allows their descendents to become entrepreneurs, which further raise the wage rate. In the steady state, $v = V$ and each dynasty's wealth converges to $\beta V/(1-\beta r) > F$; The workers and entrepreneurs are equally wealthy; the class distinction disappears.

A higher initial inequality, if it reduces X_0, can eliminate the long run inequality and the collective poverty trap.

What if $v_c \equiv (1-\beta r)F/\beta > V$? Then,

Either $v_t \leq \pi(v_t) - rF < v_c$ OR $v_t < v_c \leq \pi(v_t) - rF$.





X_t goes up and v_t goes down until $X_t = X_{t+1} = \ldots = X_\infty$.

$v_t = v_{t+1} = \ldots = v_\infty$.

A Model of Emergent Class Society: $\lambda > 0 \rightarrow C(v_t) = \text{Max}\{ 0, F - \lambda\pi(V)/r\}$.

Steady State Analysis

The Classless Society: The Steady State with Wealth Equality: $v_\infty = V$.

$$\mathbf{w_\infty = \beta V/(1-\beta r) \geq C(V) = Max\{0, F - \lambda\pi(V)/r\}.}$$

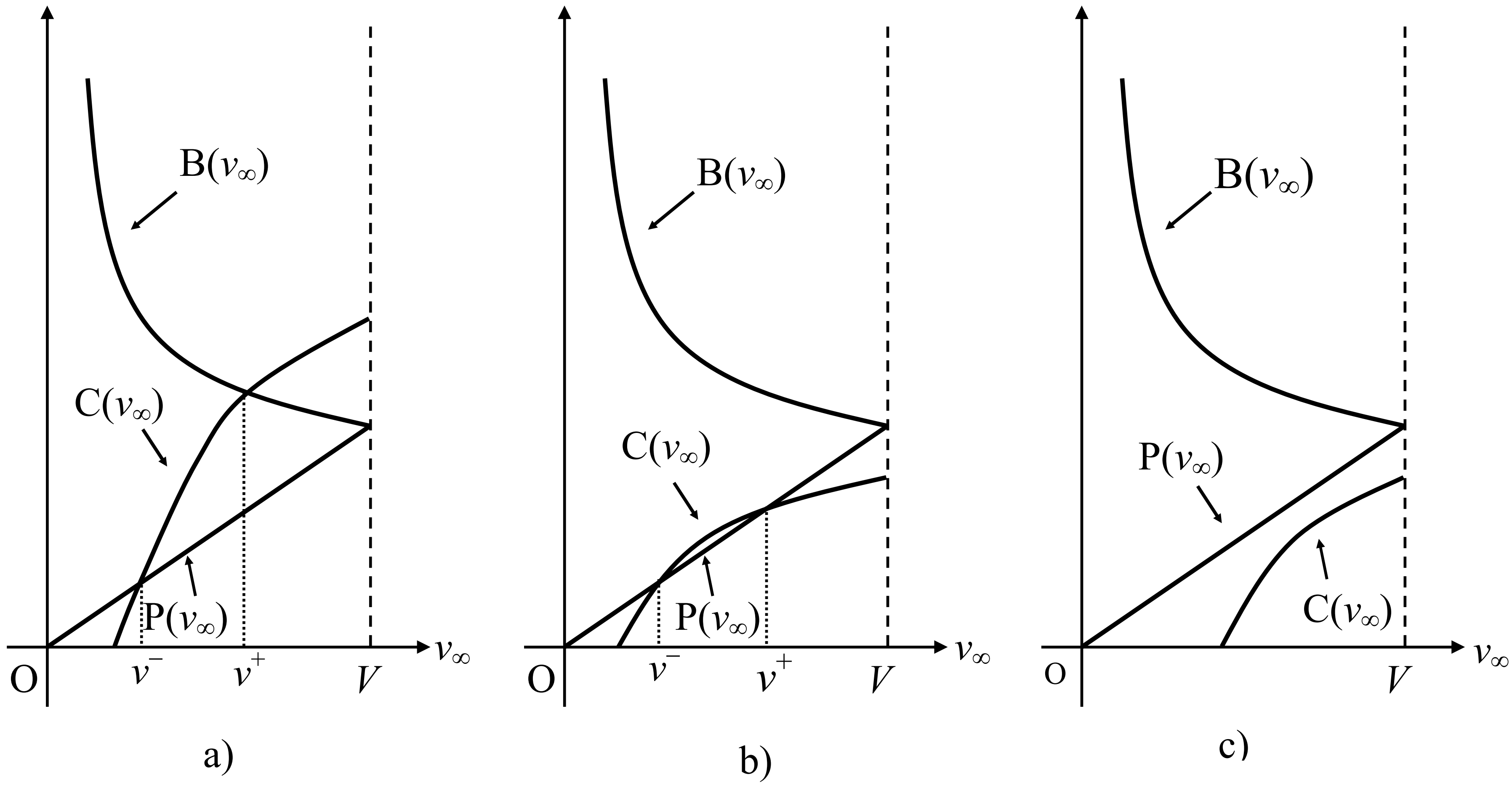
Labor Market clears because the agents are indifferent.

The Class Society: The Steady States with Wealth Inequality: $v_\infty < V$

Bourgeoisie's wealth: $\mathbf{w^B_\infty = B(v_\infty) \equiv \beta(\pi(v_\infty) - rF)/(1-\beta r) \geq C(v_\infty)}$,
Proletariat's wealth; $\mathbf{w^P_\infty = P(v_\infty) \equiv \beta v_\infty/(1-\beta r) < C(v_\infty)}$,

Labor Market Equilibrium; $X_\infty/(1-X_\infty) = n(v_\infty)$

Three Generic Cases

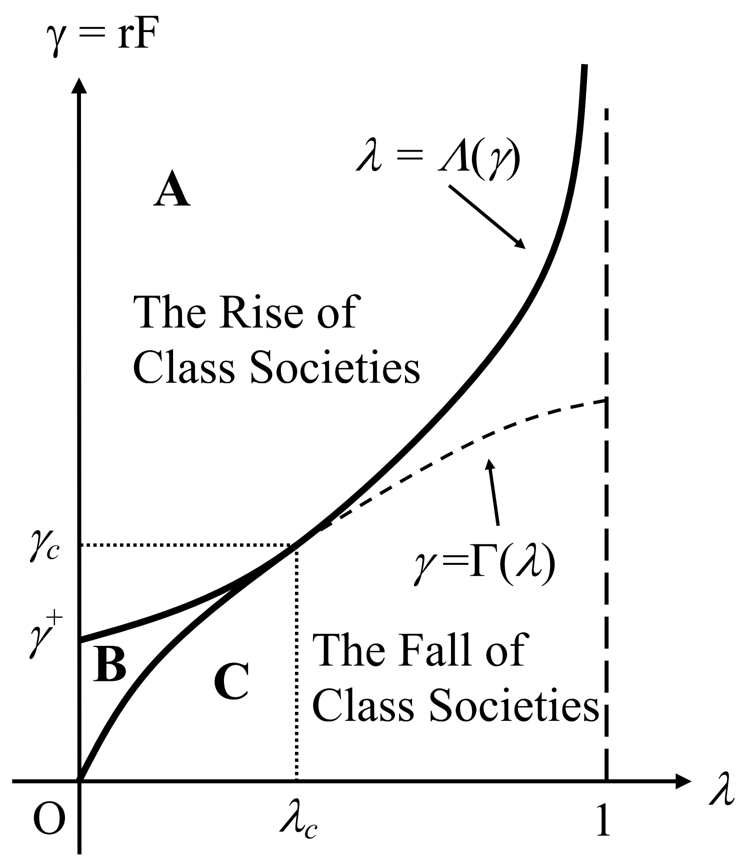


A: Unequal Steady States Only (Symmetry Breaking):
Long Run Inequality for any Initial Distribution.
Emergent Class Structure
One-Time Redistribution Ineffective

C: Equal Steady State Only:
Long Run Equality for any Initial Distribution

B: Equal and Unequal Steady States Co-Exist.

History Dependence
Initial Distribution Matters;
One-Time Redistribution Effective



Parameter Configurations

An Extension: Self-Employment

Dual Nature of Self-Employment

offers the poor an alternative to working for the rich employer

offers the rich an alternative to investing to the job-creating project

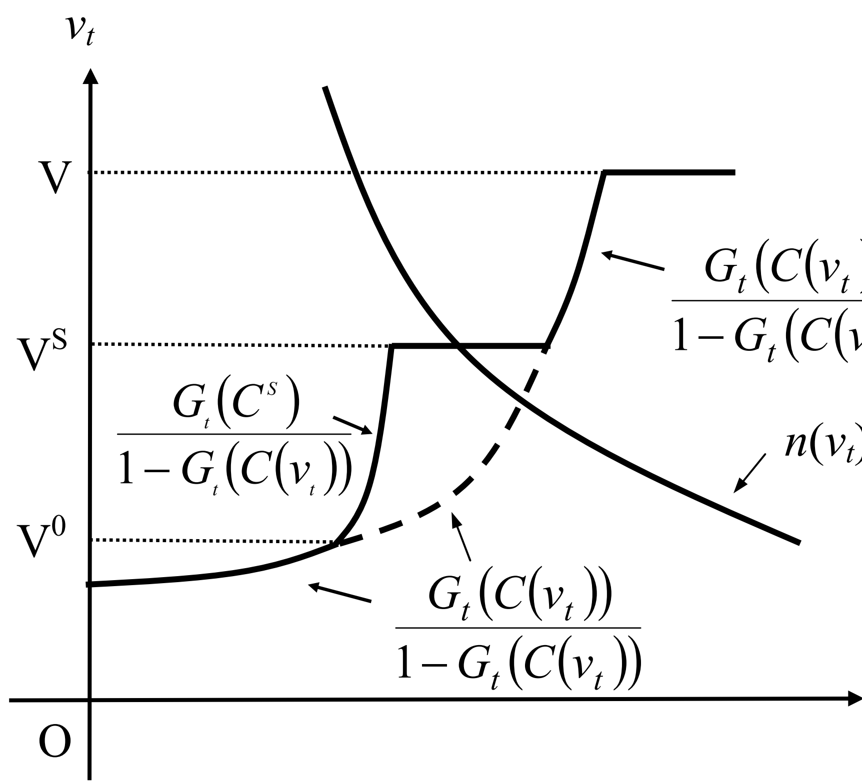
Self-Employment Technology: Invest F^S at the beginning of the period, earn π^S at the end of the period. $\lambda^S\pi^S$ is the default cost.

$V^S \equiv \pi^S - rF^S$: the net income of the self-employed

$C^S \equiv \text{Max}\{0, F^S - \lambda^S\pi^S/r\}$ the net worth required for self-employment.

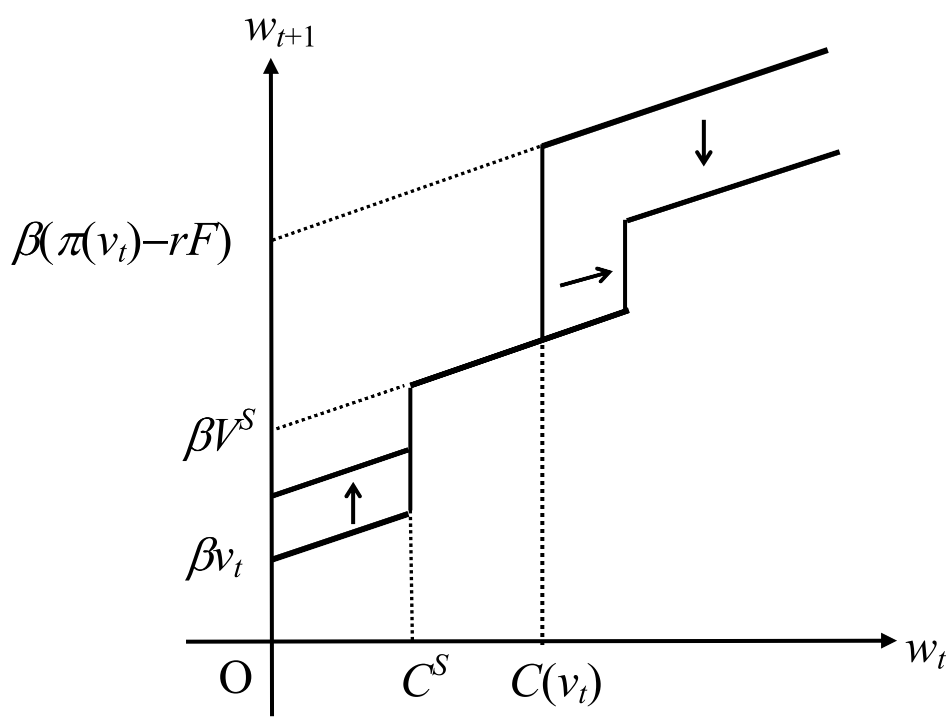
(A1)	$V^S < V$;	being an employer preferable to being self-employed
(A2)	$C^S < C(V^S)$;	self-employment can be a viable alternative.
(A3)	$C^S \leq P(V^S)$.	sustainability of the self-employed status.

Labor Market Equilibrium with Self-Employment



Wealth Accumulation with Self-Employment

$$w_{t+1} = \begin{cases} \beta(v_t + rw_t) & \text{if } w_t < C^S \\ \beta(V^S + rw_t) & \text{if } C^S \leq w_t < C(v_t) \\ \beta(\pi(v_t) - rF + rw_t) & \text{if } w_t \geq C(v_t). \end{cases}$$



The Classification of the Steady States:

1-Class Steady State without Active Self-Employment: ($v_\infty = V$).
2-Class Steady States without Active Self-Employment: ($v_\infty < V$).
1-Class Steady State with Active Self-Employment: (self-employed only)
2-Class Steady States with Active Self-Employment; ($v_\infty = V^S$)
3-Class Steady States; $v_\infty \in (V^0, V^S)$, with 3-point wealth distributions.

The Steady States in the Model with Self-Employment

		No Active Self-Employment		Active Self-Employment		
		One-Class	Two-Class	One-Class	Two-Class	Three-Class
A	I	$\varnothing$	$(v^-, v^+]$	$\varnothing$	$\varnothing$	$\varnothing$
A	IIa	$\varnothing$	$[V^S, v^+]$	V^S	V^S	$\varnothing$
A	IIb	$\varnothing$	$(v^-, V') \cap [V^S, v^+]$	V^S	V^S	(V'', V')
A	**IIIa**	$\varnothing$	$\varnothing$	$\mathbf{V^S}$	$\varnothing$	$\varnothing$
A	IIIb	$\varnothing$	(v^-, V')	V^S	$\varnothing$	(V'', V')
A	IIIc	$\varnothing$	$(v^-, v^+]$	V^S	$\varnothing$	$(V'', v^+]$
B	I	V	(v^-, v^+)	$\varnothing$	$\varnothing$	$\varnothing$
B	IIa	V	$[V^S, v^+)$	V^S	V^S	$\varnothing$
B	IIb	V	$(v^-, V') \cap [V^S, v^+)$	V^S	V^S	(V'', V')
B	**IIIa**	**V**	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
B	IIIb	V	(v^-, V')	$\varnothing$	$\varnothing$	$\varnothing$
B	IIIc	V	(v^-, v^+)	$\varnothing$	$\varnothing$	$\varnothing$
C		V	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

An Extension: Investment Without Diminishing Returns

Employers: Invest $K_t \geq F$, employ N_t at the beginning of period; produced $\Phi(N_t, K_t)$ units of the output at the end of period. Φ is a CRS, with $\Phi(N_t, K_t) = 0$ if $K_t < F$.

Let $k_t \equiv K_t/F$, $n_t \equiv N_t/k_t$, and $\phi(n_t) \equiv \Phi(n_t, F)$.

For $k_t \geq 1$, $\text{Max}_N\{\Phi(N, K) - vN\} = \text{Max}_n\{\phi(n) - vn\}k = \{\phi(n(v)) - vn(v)\}k = \pi(v)k$, where n(v) and $\pi(v)$ are defined as before.

k: the scale of operation, the investment measured in multiples of F

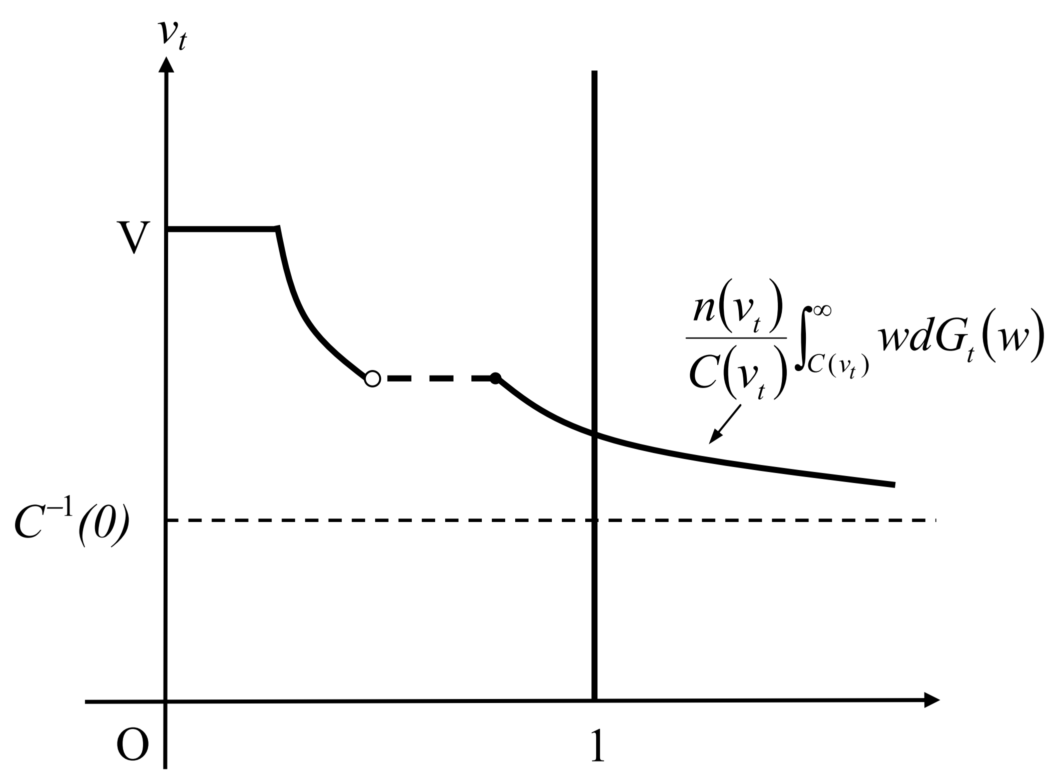
$\pi(v)$: the equilibrium profit per unit of operation.

We allow for the employer to supply one unit of labor (to avoid IRS)

Borrowing Constraint: $w_t \geq [F - \lambda\pi(v_t)/r]k_t = C(v_t)k_t$,

Labor Market Equilibrium: $\dfrac{n(v_t)}{C(v_t)} \int_{C(v_t)}^{\infty} w dG_t(w) \geq 1$; $0 < C(v_t) \leq C(V)$,

Labor Market Equilibrium without Diminishing Returns



Household Wealth Dynamics without Diminishing Returns