Notes on a Type of Learning and the Uncovered Interest Parity Puzzle
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Following is an exploration of the interest rate and exchange rate consequences of the idea that people don’t know whether the shock that drives the interest rate is permanent or temporary. Thus, suppose there is a positive money supply shock that drives down the domestic interest rate. Under UIP this must create an anticipated appreciation of the currency, or else all agents will sell assets in the domestic economy and buy foreign assets. In order for people to anticipate an appreciation, there must be an immediate depreciation. How much the depreciation must be depends on the persistence of the underlying shock. If the money shock is completely persistent, then the long-run effect on the exchange rate is to depreciate, and so the depreciation in the present period must be very great, to permit the eventual appreciation. If the shock is temporary, then the long-run exchange rate is unchanged, and the required current depreciation is very small.

Now, suppose that people don’t know whether the shock is permanent or temporary. Initially, when they see the interest rate move, they will assume that it reflects whatever shock usually moves the exchange rate. Suppose, for example that the shock moving the money supply is usually temporary, but that the persistent one hits. In this case, people will expect there to be no long-run change in the exchange rate, and so there will only be a small depreciation. However, as time passes, and they see the money supply remaining high, they will slowly revise their beliefs about the long run exchange rate and this will require a higher current depreciation. In this way it is possible, in the wake of a drop in the interest rate, for the exchange rate to depreciate over time.

This note formalizes this argument, and then goes on to show that this line of reasoning does not provide a way to understand the violations of UIP documented in the data. It is still the case that if the domestic interest rate is higher than the foreign, then on average the currency must depreciate afterward. In the data, the result often goes the other way: when the domestic interest rate is high, on average there is an appreciation of the exchange rate (the ‘UIP puzzle’). So, uncertainty about the persistence of the shocks underlying movements in the exchange rate does not offer a possible explanation of the UIP puzzle.

1. The Model

Consider the following model:

\[ E_t e_{t+1} - e_t - R_t = 0 \] (UIP)
\[ \Delta m_t - \Delta p_t + \alpha \Delta R_t = 0 \] (MM)
\[ \Delta p_t - (1 - \lambda) \Delta p_{t-1} - \lambda \Delta m_t = 0 \], (Sticky Prices)

where \( m_t \) denotes the log of the money supply, \( p_t \) denotes the log price level, \( e_t \) denotes the log of dollars per unit of foreign currency, and \( R_t \) denotes the net domestic rate of interest. The first equation is the UIP condition, the second is the money demand equation with output set to zero and interest elasticity equal to \( \alpha \). This equation has been quasi-differenced using the operator, \( \Delta \). Thus, \( \Delta p_t = p_t - \delta p_{t-1} \), and \( \delta \) is a number close to, but not necessarily equal to, unity.
The log money stock, \( m_t \), is driven by a permanent, \( P \), and a transitory component, \( T \):
\[
m_t = m_t^T + m_t^P,
\]
where
\[
m_t^T = \rho m_{t-1}^T + \varepsilon_t
\]
\[
\Delta m_t^P = \phi \Delta m_{t-1}^P + u_t.
\]
When \( \delta = 1 \), this is the classic permanent/temporary decomposition. The univariate time series representation of \( m_t \) is obtained by first multiplying \( m_t \) by \( (1 - \rho L)(1 - \phi L) \Delta \):
\[
(1 - \rho L)(1 - \phi L) \Delta m_t = (1 - \phi L)(\varepsilon_t - \delta \varepsilon_{t-1}) + (1 - \rho L) u_t
\]
(1.2)
\[
\Delta \varepsilon_t = \eta_t + \theta_1 \eta_{t-1} + \theta_2 \eta_{t-2}.
\]
(1.3)
say, where \( \eta_t \) is the one-step-ahead forecast error in \( \Delta m_t \) based on using only past \( \Delta m_t \) to forecast \( \Delta m_t \). (The representation involving \( \eta_t \), (1.3), is called the Wold representation of \( \Delta m_t \).) The following section derives formulas that can be used to compute \( \theta_1, \theta_2 \) and the variance of \( \eta_t, \sigma^2_\eta \), using as input \( \delta, \phi, \rho, \sigma^2_\varepsilon, \sigma^2_u \). That this is a Wold representation requires that the roots of the moving average representation lie inside the unit circle. The section after that solves the model, and simulates it.

2. The Univariate Representation of Money Growth

Write the error term in (1.2) in detail:
\[
\varepsilon_t - \delta \varepsilon_{t-1} - \phi \varepsilon_{t-2} + (1 - \rho L) u_t = \varepsilon_t - (\delta + \phi) \varepsilon_{t-1} + \delta \phi \varepsilon_{t-2} + u_t - \rho u_{t-1}
\]
Let \( c_\varepsilon (\tau) \) and \( c_u (\tau) \) denote the covariance function of the term involving \( \varepsilon_t \)'s and \( u_t \)'s, respectively. Note that the former is zero for all \( \tau > 2 \) and the latter is zero for all \( \tau > 1 \). Then,
\[
c_\varepsilon (0) = E [\varepsilon_t - (\delta + \phi) \varepsilon_{t-1} + \delta \phi \varepsilon_{t-2}]^2
= \sigma^2_\varepsilon [1 + (\delta + \phi)^2 + (\delta \phi)^2]
\]
\[
c_\varepsilon (1) = E [\varepsilon_t - (\delta + \phi) \varepsilon_{t-1} + \delta \phi \varepsilon_{t-2}] [\varepsilon_{t-1} - (\delta + \phi) \varepsilon_{t-2} + \delta \phi \varepsilon_{t-3}]
= -[(\delta + \phi) + \delta \phi (\delta + \phi)] \sigma^2_\varepsilon = - (\delta + \phi) (1 + \delta \phi) \sigma^2_\varepsilon
\]
\[
c_\varepsilon (2) = \delta \phi \sigma^2_\varepsilon.
\]
\[
c_u (0) = \sigma^2_u (1 + \rho^2)
\]
\[
c_u (1) = -\rho \sigma^2_u.
\]
Then, the covariance function of the whole error is \( c (\tau) \) for \( \tau \geq 0 \):
\[
c (\tau) = c_\varepsilon (\tau) + c_u (\tau), \text{ all } \tau.
\]
With \( c (\tau) \) in hand, we can compute the parameters of the \( \eta_t \) process. We have that:
\[
c (0) = E [\eta_t + \theta_1 \eta_{t-1} + \theta_2 \eta_{t-2}]^2
= \sigma^2_\eta [1 + \theta_1^2 + \theta_2^2]
\]
\[
c (1) = E [\eta_t + \theta_1 \eta_{t-1} + \theta_2 \eta_{t-2}] [\eta_{t-1} + \theta_1 \eta_{t-2} + \theta_2 \eta_{t-3}]
= \theta_1 (1 + \theta_2) \sigma^2_\eta
\]
\[
c (2) = \theta_2 \sigma^2_\eta.
\]
Then, solve the following two equations for \( \theta_1 \) and \( \theta_2 \):

\[
\begin{align*}
\frac{c(2)}{c(1)} &= \frac{\theta_2}{\theta_1 (1 + \theta_2)} \\
\frac{c(1)}{c(0)} &= \frac{\theta_1 (1 + \theta_2)}{1 + \theta_1^2 + \theta_2^2},
\end{align*}
\]

subject to the eigenvalue condition. Finally,

\[
\sigma^2_\eta = \frac{c(1)}{\theta_1 (1 + \theta_2)}.
\]

The eigenvalue condition is expressed as follows. Write:

\[
1 + \theta_1 L + \theta_2 L^2 = (1 - \delta_1 L)(1 - \delta_2 L).
\]

We require that \( \delta_1, \delta_2 \) be less than unity in absolute value. For every \( \rho, \delta, \sigma_\varepsilon, \sigma_u \), there is always such a \( \theta_1, \theta_2 \).

We have the following connection between the one-step-ahead forecast errors based on the univariate representation of \( m_t, \eta_t \), and the one-step-ahead forecast errors in the underlying fundamental shocks:

\[
\eta_t = -\theta_1 \eta_{t-1} - \theta_2 \eta_{t-2} + (1 - \phi L) (\varepsilon_t - \delta \varepsilon_{t-1}) + (1 - \rho L) u_t.
\]

Note that if there is a disturbance in \( u_t \), this will produce a serially correlated sequence of one-step-ahead forecast errors in \( \eta_t \)'s. The intuition is obvious. A jump in \( u_t \) will initially induce a same-magnitude jump in \( \eta_t \). But, \( u_t \) induces a persistent move in \( m_t \), via \( m_t^P \). To the extent that \( \sigma_\varepsilon > 0 \), so that a rational forecaster will attribute some possibility to the source of the disturbance being \( \varepsilon_t \), the dynamic move in \( m_t \) will be interpreted as a sequence of same-sign shocks in \( \varepsilon_t \). However, as time evolves, such a sequence becomes less likely and eventually it is ‘learned’ that the source of the shock in fact was \( u_t \). Throughout this period the rational forecaster makes same-sign one-step-ahead forecast errors in forecasting \( m_t \).

Here is a (rather extreme) example:

\[
\sigma^2_\varepsilon = 1, \quad \rho = 0, \quad \delta = 0.99, \quad \phi = 0.8, \quad \sigma^2_u = 0.01.
\]

A unit disturbance in \( u_t \) pushes up the one-step-ahead forecast error in \( \Delta m_t \) by unity. Since most of the variance in the series comes from \( \varepsilon_t \), a rational forecaster will assume the shock came from \( \varepsilon_t \). However, since in fact it came from \( u_t \), \( \Delta m_{t+1} \) will be surprisingly high too. So, there will be another shock then, as the forecaster assumes \( \varepsilon_{t+1} \) must have been high too, and so on. A graph of \( \eta_t \) in response to the unit shock in \( u_t \), as well as the response in \( \Delta m_t \) appears
Note how big and long-lasting the one-step-ahead forecast errors are. In some sense, the rational forecaster never learns! They continue to be surprised all along the way. It is interesting to see how the interest rate and exchange rate respond in this case. Note how the exchange rate
depreciates with the drop in the interest rate, and it continues to depreciate for 10 periods!

For comparison, consider the opposite extreme, where most of the variance in $\Delta m_t$ comes
from the permanent component. Thus, $\sigma_u^2 = 1$ and $\sigma_\varepsilon^2 = 0.01$. Then,

The picture now is quite different. In the period of the shock, the forecast error is just the surprise in the money growth rate. In the next period, there is a bit of a forecast error, because they assign some probability to the possibility that the shock was temporary. However, after the second period, they've caught on and there is no longer any forecast error. The rational forecaster correctly forecasts the implications of the surprise in $\Delta m_t$, because the surprise comes from the place it usually comes from. The following picture shows what happens to the exchange.
rate and the interest rate in this example:

Note that the exchange rate does depreciate for one period. But, right after that it resorts to the pattern one would expect.

3. Solving and Simulating the Model

To simulate this model, it is appropriate to do so using the univariate time series representation for $\Delta m_t$:

$$\Delta m_t = (\rho + \phi) \Delta m_{t-1} - \rho \phi \Delta m_{t-2} + \eta_t + \theta_1 \eta_{t-1} + \theta_2 \eta_{t-2}.$$ 

It is convenient to express $\Delta m_t$ as a vector first order autoregression:

$$\begin{pmatrix} \Delta m_t \\ \Delta m_{t-1} \\ \eta_t \\ \eta_{t-1} \end{pmatrix} = \begin{pmatrix} \rho + \phi & -\rho \phi & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta m_{t-1} \\ \Delta m_{t-2} \\ \eta_{t-1} \\ \eta_{t-2} \end{pmatrix} + \begin{pmatrix} \eta_t \\ 0 \\ 0 \\ \eta_t \end{pmatrix},$$

or,

$$s_t = P s_{t-1} + \begin{pmatrix} \eta_t \\ 0 \\ \eta_t \\ 0 \end{pmatrix},$$
where \( \eta_t \) is iid over time, with variance \( \sigma^2_\eta \). Let

\[
z_t = \begin{pmatrix} e_t \\ \Delta p_t \\ R_t \end{pmatrix},
\]

and write the system of three equations in matrix form as follows:

\[
E_t [\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0,
\]

where

\[
\alpha_0 = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} -1 & 0 & -\beta \\ 0 & -1 & \alpha \\ 0 & 1 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta \alpha \\ 0 & -(1 - \lambda) & 0 \end{bmatrix}
\]

\[
\beta_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\lambda & 0 & 0 \end{bmatrix}
\]

Given this solution and given a random draw on \( \eta_t \), the system can be simulated for \( T \) observations:

\[
s_t = Ps_{t-1} + \begin{pmatrix} \eta_t \\ 0 \\ 0 \end{pmatrix},
\]

\[
z_t = Az_{t-1} + Bs_t.
\]

The change in the exchange rate, \( e(t+1) - e(t) \), can be recovered from the first element in \( z_{t+1} \) and \( z_t \) and the interest rate, \( R_t \), can be obtained from the third element of \( z_t \). Then, the slope in the regression of \( e(t+1) - e(t) \) on \( R_t \) can be computed from:

\[
\hat{\beta} = \frac{\text{cov}(e(t+1) - e(t), R_t)}{\text{var}(R_t)}.
\]

This was done for the following parameterization:

\[
\alpha = 0.1, \quad \sigma^2_\varepsilon = 0.1, \quad \rho = 0, \quad \delta = 0.99, \quad \phi = 0.9, \quad \sigma^2_u = 0.01, \quad \lambda = 0.01
\]
After doing 10,000 replications with $T = 200$ data points, the following histogram of $\hat{\beta}$’s resulted:

Interestingly, here there is actually an upward bias. The mean, across 10,000 replications, of $\hat{\beta}$, is 1.42. The standard deviation huge, 0.85. However, this should not be interpreted to suggest that negative $\hat{\beta}$’s are possible. As the histogram makes clear, the distribution is skewed to the right. Very high numbers are possible, but not very low numbers, such as the ones estimated in the data. Interestingly,

Note that there is little bias here. Consider now the ‘best case scenario’ for the big bias, the example in which

$$\beta = .99, \delta = 0.99, \alpha = .1, \lambda = 0.01, \rho = 0, \phi = 0.8, \sigma^2_u = 0.01, \sigma^2_\varepsilon = 1$$

In this case, the mean (across 10,000 simulations) of $\hat{\beta}$ is 1.20, and the standard deviation is
0.51. The histogram of the $\hat{\beta}$’s is

The point is that the considerations raised here do nothing to resolve the UIP puzzle.