Foundations for the New Keynesian Model

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Objective

- Describe a very simple model economy with no monetary frictions.
  - Describe its properties.
  - ‘markets work well’

- Modify the model to include price setting frictions.
  - Now markets won’t necessarily work so well, unless monetary policy is good.

Model

- Household preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\},$$

$$\tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau},\ \varepsilon_t^{\tau} \sim iidN(0, \sigma_\varepsilon^2)$$

Production

- Final output requires lots of intermediate inputs:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}},\ \varepsilon > 1$$

- Production of intermediate inputs:

$$Y_{i,t} = e^{a_t} N_{i,t},\ \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a,\ \varepsilon_t^a \sim iidN(0, \sigma_a^2)$$

- Constraint on allocation of labor:

$$\int_0^1 N_{it} di = N_t$$

Efficient Allocation of Total Labor

- Suppose total labor, N_t , is fixed.

- What is the best way to allocate N_t among the various activities, $0 \leq i \leq 1$?

- Answer:
  - allocate labor equally across all the activities

$$N_{it} = N_t, \text{ all } i$$

Suppose Labor *Not* Allocated Equally

- Example:

$$N_{it} = \begin{cases} 2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\ 2(1-\alpha)N_t & i \in \left[\frac{1}{2}, 1\right] \end{cases}, \; 0 \leq \alpha \leq 1.$$

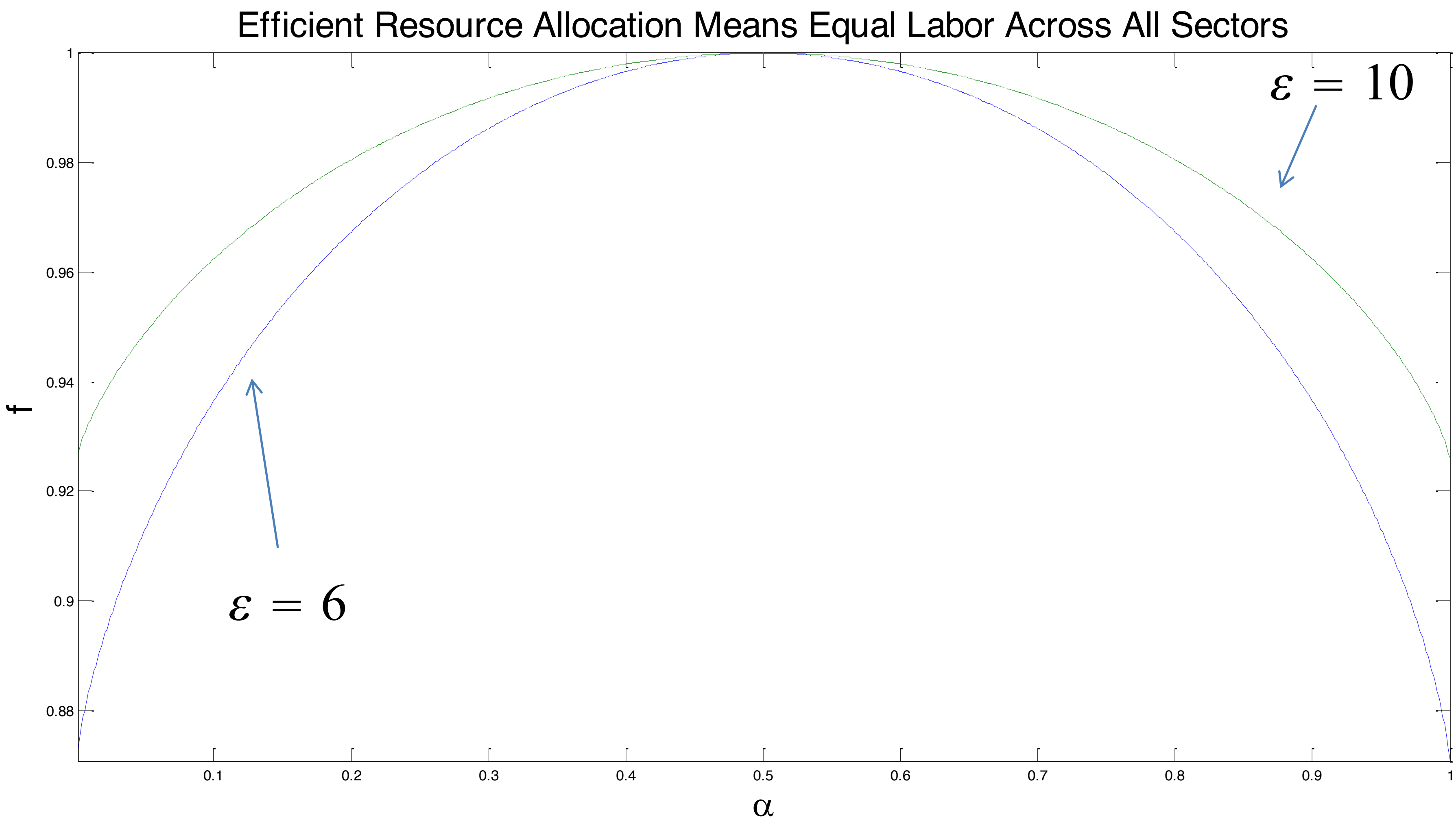
- Note that this is a particular distribution of labor across activities:

$$\int_0^1 N_{it} di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2(1-\alpha)N_t = N_t$$

Labor *Not* Allocated Equally, cnt'd

$$
\begin{aligned}
Y_t &= \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\int_0^{\frac{1}{2}} Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} \left[\int_0^{\frac{1}{2}} N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} \left[\int_0^{\frac{1}{2}} (2\alpha N_t)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 (2(1-\alpha)N_t)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} N_t \left[\int_0^{\frac{1}{2}} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} N_t \left[\frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} N_t f(\alpha)
\end{aligned}
$$

$$f(\alpha) = \left[\frac{1}{2}(2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2}(2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$



Economy with Efficient *N* Allocation

- Efficiency dictates

$$N_{it} = N_t \text{ all } i$$

- So, with efficient production:

$$Y_t = e^{a_t} N_t$$

- Resource constraint:

$$C_t \leq Y_t$$

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right),\ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau},\ \varepsilon_t^{\tau} \sim iid,$$

Efficient Determination of Labor

- Lagrangian:

$$\max_{C_t,N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \overbrace{u(C_t,N_t,\tau_t)}^{=\log C_t - \exp(\tau_t)\frac{N_t^{1+\varphi}}{1+\varphi}} + \lambda_t [e^{a_t} N_t - C_t] \right\}$$

- First order conditions:

$$u_c(C_t,N_t,\tau_t) = \lambda_t,\ u_n(C_t,N_t,\tau_t) + \lambda_t e^{a_t} = 0$$

- or:

$$u_{n,t} + u_{c,t} e^{a_t} = 0$$

$$\overbrace{\frac{-u_{n,t}}{u_{c,t}}}^{\text{marginal cost of labor in consumption units} = \frac{-\frac{du}{dN_t}}{\frac{du}{dC_t}} = \frac{dC_t}{dN_t}} = \overbrace{e^{a_t}}^{\text{marginal product of labor}}$$

Efficient Determination of Labor, cont'd

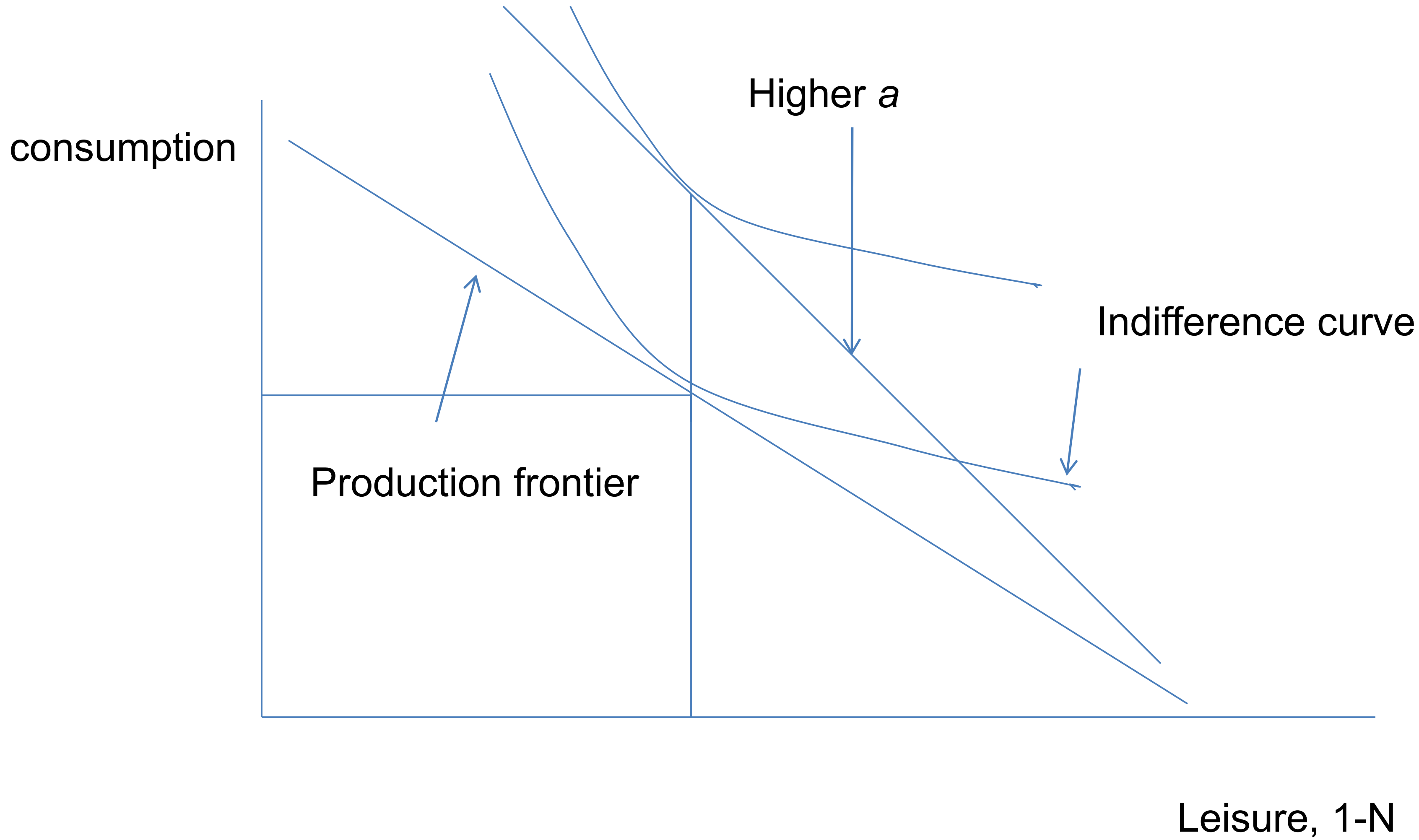
- Solving the fonc's:

$$
\begin{aligned}
\frac{-u_{n,t}}{u_{c,t}} &= e^{a_t} \\
C_t \exp(\tau_t) N_t^{\varphi} &= e^{a_t} \\
e^{a_t} N_t \exp(\tau_t) N_t^{\varphi} &= e^{a_t}
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow N_t &= \exp\left(\frac{-\tau_t}{1+\varphi}\right) \\
\rightarrow C_t &= \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)
\end{aligned}
$$

- Note:
  - Labor responds to preference shock, *not* to tech shock

Response to a Jump in *a*



Decentralizing the Model

- Give households budget constraints and place them in markets.

- Give the production functions to firms and suppose that they seek to maximize profits.

Households

- Solve:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\},$$

- Subject to:

$$C_t + \overbrace{B_{t+1}}^{\text{bonds purchases in } t} \leq \overbrace{w_t}^{\text{wage rate}} N_t + \overbrace{\pi_t}^{\text{profits}} + \overbrace{r_{t-1}}^{\text{(real) interest on bonds}} B_t$$

- First order conditions:

$$\frac{-u_{n,t}}{u_{c,t}} = C_t \exp(\tau_t) N_t^{\varphi} = w_t$$ ‘marginal cost of working equals marginal benefit’

$$u_{c,t} = \beta E_t u_{c,t+1} r_t$$ ‘marginal cost of saving equals marginal benefit’

Final Good Firms

- Final good firms buy $Y_{i,t}$, $i \in (0,1)$, at given prices, $P_{i,t}$, to maximize profits:

$$Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Subject to

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

- Fonc's:

$$P_{i,t} = \left(\frac{Y_t}{Y_{i,t}} \right)^{\frac{1}{\varepsilon}}$$

$$\rightarrow Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t, \; 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$

Intermediate Good Firms

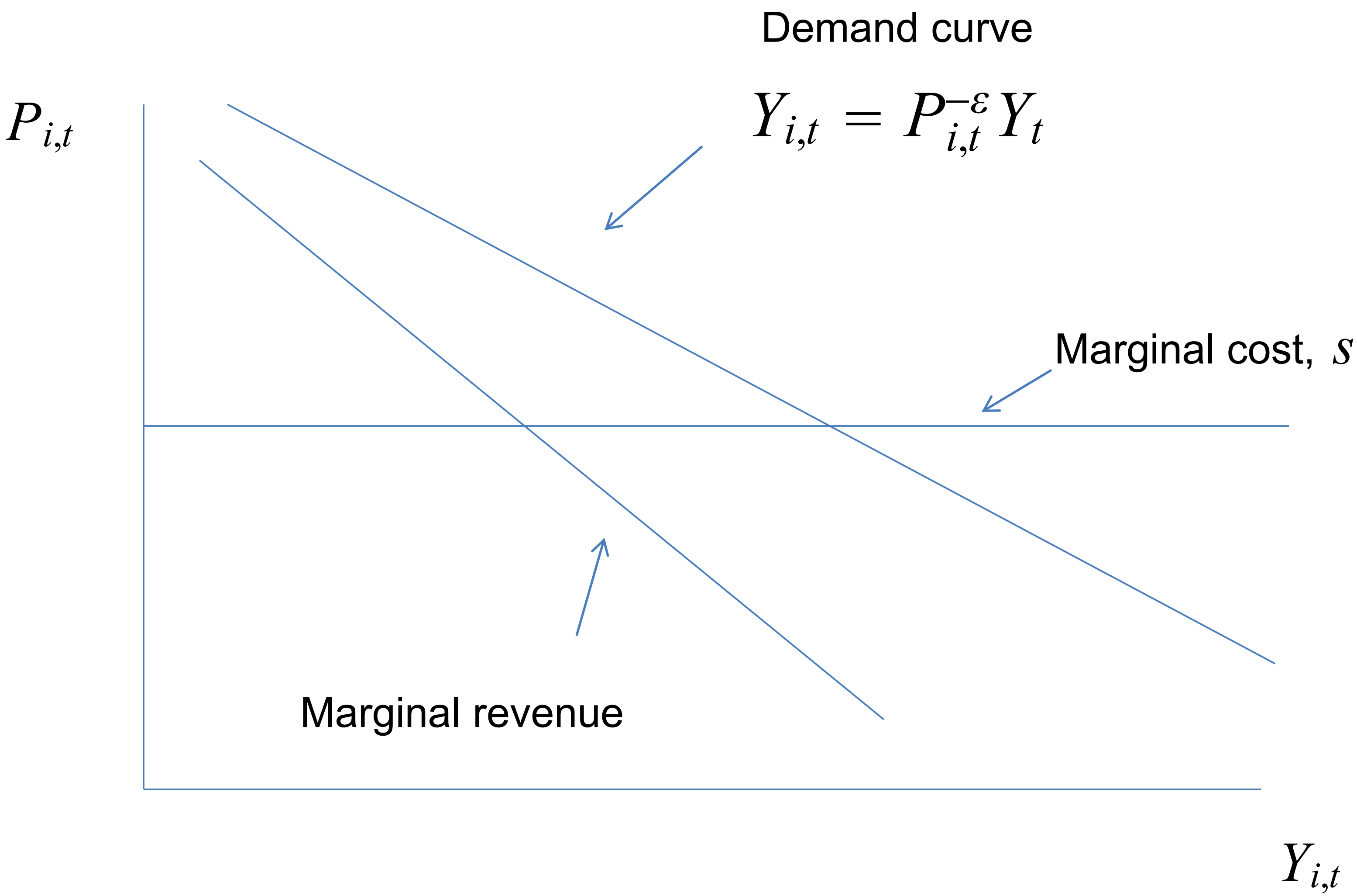
- For each $Y_{i,t}$ there is a single producer who is a monopolist in the product market and hires labor, $N_{i,t}$ in competitive labor markets.

- Marginal cost of production:

$$\text{(real) marginal cost} = s_t = \frac{\frac{dCost}{dworker}}{\frac{doutput}{dworker}} = \frac{\left(1 - \overbrace{\nu}^{\text{subsidy payment to firm}}\right) w_t}{\exp(a_t)}$$

- Subsidy will be required to ensure markets work efficiently.

Intermediate Good Firms



i*th* Intermediate Good Firm

- Problem: $\max_{N_{it}} P_{it} Y_{it} - s_t Y_{it}$

- Subject to demand for $Y_{i,t}$: $Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t$

- Problem:

$$\max_{N_{it}} P_{it} P_{i,t}^{-\varepsilon} Y_t - s_t P_{i,t}^{-\varepsilon} Y_t$$

fonc : $(1-\varepsilon) P_{it}^{-\varepsilon} Y_t + \varepsilon s_t P_{i,t}^{-\varepsilon-1} Y_t = 0$

$P_{it} = \frac{\varepsilon}{\varepsilon - 1} s_t$ ‘price is markup over marginal cost’

- Note: all prices are the same, so resources allocated efficiently across intermediate good firms.

$P_{i,t} = P_{j,t} = 1,$ because $1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$

Equilibrium

- Pulling things together:

$$1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1-\nu) w_t}{\exp(a_t)}$$

$$\overbrace{=}^{\text{household fonc}} \quad \frac{\varepsilon(1-\nu)}{\varepsilon - 1} \frac{\frac{-u_{n,t}}{u_{c,t}}}{\exp(a_t)}$$

$$\overbrace{=}^{\text{if } \frac{\varepsilon(1-\nu)}{\varepsilon-1} = 1} \quad \frac{\frac{-u_{n,t}}{u_{c,t}}}{\exp(a_t)}.$$

If proper subsidy is provided to monopolists, employment is efficient:

$$\text{if } 1 - \nu = \frac{\varepsilon - 1}{\varepsilon}, \text{ then } \frac{-u_{n,t}}{u_{c,t}} = \exp(a_t)$$

Equilibrium Allocations

- With efficient subsidy,

$$\frac{-u_{n,t}}{u_{c,t}} \overbrace{=}^{\text{functional form}} C_t \exp(\tau_t) N_t^{\varphi} \overbrace{=}^{\text{resource constraint}} \exp(a_t)\exp(\tau_t) N_t^{1+\varphi} = \exp(a_t)$$

$$\rightarrow N_t = \exp\left(\frac{-\tau_t}{1+\varphi}\right)$$

$$C_t = e^{a_t} N_t = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)$$

- Bond market clearing implies:

$B_t = 0$ always

Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t$$

$$\rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp\left[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right]}$$

$$= \frac{1}{\beta \exp\left[E_t\left(-\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right) + \frac{1}{2}V\right]}, \; V = \sigma_a^2 + \left(\frac{1}{1+\varphi}\right)^2 \sigma_\lambda^2$$

$$\log r_t = -\log\beta + E_t\left(\overbrace{\Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1+\varphi}}^{c_{t+1}-c_t}\right) + \frac{1}{2}V$$

Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t$$

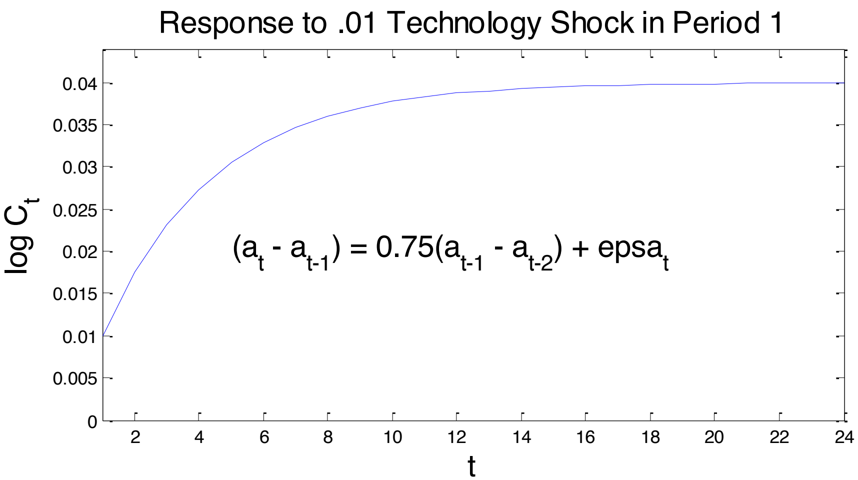
$$\rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp\left[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right]}$$

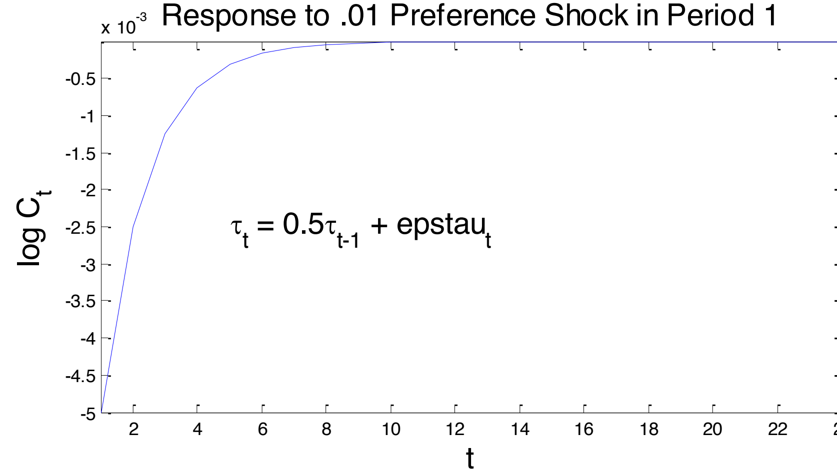
$$= \frac{1}{\beta \exp\left[E_t\left(-\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right) + \frac{1}{2}V\right]}, \; V = \sigma_a^2 + \left(\frac{1}{1+\varphi}\right)^2 \sigma_\lambda^2$$

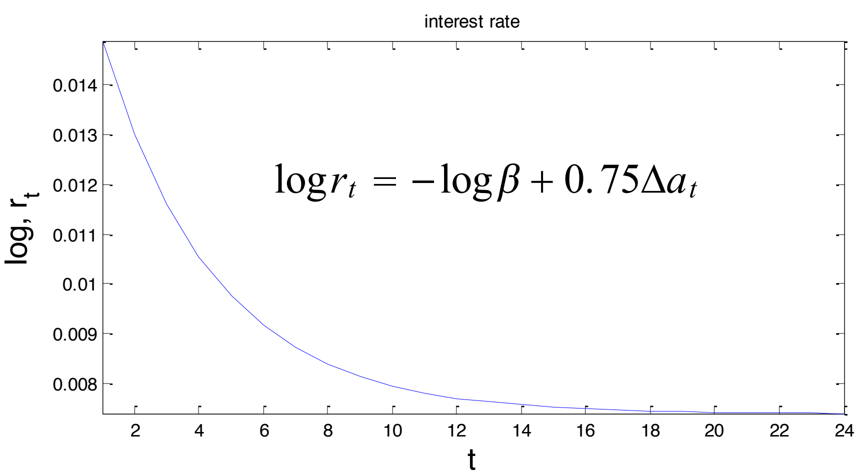
$$\log r_t = -\log\beta + E_t\left(\overbrace{\Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1+\varphi}}^{c_{t+1} - c_t}\right) + \frac{1}{2}V$$

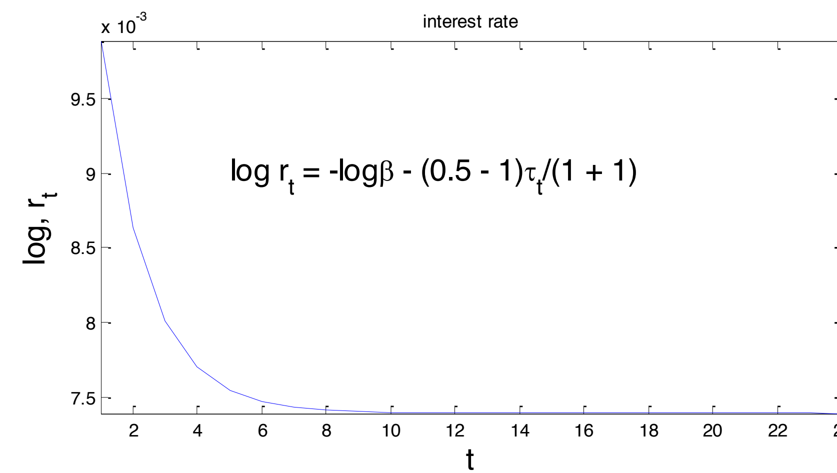
$$\overbrace{=}^{\text{using assumptions about } \Delta a_t \text{ and } \tau_t} -\log\beta + \rho\Delta a_t - \frac{(\lambda-1)\tau_t}{1+\varphi} + \frac{1}{2}V$$

Dynamic Properties of the Model









Key Features of Equilibrium Allocations

- Allocations *efficient* (as long as monopoly power neutralized)
- Employment does not respond to technology
  - Improvement in technology raises marginal product of labor and marginal cost of labor by same amount.
- First best consumption not a function of intertemporal considerations
  - Discount rate irrelevant.
  - Anticipated future values of shocks irrelevant.
- Natural rate of interest steers consumption and employment towards their natural levels.

Introducing Price Setting Frictions (Clarida-Gali-Gertler Model)

- Households maximize:

$$E_0 \sum_{t=0}^{\infty} \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \; \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}, \; \varepsilon_t^{\tau} \sim iid,$$

- Subject to:

Profits, net of taxes raised by Government to finance subsidies.

$$P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + T_t$$

- Intratemporal first order condition:

$$C_t \exp(\tau_t) N_t^{\varphi} = \frac{W_t}{P_t}$$

Household Intertemporal FONC

- Condition:

$$1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \frac{R_t}{1+\pi_{t+1}}$$

– or

$$\begin{aligned} 1 &= \beta E_t \frac{C_t}{C_{t+1}} \frac{R_t}{1+\pi_{t+1}} \\ &= \beta E_t \exp[\log(R_t) - \log(1+\pi_{t+1}) - \Delta c_{t+1}] \\ &\simeq \beta \exp[\log(R_t) - E_t \pi_{t+1} - E_t \Delta c_{t+1}],\ c_t \equiv \log(C_t) \end{aligned}$$

– take log of both sides:

$$0 = \log(\beta) + r_t - E_t \pi_{t+1} - E_t \Delta c_{t+1},\ r_t = \log(R_t)$$

– or

$$c_t = -\log(\beta) - [r_t - E_t \pi_{t+1}] + c_{t+1}$$

Final Good Firms

- Buy $Y_{i,t},\ i \in [0,1]$ at prices $P_{i,t}$ and sell Y_t for P_t
- Take all prices as given (competitive)
- Profits:

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Production function:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} di,\ \varepsilon > 1,$$

- First order condition:

$$Y_{i,t} = Y_t \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \qquad \rightarrow P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}$$

Intermediate Good Firms

- Each ith good produced by a single monopoly producer.
- Demand curve:

$$Y_{i,t} = Y_t\left(\frac{P_{i,t}}{P_t}\right)^{-\varepsilon}$$

- Technology:

$$Y_{i,t} = \exp(a_t)N_{i,t},\ \Delta a_t = \rho\Delta a_{t-1} + \varepsilon_t^a,$$

- Calvo Price-setting Friction

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1-\theta \\ P_{i,t} & \text{with probability } \theta \end{cases},$$

Marginal Cost

real marginal cost $=$ $$s_t = \frac{\frac{dCost}{dworker}}{\frac{dOutput}{dworker}} = \frac{(1-\nu)W_t/P_t}{\exp(a_t)}$$

$$= \frac{\overbrace{(1-\nu)}^{=\frac{\varepsilon-1}{\varepsilon}\text{ in efficient setting}} \quad C_t \exp(\tau_t) N_t^{\varphi}}{\exp(a_t)}$$

The Intermediate Firm's Decisions

- i*th* firm is required to satisfy whatever demand shows up at its posted price.

- Its only real decision is to adjust price whenever the opportunity arises.

Intermediate Good Firm

- Present discounted value of firm profits:

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\upsilon_{t+j}}^{\text{marginal value of dividends to household}=u_{c,t+j}/P_{t+j}} \overbrace{\left[\overbrace{P_{i,t+j} Y_{i,t+j}}^{\text{revenues}} - \overbrace{P_{t+j} s_{t+j} Y_{i,t+j}}^{\text{total cost}} \right]}^{\text{period } t+j \text{ profits sent to household}}$$

- Each of the $1-\theta$ firms that can optimize price choose $\tilde{P}_t$ to optimize

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\theta^j}^{\substack{\text{in selecting price, firm only cares about} \\ \text{future states in which it can't reoptimize}}} \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}].$$

Intermediate Good Firm Problem

- Substitute out the demand curve:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}]$$

$$= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon}].$$

- Differentiate with respect to $\tilde{P}_t$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [(1-\varepsilon)(\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1}] = 0,$$

- or

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

Intermediate Good Firm Problem

- Objective:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j P_{t+j}^{\varepsilon} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_t},\ X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j} \bar{\pi}_{t+j-1} \cdots \bar{\pi}_{t+1}},\ j \geq 1 \\ 1,\ j = 0. \end{cases} \quad ,\ X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}},\ j > 0$$

Intermediate Good Firm Problem

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \tfrac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0$$

- Solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \tfrac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

- But, still need expressions for K_t, F_t.

$$
\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left(\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta \overbrace{E_t E_{t+1}}^{=E_t \text{ by LIME}} \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \overbrace{E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j}}^{\text{exactly } K_{t+1}!} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}
\end{aligned}
$$

- From previous slide:

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

- Substituting out for marginal cost:

$$\frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\overbrace{W_t / P_t}^{d\text{Cost}/d\text{labor}}}{\underset{\displaystyle \overbrace{\exp(a_t)}}{d\text{Output}/d\text{labor}}}$$

$$= \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\overbrace{\exp(\tau_t) N_t^{\varphi} C_t}^{= \frac{W_t}{P_t} \text{ by household optimization}}}{\exp(a_t)}.$$

In Sum

- solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},$$

- Where:

$$K_t = (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t) N_t^{\varphi} C_t}{\exp(a_t)} + \beta\theta E_t \left(\frac{1}{\bar{\bar{\pi}}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

$$F_t \equiv E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta\theta E_t \left(\frac{1}{\bar{\bar{\pi}}_{t+1}} \right)^{1-\varepsilon} F_{t+1}$$

To Characterize Equilibrium

- Have equations characterizing optimization by firms and households.
- Still need:
  - Expression for all the prices. Prices, $P_{i,t},\ 0 \leq i \leq 1$, will all be different because of the price setting frictions.
  - Relationship between aggregate employment and aggregate output not simple because of price distortions:

$$Y_t \neq e^{a_t} N_t, \text{ in general}$$

- This part of the analysis is the reason why it made Calvo famous – it's not easy.

Aggregate Price Index

- Rewrite the aggregate price index.
  - let $p \in (0, \infty)$ the set of logically possible prices for intermediate good producers.
  - let $g_t(p) \geq 0$ denote the measure (e.g., 'number') of producers that have price, p, in t
  - let $g_{t-1,t}(p) \geq 0$, denote the measure of producers that had price, p, in $t-1$ and could not reoptimize in t
- Then,

$$P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} = \left(\int_0^\infty g_t(p)\, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

- Note:

$$P_t = \left((1-\theta)\, \tilde{P}_t^{1-\varepsilon} + \int_0^\infty g_{t-1,t}(p)\, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

Aggregate Price Index

- Calvo randomization assumption:

$$
\begin{aligned}
\overbrace{g_{t-1,t}(p)}^{\text{measure of firms that had price, } p, \text{ in } t-1 \text{ and could not change}} \\
= \theta \times \overbrace{g_{t-1}(p)}^{\text{measure of firms that had price } p \text{ in } t-1}
\end{aligned}
$$

- Then,

$$
\begin{aligned}
P_t &= \left((1-\theta)\tilde{P}_t^{1-\varepsilon} + \int_0^\infty g_{t-1,t}(p)\, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}} \\
&= \left((1-\theta)\tilde{P}_t^{1-\varepsilon} + \theta \overbrace{\int_0^\infty g_{t-1}(p)\, p^{(1-\varepsilon)} dp}^{=P_{t-1}^{1-\varepsilon}} \right)^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

Expression for $\tilde{p}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[(1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

  – Only two variables here!

- Divide by P_t:

$$1 = \left[(1-\theta)\tilde{p}_t^{(1-\varepsilon)} + \theta\left(\frac{1}{\bar{\pi}_t}\right)^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{p}_t = \left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}$$

Relation Between Aggregate Output and Aggregate Inputs

- Technically, there is no ‘aggregate production function’ in this model
  - If you know how many people are working, N, and the state of technology, a, you don’t have enough information to know what Y is.
  - Price frictions imply that resources will not be efficiently allocated among different inputs.
    - Implies Y low for given a and N. How low?
    - Tak Yun (JME) gave a simple answer.

Tak Yun Algebra

$$Y_t^* = \int_0^1 Y_{i,t} di \left(= \int_0^1 A_t N_{i,t} di \overbrace{=}^{\text{labor market clearing}} A_t N_t \right)$$

$$\overbrace{=}^{\text{demand curve}} Y_t \int_0^1 \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} \int_0^1 (P_{i,t})^{-\varepsilon} di$$

Calvo insight

$$= Y_t P_t^{\varepsilon} (P_t^*)^{-\varepsilon}$$

- Where: $P_t^* \equiv \left[\int_0^1 P_{i,t}^{-\varepsilon} di \right]^{\frac{-1}{\varepsilon}} = [(1-\theta)\tilde{P}_t^{-\varepsilon} + \theta(P_{t-1}^*)^{-\varepsilon}]^{\frac{-1}{\varepsilon}}$

Relationship Between Agg Inputs and Agg Output

- Rewriting previous equation:

$$Y_t = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} Y_t^*$$

$$= p_t^* e^{a_t} N_t,$$

- 'efficiency distortion':

$$p_t^* : \begin{cases} \leq 1 \\ = 1 \quad P_{i,t} = P_{j,t}, \text{ all } i,j \end{cases}$$

Tack Yun Distortion

- Let $f(x) = x^4$, a convex function. Then,

$$\text{convexity: } \alpha x_1^4 + (1-\alpha) x_2^4 > (\alpha x_1 + (1-\alpha) x_2)^4$$

  for $x_1 \neq x_2$, $0 < \alpha < 1$.

- Applying this idea to prices:

$$\text{convexity: } \int_0^1 \left(P_{i,t}^{(1-\varepsilon)}\right)^{\frac{\varepsilon}{\varepsilon-1}} di \ \geq \ \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\Longleftrightarrow \left(\int_0^1 P_{i,t}^{-\varepsilon} di\right) \ \geq \ \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\Longleftrightarrow \overbrace{\left(\int_0^1 P_{i,t}^{-\varepsilon} di\right)^{\frac{-1}{\varepsilon}}}^{P_t^*} \ \leq \ \overbrace{\left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{1}{1-\varepsilon}}}^{P_t}$$

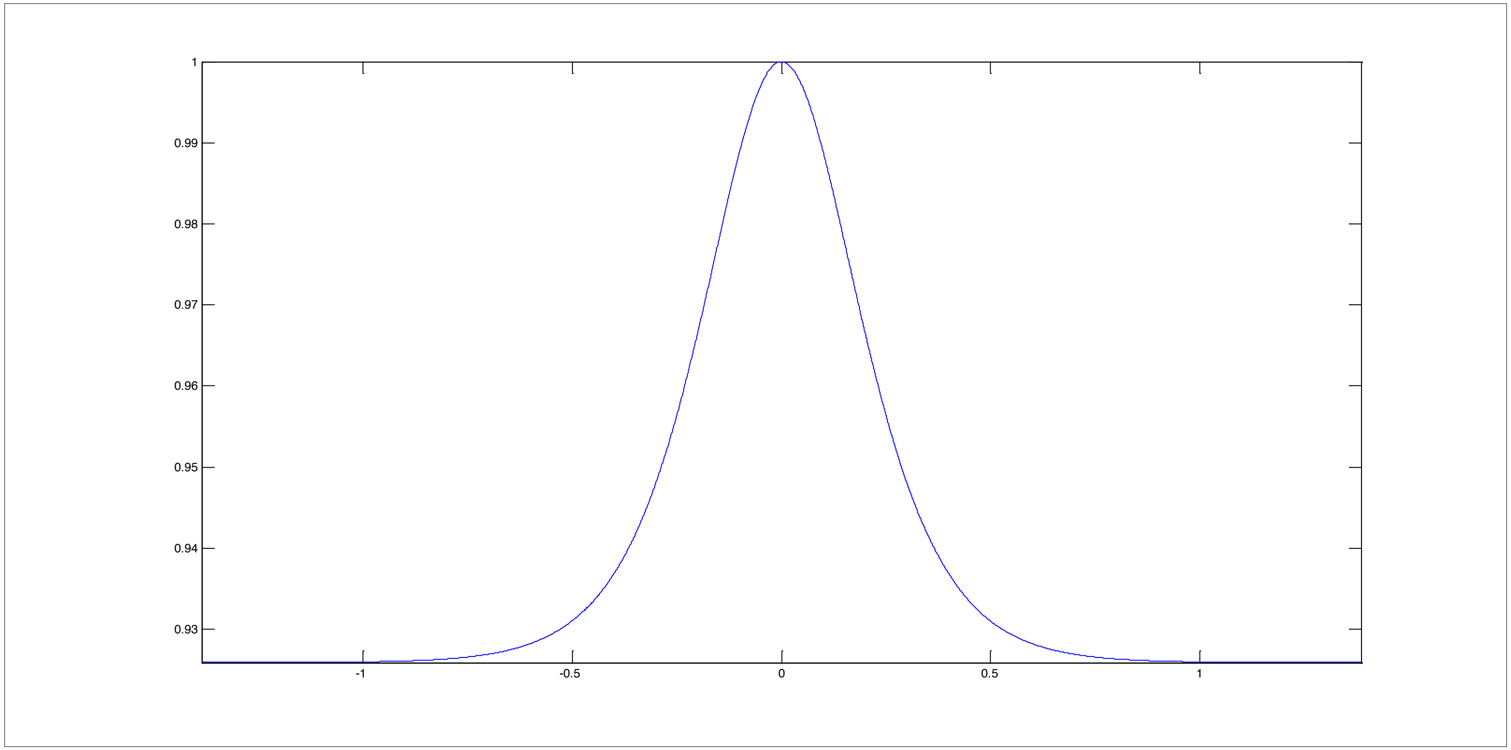
Example of Efficiency Distortion

$$P_{j,t} = \begin{cases} P^1 & 0 \le j \le \alpha \\ P^2 & \alpha \le j \le 1 \end{cases} . \; p_t^* = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} = \left(\frac{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}}{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}} \right)^{\varepsilon}$$

Example of Efficiency Distortion

$$P_{j,t} = \begin{cases} P^1 & 0 \le j \le \alpha \\ P^2 & \alpha \le j \le 1 \end{cases} \quad . \; p_t^* = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} = \left(\frac{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}}{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}\right)^{\varepsilon}$$

$\alpha = 0.5, \varepsilon = 10$



$\log P^1/P^2$

Collecting Equilibrium Conditions

- Price setting:

$$K_t = (1-\nu)\frac{\varepsilon}{\varepsilon-1}\frac{\exp(\tau_t)N_t^{\varphi}C_t}{A_t} + \beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon}K_{t+1} \quad (1)$$

$$F_t = 1 + \beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon-1}F_{t+1} \quad (2)$$

- Intermediate good firm optimality and restriction across prices:

$$\overbrace{\frac{K_t}{F_t}}^{=\tilde{p}_t \text{ by firm optimality}} = \overbrace{\left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}}^{=\tilde{p}_t \text{ by restriction across prices}} \quad (3)$$

Equilibrium Conditions

- Law of motion of (Tak Yun) distortion:

$$p_t^* = \left[(1-\theta)\left(\frac{1-\theta \bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \quad (4)$$

- Household Intertemporal Condition:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

- Aggregate inputs and output:

$$C_t = p_t^* e^{a_t} N_t \quad (6)$$

- 6 equations, 8 unknowns:

$$v,\ C_t,\ p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t$$

- System under determined!

Underdetermined System

- Not surprising: we added a variable, the nominal rate of interest.
- Also, we're counting subsidy as among the unknowns.
- Have two extra policy variables.
- One way to pin them down: compute optimal policy.

Ramsey-Optimal Policy

- 6 equations in 8 unknowns…..
  - Many configurations of the 8 unknowns that satisfy the 6 equations.
  - Look for the best configurations (Ramsey optimal)
    - Value of tax subsidy and of *R* represent optimal policy

- Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.

Ramsey Problem

$$\max_{\nu, p_t^*, C_t, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{1t} \left[\frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t]$$

$$+ \lambda_{4t} \left[(1-\nu) \frac{\varepsilon}{\varepsilon - 1} \frac{C_t \exp(\tau_t) N_t^{\varphi}}{e^{a_t}} + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[F_t \left(\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right]$$

$$+ \lambda_{6t} [C_t - p_t^* e^{a_t} N_t] \Bigg\}$$

Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\max_{\nu, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{e^{a_t} \beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t]$$

$$+ \lambda_{4t} \left[(1-\nu) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[F_t \left(\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \}$$

Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$
\begin{aligned}
& && \max_{\nu, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \\
& \text{defines } R && + \lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{e^{a_t} \beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
& && + \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \\
& \text{defines } F && + \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta\theta F_{t+1} - F_t] \\
& \text{defines tax} && + \lambda_{4t} \left[(1-\nu) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta\theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right] \\
& \text{defines } K && + \lambda_{5t} \left[F_t \left(\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \Big\}
\end{aligned}
$$

Solving the Ramsey Problem, cnt'd

- Simplified problem:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \Big\}$$

- First order conditions with respect to $p_t^*,\ \bar{\pi}_t,\ N_t$

$$p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^{\varepsilon} = \lambda_{2t},\ \bar{\pi}_t = \left[\frac{(p_{t-1}^*)^{\varepsilon-1}}{1-\theta+\theta(p_{t-1}^*)^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}},\ N_t = \exp\left(-\frac{\tau_t}{\varphi+1} \right)$$

- Substituting the solution for inflation into law of motion for price distortion:

$$p_t^* = \left[(1-\theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}.$$

Solution to Ramsey Problem

Eventually, price distortions eliminated, regardless of shocks

$$p_t^* = \left[(1-\theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}$$

When price distortions gone, so is inflation.

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

Efficient ('first best') allocations in real economy

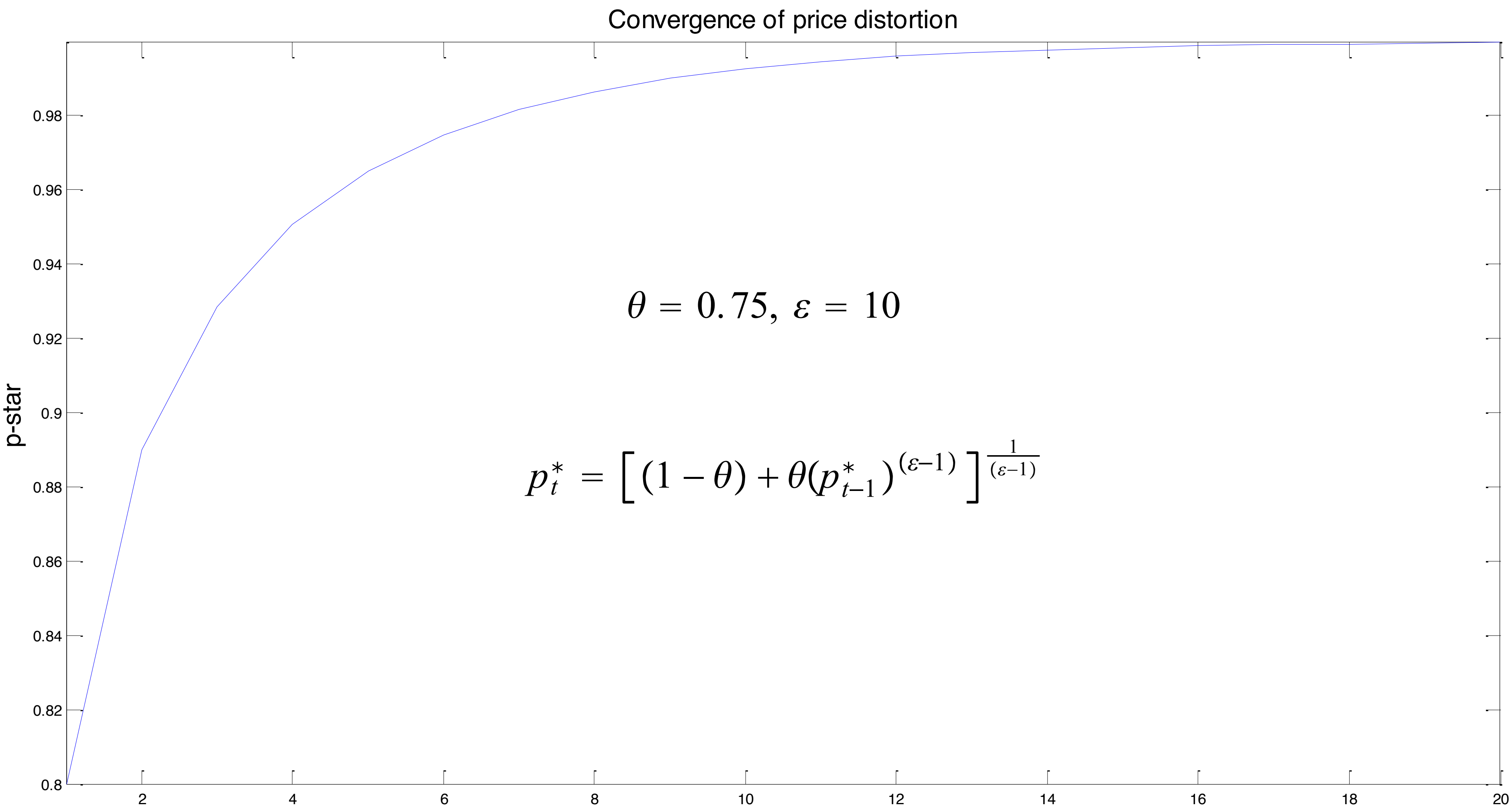
$$N_t = \exp\left(-\frac{\tau_t}{1+\varphi} \right)$$

$$1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = p_t^* e^{a_t} N_t.$$

Consumption corresponds to efficient allocations in real economy, eventually when price distortions gone

Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide



- The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., ‘first best allocations’)

- Refer to the Ramsey allocations as the ‘natural allocations’….
  – Natural consumption, natural rate of interest, etc.