Solving DSGE Models by Linearization

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http://faculty.wcas.northwestern.edu/~lchrist/d16/d1614/generallinearizationmethods.pdf

Solving the Model by First Order Perturbation (linearization)

- Express the equilibrium conditions for the $n \times 1$ vector of variables as follows:

$$E_t v\left(Z_{t-1}, Z_t, Z_{t+1}, s_t, s_{t+1}\right) = \underbrace{0}_{n \times 1},$$

  - Z_t ~ $n \times 1$ vector of the time t endogenous variables.
  - s_t ~ column vector of (zero mean) shocks, with law of motion:

$$s_t = P s_{t-1} + \epsilon_t$$

- In our example,
  - s_t ~ 2×1 vector composed of technology, a_t, and the labor supply shock, τ_t.
  - Z_t ~ 12×1 vector composed of the 12 endogenous variables $(n = 12)$.
  - v ~ the 12 nonlinear equations of the model, including monetary policy rule.

Solving the Model by First Order Perturbation (linearization)

- First step: find *steady state*, Z such that

$$v\left(Z,Z,Z,0,0\right)=0.$$

- Step two: replace v by

$$\alpha_0 z_{t+1}+\alpha_1 z_t+\alpha_2 z_{t-1}+\beta_0 s_{t+1}+\beta_1 s_t,$$

where

$$\begin{aligned}
z_t &\equiv Z_t - Z \\
\alpha_i &= \frac{dv\left(Z_{t-1},Z_t,Z_{t+1},s_t,s_{t+1}\right)}{dZ'_{t+1-i}},\ i=0,1,2, \\
\beta_i &= \frac{dv\left(Z_{t-1},Z_t,Z_{t+1},s_t,s_{t+1}\right)}{ds'_{t+1-i}},\ i=0,1.
\end{aligned}$$

where derivatives evaluated at $Z_{t-1}=Z_t=Z_{t+1}=Z$, $s_t=s_{t+1}=0$.

Simulation

- System of (linearized) equilibrium conditions:

$$\begin{aligned} E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] &= 0 \\ s_t - P s_{t-1} - \epsilon_t &= 0. \end{aligned}$$

- Would like to determine the response of z_t to a realization of shocks up to time t (simulation).
- Problem: in equilibrium conditions, z_t is a function of past *and the future*. (Not convenient!).
- Need an expression of the following form:

$$z_t = A z_{t-1} + B s_t \text{ (**)}$$

- Previous expression convenient for simulation.
  - draw a sequence, $\epsilon_0, \epsilon_1, ..., \epsilon_T$ using a computer random number generator (e.g., randn.m in MATLAB).
  - compute a sequence, $s_0, s_1, ..., s_T$ using the law of motion for the shocks, and s_{-1}.
  - compute a sequence, $z_0, z_1, ..., z_T$ using (**).

How to Construct A, B?

- Equilibrium conditions:

$$\begin{aligned} E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] &= 0 \\ s_t - P s_{t-1} - \epsilon_t &= 0. \end{aligned}$$

- How to find A and B such that when (**) is used to do simulation, the equilibrium conditions are satisfied?
  - Answer (easy to verify): A and B in (**)

$$z_t = A z_{t-1} + B s_t \text{ (**)}$$

must satisfy:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0,$$

and

$$(\beta_0 + \alpha_0 B) P + [\beta_1 + (\alpha_0 A + \alpha_1) B] = 0$$

How to Construct A, B?

- Equilibrium conditions:

$$\begin{aligned} E_t\left[\alpha_0 z_{t+1}+\alpha_1 z_t+\alpha_2 z_{t-1}+\beta_0 s_{t+1}+\beta_1 s_t\right] &= 0 \\ s_t - Ps_{t-1}-\epsilon_t &= 0 \\ z_t = Az_{t-1}+Bs_t \text{ (**)} \end{aligned}$$

- Solve for A, B :

$$\begin{aligned} \alpha_0 A^2+\alpha_1 A+\alpha_2 I &= 0 \\ (\beta_0+\alpha_0 B)P+[\beta_1+(\alpha_0 A+\alpha_1)B] &= 0. \end{aligned}$$

- Problem: more than one matrix A solves the matrix polynomial.
  - If there is exactly one A which has eigenvalues all less than unity in absolute value, then pick that one and then solve for B.

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Things That Can go Wrong With Linearization Strategy

- *More than one matrix A satisfying eigenvalue condition*: multiple solutions (*indeterminacy* of the steady state equilibrium in the nonlinear system).
  - Some potentially interesting economics.
  - The standard (e.g., no networks not working capital) New Keynesian model when the Taylor principle is *not* satisfied:

    $$R_t/R = (R_{t-1}/R)^{\rho} \exp\left[(1-\rho)\,\phi_{\pi}(\bar{\pi}_t - \bar{\pi}) + u_t\right],\ (0 < \phi_{\pi} < 1)$$

    or,

    $$r_t = \rho r_{t-1} + (1-\rho)\,\phi_{\pi}(\bar{\pi}_t - \bar{\pi}),\ r_t \equiv \log(R_t) - \log(R).$$

- *No matrix A satisfying eigenvalue restriction*: any equilibrium leaves a neighborhood of steady state if you start even only slightly away from steady state.
  - Linearization not useful in this case, since there is no equilibrium that remains arbitrarily close to steady state.