Log Linearized Phillips Curve for Simple New Keynesian Model with No Capital, Networks, or Working Capital

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Equilibrium Conditions

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} \text{ (1)}, \; F_t = 1 + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \text{ (2)}$$

$$\frac{K_t}{F_t} = \left[\frac{1 - \theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} \text{ (3)},$$

$$p_t^* = \left[(1-\theta) \left(\frac{1 - \theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \text{ (4)}$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \text{ (5)}, \; C_t = \exp(a_t) p_t^* N_t \text{ (6)}$$

with the understanding,

$$s_t = \frac{(1-\nu)\exp(\tau_t) C_t N_t^{\varphi}}{A_t}$$

Steady State

Conditional on $\bar{\pi}$, ν

$$K = \frac{\frac{\varepsilon}{\varepsilon-1}\overbrace{s}^{=\frac{(1-\nu)\exp(\tau)CN^{\varphi}}{e^{a}}}}{1-\beta\theta\bar{\pi}^{\varepsilon}} \text{ (1)}, \; F = \frac{1}{1-\beta\theta\bar{\pi}^{\varepsilon-1}} \text{ (2)}$$

$$\frac{K}{F} = \left[\frac{1-\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \text{ (3)},$$

$$p^{*} = \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}^{\varepsilon}}{p^{*}}\right]^{-1} \text{ (4)}$$

$$1 = \beta\exp(-\Delta a)\frac{R}{\bar{\pi}} \text{ (5)}, \; C = \exp(a)\,p^{*}N \text{ (6)},$$

which is six equations in six unknowns: K, F, C, N, p^{*}, R.

Log Linearization

- Hat notation:

$$\hat{x}_t \equiv \frac{dx_t}{x} = \frac{x_t - x}{x} \rightarrow dx_t = \hat{x}_t x.$$

- Log linearize equation (1) about steady state:

$$\begin{aligned}
K_t &= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} \\
\hat{K}_t K &= \frac{\varepsilon}{\varepsilon - 1} \hat{s}_t s + \beta\theta\varepsilon\bar{\pi}^{\varepsilon-1} \hat{\bar{\pi}}_{t+1} \bar{\pi} K + \beta\theta\bar{\pi}^{\varepsilon} \hat{K}_{t+1} K \\
\hat{K}_t &= \frac{\varepsilon}{\varepsilon - 1} \hat{s}_t \overbrace{\frac{s}{K}}^{= \frac{1-\beta\theta\bar{\pi}^{\varepsilon}}{\frac{\varepsilon}{\varepsilon-1}}} + \beta\theta\bar{\pi}^{\varepsilon} \left(\varepsilon\hat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right) \\
\hat{K}_t &= \left(1 - \beta\theta\bar{\pi}^{\varepsilon}\right) \hat{s}_t + \beta\theta\bar{\pi}^{\varepsilon} \left(\varepsilon\hat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right)
\end{aligned}$$

Phillips Curve

- Linearizing (1), (2) and (3), about steady state,

$$\hat{K}_t = \left(1 - \beta\theta\bar{\pi}^{\varepsilon}\right)\hat{s}_t + \beta\theta\bar{\pi}^{\varepsilon} E_t\left(\varepsilon\widehat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right) \text{ (a)}$$

$$\hat{F}_t = \beta\theta\bar{\pi}^{\varepsilon-1} E_t\left(\left(\varepsilon - 1\right)\widehat{\bar{\pi}}_{t+1} + \hat{F}_{t+1}\right) \text{ (b)}$$

$$\hat{K}_t = \hat{F}_t + \frac{\theta\bar{\pi}^{(\varepsilon-1)}}{1 - \theta\bar{\pi}^{(\varepsilon-1)}}\widehat{\bar{\pi}}_t. \text{ (c)}$$

- Substitute out for $\hat{K}_t$ in (a) using (c) and then substitute out for $\hat{F}_t$ from (b) to obtain the equation on the next slide.

Phillips Curve

- Performing the substitutions described on the previous slide:

$$\beta\theta\bar{\pi}^{\varepsilon-1}E_t\left((\varepsilon-1)\,\widehat{\bar{\pi}}_{t+1}+\hat{F}_{t+1}\right)$$
$$+\frac{\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta\bar{\pi}^{(\varepsilon-1)}}\widehat{\bar{\pi}}_t=(1-\beta\theta\bar{\pi}^{\varepsilon})\,\hat{s}_t$$
$$+\beta\theta\bar{\pi}^{\varepsilon}E_t\left(\varepsilon\widehat{\bar{\pi}}_{t+1}+\hat{F}_{t+1}+\frac{\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta\bar{\pi}^{(\varepsilon-1)}}\widehat{\bar{\pi}}_{t+1}\right).$$

Phillips Curve

- Collecting terms,

$$\overbrace{\hat{\bar{\pi}}_t = \frac{\left(1-\theta\bar{\pi}^{(\varepsilon-1)}\right)\left(1-\beta\theta\bar{\pi}^{\varepsilon}\right)}{\theta\bar{\pi}^{(\varepsilon-1)}}\hat{s}_t + \beta E_t \hat{\bar{\pi}}_{t+1}}^{\text{familiar Phillips curve}}$$

$$+\left(1-\bar{\pi}\right)\left(1-\theta\bar{\pi}^{(\varepsilon-1)}\right)\beta$$

$$\times E_t\left(\hat{F}_{t+1} + \left(\varepsilon + \frac{\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta\bar{\pi}^{(\varepsilon-1)}}\right)\hat{\bar{\pi}}_{t+1}\right).$$

- Don't actually get standard Phillips curve unless $\bar{\pi} = 1$.
  - More generally, get standard Phillips curve as long as there are no price distortions in steady state.
- Going for the Phillips curve in terms of the output gap.

Linearized Marginal Cost

- Real Marginal Cost:

$$
\begin{aligned}
s_t &= \frac{(1-\nu)\exp(\tau_t)\, C_t N_t^{\varphi}}{A_t} = \frac{(1-\nu)\exp(\tau_t)\, C_t \left(\frac{C_t}{A_t p_t^*}\right)^{\varphi}}{A_t} \\
&= (1-\nu)\exp(\tau_t)\left(\frac{C_t}{A_t}\right)^{1+\varphi}\left(\frac{1}{p_t^*}\right)^{\varphi} \\
&= (1-\nu)\left(\frac{C_t}{A_t \exp\left[-\frac{\tau_t}{1+\varphi}\right]}\right)^{1+\varphi}\left(\frac{1}{p_t^*}\right)^{\varphi} \\
&= (1-\nu)\, X_t^{1+\varphi}\left(\frac{1}{p_t^*}\right)^{\varphi},
\end{aligned}
$$

where

$$
X_t = \frac{\text{Actual consumption}}{\text{Ramsey consumption}} = \text{"output gap"}
$$

Log linearizing Marginal Cost

- We have,

$$s_t = (1-\nu) X_t^{1+\varphi} (p_t^*)^{-\varphi}$$

- Then,

$$\begin{aligned} s\hat{s}_t &= (1+\varphi)(1-\nu) X^{\varphi} (p^*)^{-\varphi} X\hat{X}_t \\ &\quad -\varphi(1-\nu) X^{1+\varphi} (p^*)^{-\varphi-1} p^* \widehat{p_t^*} \\ &= (1+\varphi)(1-\nu) X^{1+\varphi} (p^*)^{-\varphi} x_t \\ &\quad -\varphi(1-\nu) X^{1+\varphi} (p^*)^{-\varphi} \widehat{p_t^*} \end{aligned}$$

- Dividing by s :

$$\begin{aligned} \hat{s}_t &= (1+\varphi) \frac{(1-\nu) X^{1+\varphi} (p^*)^{-\varphi}}{s} x_t \\ &\quad -\varphi \frac{(1-\nu) X^{1+\varphi} (p^*)^{-\varphi}}{s} \widehat{p_t^*} \\ &= (1+\varphi) x_t - \varphi \widehat{p_t^*} \end{aligned}$$

Phillips Curve in Terms of Output Gap

- Collecting terms,

$$\hat{\bar{\pi}}_t = \frac{\left(1-\theta\bar{\pi}^{(\varepsilon-1)}\right)\left(1-\beta\theta\bar{\pi}^{\varepsilon}\right)}{\theta\bar{\pi}^{(\varepsilon-1)}}\overbrace{\left[\left(1+\varphi\right)x_t-\varphi\widehat{p_t^*}\right]}^{=\hat{s}_t}$$
$$+\beta E_t\hat{\bar{\pi}}_{t+1}$$
$$+\left(1-\bar{\pi}\right)\left(1-\theta\bar{\pi}^{(\varepsilon-1)}\right)\beta$$
$$\times E_t\left(\hat{F}_{t+1}+\left(\varepsilon+\frac{\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta\bar{\pi}^{(\varepsilon-1)}}\right)\hat{\bar{\pi}}_{t+1}\right).$$

- Phillips curve in principle quite complicated since it includes $\hat{F}_{t+1}$ and $\widehat{p_t^*}$, and their laws of motion!
- Not surprising since prices appear in several equations: (1),(2),(3),(4).

Where Does the Standard Phillips Curve Come From?

- In the special case, $\bar{\pi} = 1$, then

$$p^* = \left[(1-\theta) \left(\frac{1 - \theta \bar{\pi}^{(\varepsilon - 1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon - 1}} + \frac{\theta \bar{\pi}^{\varepsilon}}{p^*} \right]^{-1}$$

$$\rightarrow \quad p^* = 1,$$

  so with zero inflation, then there is no dispersion in steady state (i.e., the cross-industry resource allocation problem is solved).
- Also, when $\bar{\pi} = 1$ then first order approximation to law of motion of Tack Yun distortion, (4), is:

$$\widehat{p_t^*} = \theta \times \widehat{p_{t-1}^*},$$

  so eventually $\widehat{p_t^*} = 0$ (easy to verify).
- Conclude: when $\bar{\pi} = 1$ then $p_t^* = p^* = 1$, to a first order approximation.

Where Does the Standard Phillips Curve Come From?

- In steady state,

$$
\begin{aligned}
K &= \frac{\frac{\varepsilon}{\varepsilon-1}s}{1-\beta\theta\bar{\pi}^{\varepsilon}} \text{ (1)},\ F = \frac{1}{1-\beta\theta\bar{\pi}^{\varepsilon-1}} \text{ (2)} \\
\frac{K}{F} &= \left[\frac{1-\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \quad \text{(3)}
\end{aligned}
$$

so

$$
\begin{aligned}
s &= \left(1-\beta\theta\bar{\pi}^{\varepsilon}\right)\frac{\varepsilon-1}{\varepsilon}\left[\frac{1-\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}\frac{1}{1-\beta\theta\bar{\pi}^{\varepsilon-1}} \\
&= \frac{\varepsilon-1}{\varepsilon}, \text{ when } \bar{\pi}=1.
\end{aligned}
$$

Where Does the Standard Phillips Curve Come From?

- From before, when $\bar{\pi} = 1$, then
$$s = \frac{\varepsilon - 1}{\varepsilon}.$$
- At the same time,
$$\begin{aligned} s &= (1-\nu) X^{1+\varphi} (p^*)^{-\varphi} \\ &= (1-\nu) X^{1+\varphi}, \end{aligned}$$
when $\bar{\pi} = 1$.
- Thus, if $1 - \nu = (\varepsilon - 1)/\varepsilon$, then output gap is zero in steady state:
$$X = 1.$$
- Conclude: achieve first best efficiency in steady state with
$$\begin{aligned} 1-\nu &= (\varepsilon - 1)/\varepsilon, \text{ guarantees efficient level of employment} \\ \bar{\pi} &= 1 \text{ guarantees efficient cross-sectoral allocation.} \end{aligned}$$

Where Does the Standard Phillips Curve Come From?

- Answer:

$$\bar{\pi} = 1.$$

- In this case, obtain

$$\hat{\bar{\pi}}_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}(1+\varphi)x_t + \beta E_t \hat{\bar{\pi}}_{t+1}$$

- Also,

$$\hat{\bar{\pi}}_t = \frac{\bar{\pi}_t - \bar{\pi}}{\bar{\pi}} = \frac{\bar{\pi}_t - 1}{1} = \pi_t,$$

where π_t denotes net inflation. So, with $\bar{\pi} = 1$,

$$\pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}(1+\varphi)x_t + \beta E_t \pi_{t+1},$$

where

$$x_t = \log(X_t) - \log(X).$$

Where Does the Standard Phillips Curve Come From?

- In practice, sometimes also suppose that

$$X = 1,$$

which requires

$$1 - \nu = (\varepsilon - 1)/\varepsilon.$$

- In this case,

$$x_t = \log(X_t) = \log C_t - \log C_t^*,$$

where C_t^* is natural consumption (i.e., Ramey consumption) and C_t is actual consumption.

- This reflects the sunny disposition in traditional New Keynesian literature, that after the storm has subsided (i.e., all the shocks have settled down) then everything will be well (ie., first best).
- A darker cloud has settled over these models, in the form of secular stagnation, but that is another story.