‘Gertler-Kiyotaki: Banking, Liquidity and Bank Runs in an Infinite Horizon Economy’, AER (2015)

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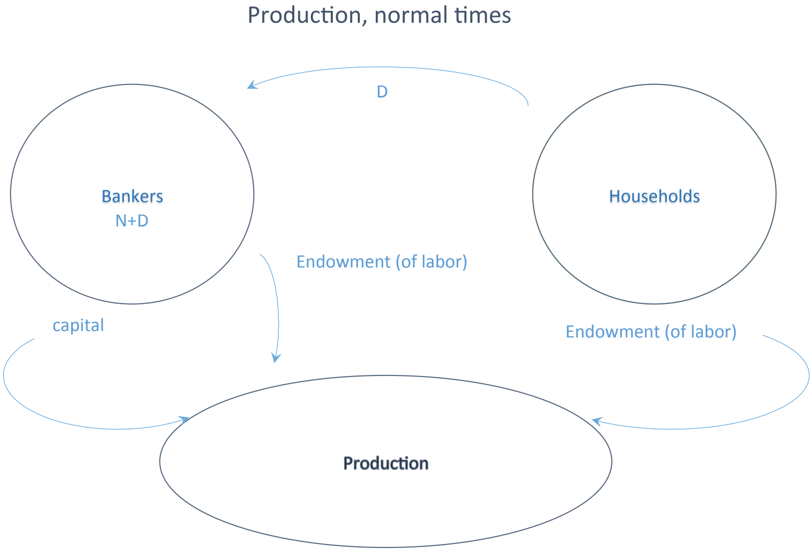
Background

- In the period, 2007 - 2009(?), it appeared that there was a big ‘bank run’, in that financial institutions could not roll over the liabilities they had issued to finance long-term mortgage backed securities.
- This bank run was thought to have disrupted the financial system and dealt a blow to the real economy.
- Gertler-Kiyotaki (GK) designed a model that could in principle capture this vision.
- The model draws on the ideas of Diamond and Dybvig and Cole and Kehoe.

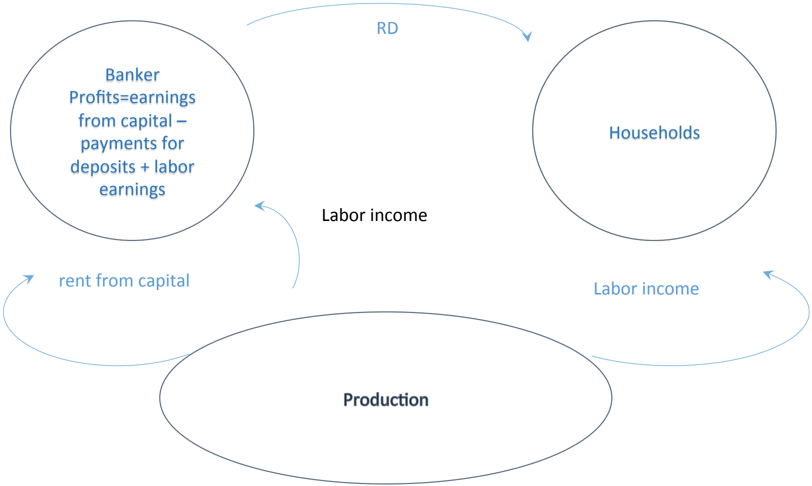
Banks

- The commercial banking system supervised by the Fed does not have runs.
  - Deposit insurance
  - Fed lender of last resort activities.
- Banks in the model: shadow banks (banks not supervised by Fed)
- Households' in the model: combination of actual households and commercial banking system.
- Gertler-Kiyotaki-Prestipino (*Handbook of Macroeconomics*):
  - extend the model to be explicit about the distinction between commercial banks, shadow banks and households.

Model In 'Normal' Times (no bank runs)



Income, normal times



Capital

- Abstract from capital accumulation
  - Total capital in the economy is fixed at $\bar{K}$ (=1).
  - No depreciation of capital.
- Ownership of Capital
  - Normally banks own most of the capital because they are efficient at operating it.
    - alternatively, the people they lend to are good at running it.
  - Households can also hold capital, but they are not efficient at it.
    - Stand in for idea that the banking system is specialized.
    - When institutions involved in management of capital are damaged and other institutions must take over, there is a loss (in practice, perhaps only a temporary loss).

Financial Frictions in Banks

- 'Normal' agency problem: banks can run away with fraction, θ, of assets. If they run away, we say they *default*.
- If banks try to issue too many deposits, depositors worry (correctly) that the bank will default and they refuse to give the bank any funds. Knowing this, banks limit how many funds they try to issue.

Source of Bank Run: Maturity Mismatch

- Banks hold long-term assets (capital) and finance it by one-period deposits.
- In normal times, households simply *roll over* their deposits
  - the accrued principal and interest on deposits is simply converted into new deposits (rolled over), so the bank does not have to sell assets to pay the household.
- If households do not roll over their deposits, then banks must sell assets to pay the principle and interest on deposits.
  - It is possible to be in a situation where *all* households refuse to roll over: this is a *bank run*.
  - In a bank run, banking system has to sell the capital to less productive users of capital, at a loss ('firesale price').
  - Bank equity is wiped out, banking system must shut down.
    - Capital fall into less capable hands and output suffers.

What We Do

- Study an equilibrium in which the only source of uncertainty is the possibility of a bank run.
- Explore the events before, during and after a bank run.

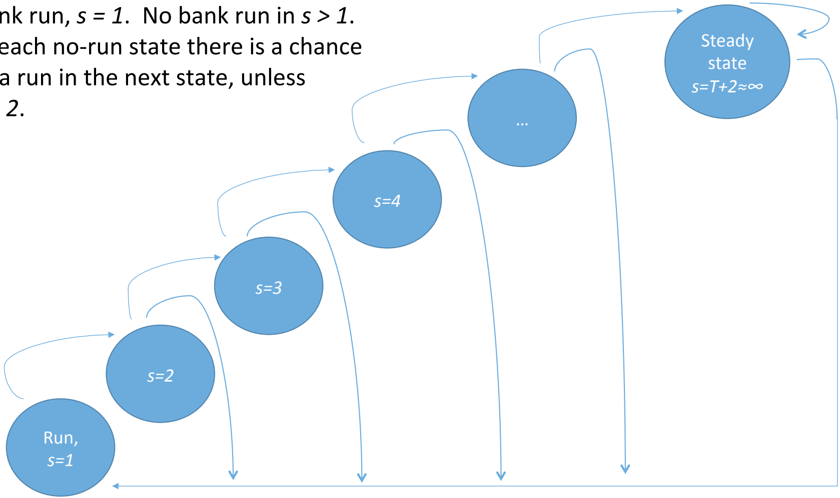
General Idea

- The state of the economy is denoted by $s \geq 1$.
  - $s = 1$: state in which a bank run occurs.
  - $s > 1$: state in which last bank run occurred $s - 1$ periods in the past.
  - s_t denotes realized value of s in period, t. Then,

$$P(s) = prob\left[s_{t+1} = 1 | s_t = s\right]$$
$$1 - P(s) = prob\left[s_{t+1} = s + 1 | s_t = s\right]$$

- $X(s)$ denotes vector of 8 endogenous variables (including $P(s)$) in state s.
- We will assume that $X(s+1) = X(s)$ for $s \geq T + 1$.
  - We say that $X(s)$ is 'in steady state' for $s \geq T + 1$.

Possible states: $s = 1, 2, 3, \ldots, T+2$.
Bank run, $s = 1$. No bank run in $s > 1$.
In each no-run state there is a chance of a run in the next state, unless $s = 2$.



Outline of Analysis

- Study $X(s)$ for all s.
- Consider a (very unlikely) sequence, $s_t = t$, for $t = 1, 2, 3, ...$
  - $t = 1$: bank run state.
  - $t = 2$: state after run.
  - $t = 3, ..., T + 1$, 'middle states' (no bank occurs, but a bank run could occur in $t + 1$)
  - $t = T + 2$, steady state.

Eight Equations Governing $X(s)$

- Bankers (where most of the action is)
  - participation constraint of banks.
  - law of motion of bank net worth.
  - probability of a bank run.
- Households
  - deposit decision.
  - capital decision.
- Other conditions:
  - resource constraint and two more equations.

Birth and Death of Bankers

- In each period the number of bankers corresponds to the number of points on the unit interval ('there is a unit measure of bankers').
- At the start of period t, $1-\sigma$ bankers are randomly selected to exit the economy ('die'), and σ remain as bankers for period t.
- Dying bankers consume their net worth in the period they exit.
  - That last meal is what they live for.
  - Bankers are risk neutral and their objective is to maximize the expected present discounted value of consumption.
- The population of bankers is constant, so $1-\sigma$ new bankers are born in each period.
  - If the period of their birth is a bank panic, then bankers delay entry for one period (technically useful assumption).
- Bankers receive a one-time payment, w_t^b, in the period that they enter. Without this 'seed money' they can't issue deposits.

Bankers' Problem in $t > 1$

- Individual banker has net worth, n_t, and issues deposits, d_t.
- Buys capital, k_t^b :
$$Q_t k_t^b = n_t + d_t,$$
where Q_t is price of capital in period when there is no bank run.
- The banker now has the option to default on its obligations to the household, or remain in business and possibly default later.
- First, consider no default. Then, default option.

Banker in $t > 1$ Who Chooses Not to Default

- Period$t+1$ assets of banker that does not choose default in period$t$:

$k_t^b\left(Z_{t+1}+Q_{t+1}\right)-\bar{R}_t d_t$	if no bank run in $t+1$
$k_t^b\left(Z_{t+1}+Q_{t+1}^*\right)-\bar{R}_t x_{t+1} d_t$	if bank run in $t+1$

,

where $Q_{t+1}^* < Q_{t+1}$ and $Q_t k_t^b = n_t + d_t$.

- In the equilibria we consider, bank assets are wiped out in a bank run state.
  - So, in a period$t+1$ bank run the recovery rate, x_{t+1}, for depositors is:

$$x_{t+1} = \frac{k_t^b\left(Z_{t+1}+Q_{t+1}^*\right)}{\bar{R}_t d_t} < 1.$$

That is, in a bank run state households recover less than $\bar{R}_t d_t$.

Banker in $t > 1$ Who Chooses to Default

- Immediately after receiving deposits, the banker has the option to 'default'.
- Takes a fraction, θ, of the assets:

$$\theta Q_t k_t^b,$$

and leaves $(1-\theta) Q_t k_t^b$ for the depositors.

- Defaulting banker exits from banking, and so consumes $\theta Q_t k_t^b$ in a default. Thus,

$$\text{value to banker of default in period } t: \; \theta Q_t k_t^b.$$

- Banker chooses default if, and only if, the default option dominates value of not defaulting in period t, V_t :

$$\theta Q_t k_t^b > V_t.$$

Value to Banker of Not Defaulting

- V_t value of not defaulting in t and keeping open the option of defaulting in $t+1$.

$$
\begin{aligned}
V_t \;&=\; \max_{d_t} E_t[\beta \overbrace{(1-\sigma)}^{\text{exogenous probability of exiting in } t+1} n_{t+1} \\
&\quad + \beta\sigma \overbrace{\max\left(V_{t+1}, \theta Q_{t+1} k_{t+1}^b\right)}^{\text{a banker has option to default in } t+1}]
\end{aligned}
$$

Participation Constraint in $t > 1$

- A bank must declare how many deposits, d_t, it intends to take before households place their deposits in that bank.
  - The banker finds it optimal not to declare a value of d_t which implies that it would be desirable for the banker to default.
    - This is because the banker understands that households would place zero deposits in such a bank.
- Thus, banks choose d_t subject to:

$$\theta Q_t k_t^b \leq V_t.$$

Bankers' Constrained Problem in $t > 1$

- Banker problem:

$$V_t = \max_{d_t} E_t \left[\beta (1-\sigma) n_{t+1} + \beta\sigma V_{t+1} \right],$$

$$\text{subject to:} \quad \theta Q_t k_t^b \leq V_t, \text{ for each } t.$$

- Different bankers have different levels of net worth, n_t, because each has a different age. By scaling, can see that aggregation is simple in this model:

$$\psi_t \equiv \frac{V_t}{n_t},\ \phi_t \equiv \frac{Q_t k_t^b}{n_t}.$$

- Note that d_t can be replaced by leverage, ϕ_t, in this scaled notation, so that

$$\psi_t = \max_{\phi_t} E_t \left\{ \left[\beta (1-\sigma) + \beta\sigma\psi_{t+1} \right] \frac{n_{t+1}}{n_t} \right\}$$

$$\text{subject to:} \quad \theta\phi_t \leq \psi_t$$

Bankers' Constrained Problem in $t > 1$

- Banker problem:

$$
\begin{aligned}
\psi_t &= \max_{\phi_t} E_t \left\{ \left[\beta(1-\sigma) + \beta\sigma\psi_{t+1}\right] \frac{n_{t+1}}{n_t} \right\} \\
\text{subject to:} \quad & \theta\phi_t \le \psi_t
\end{aligned}
$$

- Law of motion of individual banker net worth:

$$
n_{t+1} = \begin{cases} \overbrace{n_t \left[\phi_t \frac{Z_{t+1} + Q_{t+1}}{Q_t} - (\phi_t - 1)\bar{R}_t\right]}^{=k_t^b (Z_{t+1}+Q_{t+1}) - d_t \bar{R}_t} & \text{no run in } t+1 \\ 0 & \text{if run} \end{cases}
$$

where $\bar{R}_t$ denotes interest rate on deposits in the event of no run.

  – Recall, $\phi_t \equiv \frac{Q_t k_t^b}{n_t}$, so $\phi_t - 1 = d_t / n_t$.

Bankers' Constrained Problem in $t > 1$

- Only uncertainty is whether or not there is a run in period $t+1$
  - period t probability that there is a run in $t+1$ is denoted P_t.
  - P_t exogenous to an individual banker.
- Then, the banker problem is

$$\begin{aligned}
\psi_t &= \max_{\phi_t} \{ (1-P_t) \left[\beta (1-\sigma) + \beta\sigma\psi_{t+1} \right] \\
&\quad \times \left[\phi_t \frac{Z_{t+1}+Q_{t+1}}{Q_t} - (\phi_t - 1) \bar{R}_t \right] \} \\
\text{subject to:} &\quad \theta\phi_t \le \psi_t
\end{aligned}$$

- Absence of n_t here implies that all bankers choose the same ϕ_t regardless of n_t.
- Also,

$$V_t = \psi_t n_t,$$

where ψ_t is independent of n_t. So, $n_t = 0$ implies $V_t = 0$.

Bankers' Constrained Problem in $t > 1$

- Assume that
$$\frac{Z_{t+1} + Q_{t+1}}{Q_t} - \bar{R}_t > 0,$$
but not too big (see GK, p. 2020). Then the banker participation constraint is always binding and they go to maximum leverage:
$$\phi_t = \frac{\psi_t}{\theta}.$$

- The banker's choice of leverage is characterized by:
$$\begin{aligned} \phi_t &= (1 - P_t) \frac{\beta}{\theta} [1 - \sigma + \sigma\theta\phi_{t+1}] \\ &\quad \times \left[\phi_t \frac{Z_{t+1} + Q_{t+1}}{Q_t} - (\phi_t - 1) \bar{R}_t\right]. \end{aligned}$$

- Must verify the above inequality numerically.

Bankers' Constrained Problem in $t > 1$

- Individual bank participation constraint:

$$\begin{aligned}\phi_t &= (1 - P_t)\,\frac{\beta}{\theta}\left[1 - \sigma + \sigma\theta\phi_{t+1}\right] \\ &\quad \times \left[\phi_t \frac{Z_{t+1} + Q_{t+1}}{Q_t} - (\phi_t - 1)\,\bar{R}_t\right].\end{aligned}$$

- Since each banker's ϕ_t is the same, it is also equal to the aggregate economy-wide leverage ratio, Φ_t, so

Participation Constraint (1)

$$\begin{aligned}\Phi_t &= (1 - P_t)\,\frac{\beta}{\theta}\left[1 - \sigma + \sigma\theta\Phi_{t+1}\right] \\ &\quad \times \left[\Phi_t \frac{Z_{t+1} + Q_{t+1}}{Q_t} - (\Phi_t - 1)\,\bar{R}_t\right]\end{aligned}$$

Banker Net Worth in No-run $t+1$

- Let $\zeta_t(n)$ denote the measure of bankers with net worth, n, in period t. Then,

$$N_t \equiv \int_0^\infty n\zeta_t(n)\,dn.$$

- Assuming no bank run in period $t+1$, the net worth of bankers in business during period t :

$$\begin{aligned}
&\int_0^\infty n\left[\phi_t \frac{Z_{t+1}+Q_{t+1}}{Q_t} - (\phi_t - 1)\bar{R}_t\right]\zeta_t(n)\,dn \\
= \;& N_t\left[\Phi_t \frac{Z_{t+1}+Q_{t+1}}{Q_t} - (\Phi_t - 1)\bar{R}_t\right] \\
= \;& K_t^b(Z_{t+1}+Q_{t+1}) - D_t\bar{R}_t
\end{aligned}$$

- A randomly selected fraction, $1-\sigma$, of bankers in business in t exit in period $t+1$ and consume their net worth:

$$C_{t+1}^b = (1-\sigma)N_t\left[\Phi_t \frac{Z_{t+1}+Q_{t+1}}{Q_t} - (\Phi_t - 1)\bar{R}_t\right]$$

Banker Net Worth in No-run $t+1$

- A measure, $1-\sigma$, of new bankers enter in period $t+1$ to replace the $1-\sigma$ that exit.
  - Entering bankers in period $t+1$ have an endowment of labor with productivity w_{t+1}^b which they supply inelastically to labor market. They are paid w_{t+1}^b.
- In case there is no bank run in $t+1$:

law of motion of aggregate net worth (2)

$$N_{t+1} = \sigma\left[K_t^b\left(Z_{t+1}+Q_{t+1}\right)-D_t\bar{R}_t\right] + \overbrace{W_{t+1}^b}^{(1-\sigma)w_{t+1}^b}$$

- Note:

$$C_{t+1}^b = \frac{1-\sigma}{\sigma}\left(N_{t+1}-W_{t+1}^b\right).$$

Bank Run in $t+1$

- We assumed above that
$$\frac{Z_{t+1}+Q_{t+1}}{Q_t}-\bar{R}_t>0.$$
- Multiply by $Q_t\left(1-K_t^h\right)$ and use $Q_t\left(1-K_t^h\right)=D_t+N_t$,
$$\overbrace{\left(Z_{t+1}+Q_{t+1}\right)\left(1-K_t^h\right)}^{\text{assets at the start of } t+1}>\bar{R}_t\left(D_t+N_t\right)\geq\bar{R}_tD_t.$$
so, in the absence of a bank run, bankers are *solvent*.
- In a bank run, depositors want to be repaid in full by setting $D_{t+1}=0$, requiring banks to sell assets.
- If all banks had to sell their assets, the price of assets, Q_{t+1}^*, could be so low that *depositors' recovery ratio in the event of a bank run*, x_{t+1}, would be less than unity:
$$x_{t+1}\equiv\frac{\left(Z_{t+1}+Q_{t+1}^*\right)\left(1-K_t^h\right)}{\bar{R}_tD_t}<1.$$

Bank Run in $t+1$

- Is $D_{t+1} = 0$ consistent with individual rationality if $x_{t+1} < 1$?
- Assume that in case of $D_{t+1} = 0$ (i.e., a bank run):
  - The $1 - \sigma$ bankers that would have entered in$t + 1$ do not enter
    - this implies $N_{t+1} = 0$.
  - Why new entrant banks stay away (GK, p. 2024):
    - "Suppose, for example, that during the run it is not possible for households to identify new banks that are financially independent of the banks being run on: new banks accordingly wait for the dust to settle and then begin issuing deposits in the subsequent period."
- It is individually rational for households to make no bank deposit in period$t + 1$ because the banks (having $V_{t+1} = 0$ because $N_{t+1} = 0$) would default on the deposit.
  - households assumed to be able to earn more by investing in capital directly, or going to a non-defaulting bank if one exists (must be verified numerically).

Sunspot Selection of Bank Run or No Bank Run in t+1

- If $x_{t+1} < 1$, then it is an equilibrium to have a bank run and it is an equilibrium that there be no bank run in $t+1$.
  - bank run: $D_{t+1} = 0$, so that households refuse to roll over their period$t$ liabilities.
- Int, determine whether a bank run occurs in $t+1$ according to a binomial variable with probability:

Probability of a Bank Run (3)

$$P_t = 1 - \min\{x_{t+1}, 1\},$$

where, recall:

$$x_{t+1} \equiv \frac{\left(Z_{t+1} + Q_{t+1}^{*}\right)\left(1 - K_t^h\right)}{\left(\Phi_t - 1\right) N_t \bar{R}_t}, \quad \left(\Phi_t - 1\right) N_t \bar{R}_t = D_t \bar{R}_t.$$

Eight Equations

- Bankers
  - participation constraint of banks. (X)
  - law of motion of bank net worth. (X)
  - probability of a bank run. (X)
- Households
  - deposit decision.
  - capital decision.
- Other conditions:
  - resource constraint and two more equations.

Households in $s > 1$

- Representative household's budget constraint:

$$C_t^h + D_t + Q_t K_t^h + f\left(K_t^h\right) = W_t^h + R_t D_{t-1} + \left(Z_t + Q_t\right) K_{t-1}^h,$$

where

$$\begin{aligned}
& C_t^h \text{ ~ household consumption} \\
& D_t \text{ ~deposits} \\
& Q_t \text{ ~price of capital} \\
& K_t^h \text{ ~household capital} \\
f\left(K_t^h\right) &= \frac{\alpha}{2}\left(K_t^h\right)^2 \text{ ~ goods required to manage } K_t^h \\
& Z_t \text{ ~productivity in period } t \text{ of capital} \\
& W_t^h \text{ ~productivity of household's labor endowment} \\
& R_t \text{ ~return on household deposits}
\end{aligned}$$

Households in $s > 1$

- Household has log utility of consumption, so the intertemporal equation associated with the deposit decision is:

household deposit decision (4)

$$(1 - P_t)\,\bar{R}_t \beta \frac{C_t^h}{C_{t+1}^h} + P_t \beta \frac{C_t^h}{C_{h,t+1}^*} \bar{R}_t x_{t+1} \quad = \quad 1,$$

where (recall)x_{t+1} denotes the recovery rate on deposits in period$t+1$:

$$x_{t+1} \equiv \frac{\left(Z_{t+1} + Q_{t+1}^*\right)\left(1 - K_t^h\right)}{\left(\Phi_t - 1\right) N_t \bar{R}_t} < 1 \text{ if } P_t > 0.$$

Household Risk Premium

- The object, $\bar{R}_t$, is the 'interest rate on deposits'.
  - $\bar{R}_t$ is what the depositor gets if all goes well, i.e., if there is no run:

$$\bar{R}_t = \frac{1}{(1-P_t)\,\beta\frac{C_t^h}{C_{t+1}^h} + P_t\beta\frac{C_t^h}{C_{h,t+1}^*}x_{t+1}}.$$

- The 'risk free rate of interest' is the interest rate on a one-period bond which pays R_t^f in period $t+1$ regardless of whether or not there is a run then:

$$R_t^f = \frac{1}{(1-P_t)\,\beta\frac{C_t^h}{C_{t+1}^h} + P_t\beta\frac{C_t^h}{C_{h,t+1}^*}}.$$

  There is zero trade in this market, if we assume that only households can trade in it and we take into account that households are identical.

Household Risk Premium

- From previous slide:

$$\bar{R}_t = \frac{1}{(1-P_t)\,\beta\frac{C_t^h}{C_{t+1}^h} + P_t\beta\frac{C_t^h}{C_{h,t+1}^*}x_{t+1}}$$

$$R_t^f = \frac{1}{(1-P_t)\,\beta\frac{C_t^h}{C_{t+1}^h} + P_t\beta\frac{C_t^h}{C_{h,t+1}^*}}.$$

- The risk premium (after some rearranging):

$$\frac{\bar{R}_t}{R_t^f} - 1 = \left(\frac{P_t}{1-P_t}\right)\frac{P_t\frac{C_{t+1}^h}{C_{h,t+1}^*}}{1+P_t\frac{C_{t+1}^h}{C_{h,t+1}^*}}$$

  This expression is intuitive:

  - Premium increasing in P_t, likelihood of run.
  - Premium increasing with the size of the consumption drop in the run state.

Households in $s > 1$

- The intertemporal equation associated with the household capital decision, K_t^h:

$$\left(1 - P_t\right)\beta \frac{C_t^h}{C_{t+1}^h} \frac{Z_{t+1} + Q_{t+1}}{Q_t + \alpha K_t^h} + P_t \beta \frac{C_t^h}{C_{h,t+1}^*} \frac{Z_{t+1} + Q_{t+1}^*}{Q_t + \alpha K_t^h} \leq 1,$$

where the weak inequality is an equality in case $K_t^h > 0$ and

$Q_t + \alpha K_t^h$: period t marginal outlay for capital acquired in t.

- Can express this in the form of a complementary slackness condition (i.e., $\lambda x = 0$ and $\lambda, x \geq 0$):

household capital decision (5)

$$\begin{aligned} 0 \;=\; & K_t^h [1 - \left(1 - P_t\right)\beta \frac{C_t^h}{C_{t+1}^h} \frac{Z_{t+1} + Q_{t+1}}{Q_t + \alpha K_t^h} \\ & - P_t \beta \frac{C_t^h}{C_{h,t+1}^*} \frac{Z_{t+1} + Q_{t+1}^*}{Q_t + \alpha K_t^h}] \end{aligned}$$

Other Conditions in $s > 1$

Resource Constraint (6)

$$\overbrace{C_t^h + C_t^b + \frac{\alpha}{2}\left(K_t^h\right)^2}^{\text{uses of output}} = \overbrace{Z_t}^{\text{Production using all capital}} + \overbrace{W_t^h}^{\text{household endowment}} + \overbrace{W_t^b}^{\text{output produced by `labor' of new-born bankers}}$$

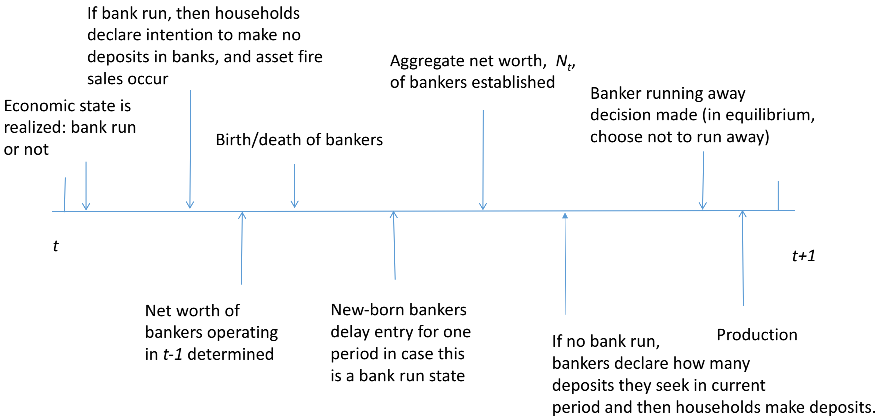
- Two other equations:

$$\textbf{(7)}\ \Phi_t = \frac{Q_t\left(1 - K_t^h\right)}{N_t},\ \textbf{(8)}\ N_t + D_t = Q_t\left(1 - K_t^h\right).$$

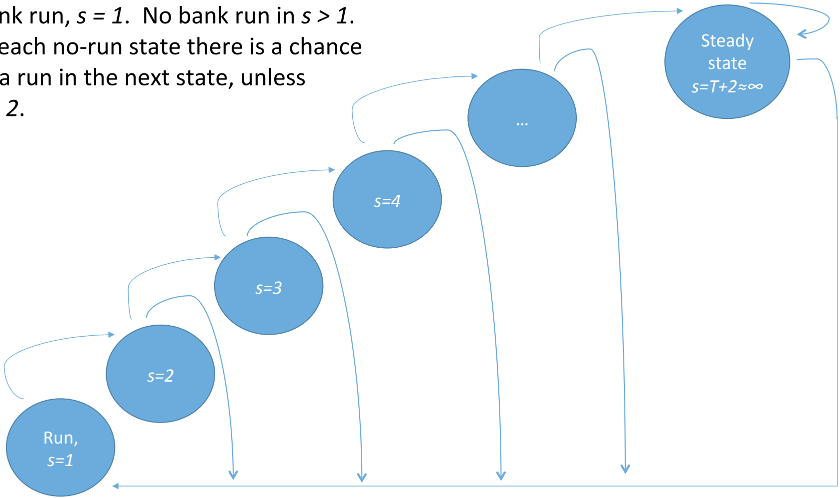
Eight Equations

- Bankers
  - participation constraint of banks.**(1)**
  - law of motion of bank net worth.**(2)**
  - probability of a bank run.**(3)**
- Households
  - deposit decision.**(4)**
  - capital decision.**(5)**
- Other conditions:
  - resource constraint.**(6)**
  - leverage definition,**(7)**, and budget constraint of bankers, **(8)**

Order of Events in One Period



Possible states: *s = 1, 2, 3,…, T+2*.
Bank run, *s = 1*. No bank run in *s > 1*.
In each no-run state there is a chance of a run in the next state, unless *s = 2*.



Solving the Model

- This means computing what happens in a bank run, one period after the most recent bank run, two periods after the most recent bank run, etc.
  For this, need to:
  - Collect and organize the equations conveniently.
  - Develop a strategy for solving those equations.
- With the model solution in hand, we can use it to generate artificial data.

Variables and Equations

- Eight equations in eight no-bank-run variables:

$$X_t = \left[K_t^h, D_t, Q_t, \bar{R}_t, P_t, C_t^h, \Phi_t, N_t \right].$$

- Use **(2)**, (**7**),**(3)**,**(6)** to determineN_t,Φ_t,P_t,C_t^h :

$$\begin{aligned}
N_t &= \sigma \left[K_{t-1}^b \left(Z_t + Q_t \right) - D_{t-1} \bar{R}_{t-1} \right] + W_t^b \\
\Phi_t &= \frac{Q_t \left(1 - K_t^h \right)}{N_t} \\
P_t &= 1 - \min \left\{ x_{t+1}, 1 \right\}, \; x_{t+1} \equiv \frac{\left(Z_{t+1} + Q_{t+1}^* \right) \left(1 - K_t^h \right)}{\left(\Phi_t - 1 \right) N_t \bar{R}_t} \\
C_t^h &= Z_t + W_t^h + W_t^b - \overbrace{C_t^b}^{= \frac{1-\sigma}{\sigma} \left(N_t - W_t^b \right)} - \frac{\alpha}{2} \left(K_t^h \right)^2,
\end{aligned}$$

for$t = 3, 4, \ldots$.

Middle Equations, $s = 3, ..., T + 1$

- Let the unknowns be:

$$z_t = \left(\begin{array}{c} Q_t \\ K_t^h \\ D_t \\ \bar{R}_t \end{array} \right).$$

- The equations,**(1)**,**(4)**,**(5)**,**(8)**, for $t = 3, 4, ..., T + 1$ can be expressed:

$$v\left(z_{t-1}, z_t, z_{t+1}\right) = 0,$$

  where $T + 1$ denotes the state before the steady state value of z_t is assumed to be reached.
- The equations for state $t = 2$ are slightly different (see below).

Middle Equations, $s = 3, ..., T+1$

- Equations that define $v(z_{t-1}, z_t, z_{t+1}) = 0, t = 3, 4, ..., T+1$:

$$\begin{aligned}
\Phi_t &= (1 - P_t)\frac{\beta}{\theta}[1 - \sigma + \sigma\theta\Phi_{t+1}] \\
&\quad \times \left[\Phi_t \frac{Z_{t+1} + Q_{t+1}}{Q_t} - (\Phi_t - 1)\bar{R}_t\right] \quad \textbf{(1)} \\
1 &= (1 - P_t)\bar{R}_t\beta\frac{C_t^h}{C_{t+1}^h} + P_t\beta\frac{C_t^h}{C_{h,t+1}^*}\bar{R}_t x_{t+1} \quad \textbf{(4)} \\
0 &= K_t^h[1 - (1 - P_t)\beta\frac{C_t^h}{C_{t+1}^h}\frac{Z_{t+1} + Q_{t+1}}{Q_t + \alpha K_t^h} \\
&\quad -P_t\beta\frac{C_t^h}{C_{h,t+1}^*}\frac{Z_{t+1} + Q_{t+1}^*}{Q_t + \alpha K_t^h}] \quad \textbf{(5)} \\
D_t + N_t &= Q_t\left(1 - K_t^h\right) \quad \textbf{(8)}
\end{aligned}$$

Steady State

- Value of z such that:

$$v(z, z, z) = 0,$$

  can be found using MATLAB's fsolve.m.
  - Problem: don't know if the solution is unique.
- Alternative (computationally cumbersome!) strategy :
  - Online notes provide an algorithm that finds the steady state by solving a set of nested one-dimensional zero-finding problems, using fzero.m.
  - In this way it is easy to verify that there is only one solution, for the given parameter values.
  - In the numerical examples we studied, we found that the steady state is unique.
- Note: the equations,v, require$Q^*, C^{h,*}$!
  - for this, must study period 1, the bank run state.

Equilibrium Conditions in $s = 1$

- Household fonc for deposits,**(4)**, bank participation constraint,**(1)**, definition of leverage,**(7),** and banker budget constraint,**(8)**, all irrelevant.
- Law of motion of net worth, equation**(2)**, becomes

$$
\begin{aligned}
N_2 &= \overbrace{W_2^b}^{\text{labor income of bankers born in period 2}} \\
&+ \overbrace{\sigma W_1^b}^{\text{labor income of bankers born in period 1 who survive to period 2}},
\end{aligned}
$$

- $P_1 = 0$ (equation**(3)**):
  - Bank liabilities in $t = 2$ are zero $(D_1 = 0)$, so the probability of a bank run in $t = 2$, P_1, is zero.

Equilibrium Conditions in $s = 1$

- Household foc for capital,**(5)**,

$$\beta \frac{C_1^{*,h}}{C_2^h} \frac{Z_2 + Q_2}{Q_1^* + \alpha} = 1,$$

  using $K^h = 1$ and $P_1 = 0$.

- Resource constraint,**(6)**

$$C_1^{*,h} + \overbrace{\frac{\alpha}{2}}^{\text{resources used by households to manage capital}}$$

$$= \overbrace{Z_1}^{\text{output produced from capital}} + \overbrace{W_1^h}^{\text{output produced by household labor}}$$

Simplifying Assumption

- We impose the following time-invariance assumption:

$$Z_t = Z,\ W_t^h = W^h,\ W_t^b = W^b,\ \text{for all } t.$$

  – For example, ifZ_t followed a growth path, $Z_t = gZ_{t-1}, g > 1$, then what happens in a bank run state depends on the calendar date when the bank run occurs.

- Equations**(5)** and**(6)**reduce to:

$$\begin{aligned} \beta \frac{C^{*,h}}{C_2^h} \frac{Z + Q_2}{Q^* + \alpha} &= 1 \\ C^{*,h} + \frac{\alpha}{2} &= Z + W^h. \end{aligned} \tag{1}$$

- Note that $C^{*,h}$ is in effect an exogenous variable. But, Q^* appears in an equation with other endogenous variables.

First State After Run, $s = 2$

- Two equations different from $t = 3, ..., T + 1$.
- Replace equation **(2)** by the value of N_2 implied by run state:

$$N_2 = (1 + \sigma) W^b.$$

- Period 2 resource constraint, (**6**), slightly adjusted to reflect banker endowments and $C_2^b = 0$:

$$C_2^h = \overbrace{Z + W^h + (1 + \sigma) W^b}^{\text{gross output}} - \frac{\alpha}{2} \left(K_2^h \right)^2$$

First State After Run, $s = 2$

- Equations **(7)** and **(3)** not affected, and can be used to compute Φ_2 and P_2 :

$$\Phi_2 = \frac{Q_2\left(1-K_2^h\right)}{N_2} \quad \textbf{(7)}$$

$$P_2 = 1-\min\left\{x_3,1\right\},\ x_3 \equiv \frac{\left(Z+Q^*\right)\left(1-K_2^h\right)}{\left(\Phi_2-1\right)N_2\bar{R}_2} \quad \textbf{(3)}$$

First State After Run, $s = 2$

- System of four equations in state $t = 2 : \tilde{v}(z_2, z_3) = 0$, where

$$
\begin{aligned}
\Phi_2 &= (1 - P_2)\,\frac{\beta}{\theta}\,[1 - \sigma + \sigma\theta\Phi_3] \\
&\quad \times \left[\Phi_2 \frac{Z + Q_3}{Q_2} - (\Phi_2 - 1)\,\bar{R}_2\right] \quad \textbf{(1)} \\
1 &= (1 - P_2)\,\bar{R}_2 \beta \frac{C_2^h}{C_{2+1}^h} + P_2 \beta \frac{C_2^h}{C_{h,2+1}^*} \bar{R}_2 x_{2+1} \quad \textbf{(4)} \\
0 &= K_2^h [1 - (1 - P_2)\,\beta \frac{C_2^h}{C_{2+1}^h} \frac{Z + Q_{2+1}}{Q_2 + \alpha K_2^h} \\
&\quad - P_2 \beta \frac{C_2^h}{C_{h,2+1}^*} \frac{Z + Q_{2+1}^*}{Q_2 + \alpha K_2^h}] \quad \textbf{(5)} \\
D_2 + N_2 &= Q_2 \left(1 - K_2^h\right) \quad \textbf{(8)}
\end{aligned}
$$

Equilibrium

- An equilibrium for this economy is a set of values for $X(s)$, $s = 1, 2, 3$, ... such that household, banker and market clearing conditions are satisfied.

Computing the Equilibrium

- One dimensional search inQ^* ($C^{*,h}$ is a function of parameters).
- Define the following mapping:

$$Q^{*\prime} = f\left(Q^*\right).$$

  - Given Q^*, find z that solves $v\left(z,z,z\right) = 0$.
  - Given Q^*,$C^{*,h}$ and z, find $z_2, z_3, ..., z_{T+1}$ that satisfy

$$\tilde{v}\left(z_2, z_3\right) = 0,\ v\left(z_{t-1}, z_t, z_{t+1}\right) = 0,\ t = 3, ..., T+1$$

    (for this, $z_t = z$, $t = 2, ..., T+1$ can be used as initial conditions).
  - Using the computed value of z_2, solve for Q^* using (1). Denote this by $Q^{*\prime}$.

Computing the Equilibrium

- Find fixed point of f :

$$Q^* = f\left(Q^*\right).$$

- The procedure suggested by GK works well. Select an initial guess for Q^*, $Q^*\left(1\right)$. Then, iterate to convergence on

$$Q^*\left(j+1\right) = f\left(Q^*\left(j\right)\right),\ j = 1, 2, \ldots\ .$$

Numerical Example

- Model parameter values

$$\begin{aligned} \alpha &= 0.00797,\ \theta = 0.1934,\ \sigma = 0.95,\ \beta = 0.99, \\ W^h &= 0.045,\ W^b = 0.0011487/50, \end{aligned}$$

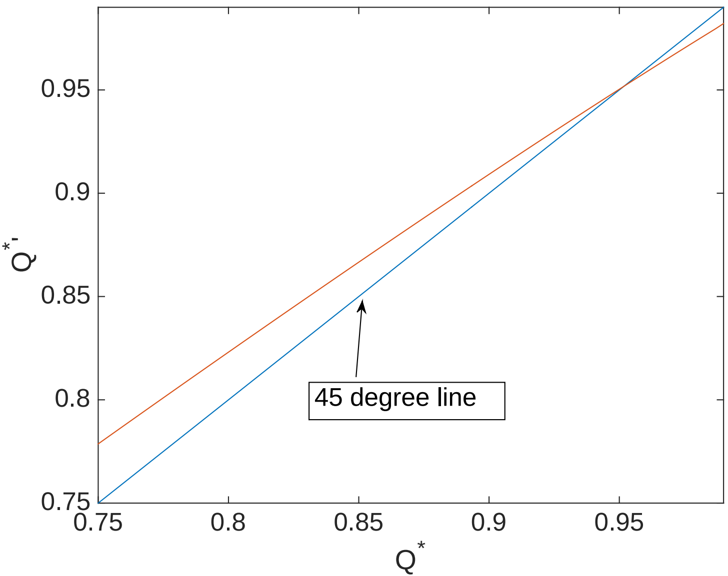
  taken from Gertler-Kiyotaki (except W^b, which is their number divided by 50 to ensure $P_\infty > 0$).

- Steady state

$$\begin{aligned} Q^* &= 0.91, \quad Q_\infty = 0.98, \quad K^h_\infty = 0.28, \quad D_\infty = 0.66 \\ \Phi_\infty &= 14.93, \quad P_\infty = 0.0075, \\ C^h_\infty &= 0.055, \quad C^b_\infty = 0.0025, \quad N_\infty = 0.047, \\ R^f_\infty &= 1/\beta - 2.5/10000, \quad \frac{\bar{R}_\infty}{R^f_\infty} - 1 = 0.57/10000 \end{aligned}$$

  risk free rate lower than $1/\beta$ because of risk of consumption-drop in run.

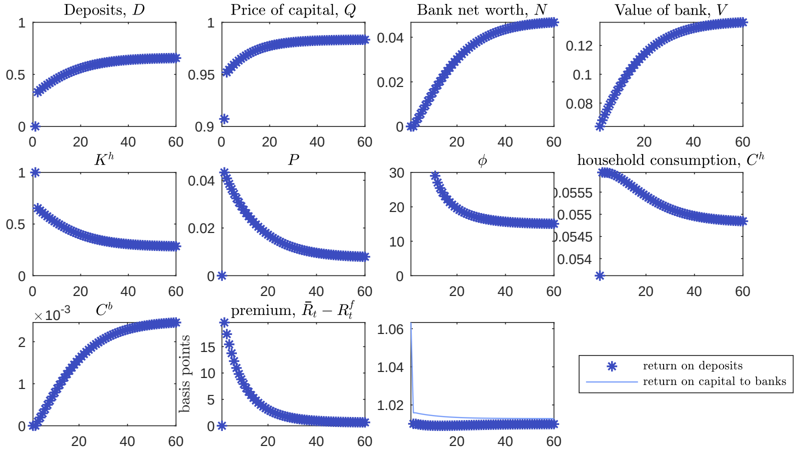
Mapping from Q^* to $Q^{*\prime}$



Model Solution

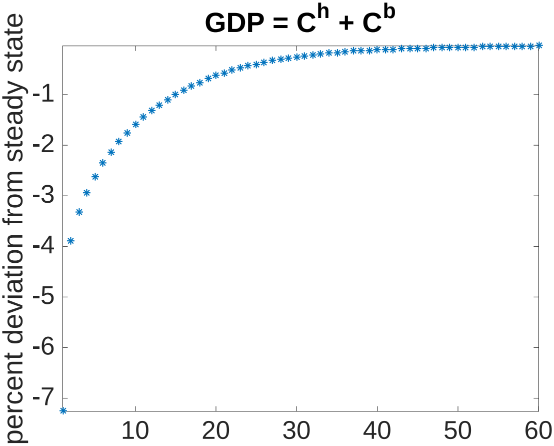
- A bank run can occur after state 1 with the probability indicated in the 2,2 diagram.
  - In case of a bank run, the economy returns to state 1.
  - In the immediate period after a bank run, no bank run is possible.
- As the banking system restarts, in state 3, the probability of another bank run is ‘high’, over 4 percent.
  - probability falls as the net worth of the banking system rises from zero.

Response of Economy to Bank Run, When no Additional Run Occurs



Domestic Product (consumption)

- In the period of the bank run, $C^h + C^b$ fall 7 percent.
  - Reflects loss of direct endowment of entering bankers and inefficient management of capital.
  - As capital flows bank into the hands of people that know how to manage it, GDP begins to rise again.
- One expects that in an economy with capital investment, the effects of a bank run might be more substantial.
- The bank run can magnify a recession triggered by a shock that would make the economy slow down anyway (see GK's AER article)


GDP = C^h + C^b
percent deviation from steady state
-1
-2
-3
-4
-5
-6
-7
10
20
30
40
50
60

Leverage

- Leverage is extremely high in immediate aftermath of crisis:

$$\Phi_2 = 7,371.5 \text{ (!)}$$

  - Reflects relaxation of participation constraint because banking (business of issuing deposits) is more profitable.
    - return on bank assets high because of anticipated strong increase in Q_t.

$$\frac{Z_{i+1} + Q_{i+1}}{Q_i} - \bar{R}_i = \begin{cases} 0.0058 & i = 1 \\ 0.0030 & i = \infty \end{cases}$$

More On Leverage

- Why does mere doubling of spread produce such a *gigantic* impact on Φ_2?
  - Solving equation **(1)** for Φ_t :

$$
\begin{aligned}
\Phi_i &= \frac{(1-P_i)\frac{\beta}{\theta}\left[1-\sigma+\sigma\theta\Phi_{i+1}\right]\bar{R}_i}{1-a_i}, \\
a_i &\equiv (1-P_i)\frac{\beta}{\theta}\left(1-\sigma+\sigma\theta\Phi_{i+1}\right)\left(\frac{Z_{i+1}+Q_{i+1}}{Q_i}-\bar{R}_i\right)
\end{aligned}
$$

- *Highly* convex in a_i :

$$
a_i = \begin{cases} 0.98 & i=2 \\ 0.04 & i=\infty \end{cases}
$$

- Is prediction for Φ_i and ψ_i (i.e., value of bank, per unit of net worth) empirically plausible?

Slow Recovery

- It takes (at least) about 10 years for the banking system and economy to fully repair themselves.
- To recover its capacity to conduct intermediation, banking system needs to have a high amount of net worth.
  - To grow its net worth, banking system retains earnings, which requires issuing a lot of deposits.
  - But, to be able to issue a lot of deposits, must have a lot of net worth!

Stochastic Equilibrium as a First Order Markov Chain

- Equilibrum represented as a 122-state Markov chain with states

$$s \in \{1, 2, ..., 122\},$$

where

$s = 1$	state in which a bank run occurs
$s = i$	state in which $i - 1$ periods have passed since last bank r
	$2 \leq i \leq 122$

Computations described above provides the 8 equilibrium variables for each state:

$$X(s) = \left[K^h(s), \ D(s), \ Q(s), \ \bar{R}(s), \ P(s), \ C^h(s), \ \Phi(s), \ N(s)\right]$$

Stochastic Equilibrium as a First Order Markov Chain

- Let the Markov transition matrix be

$$\pi_{ij} = prob\left[s_{t+1} = j | s_t = i\right],$$

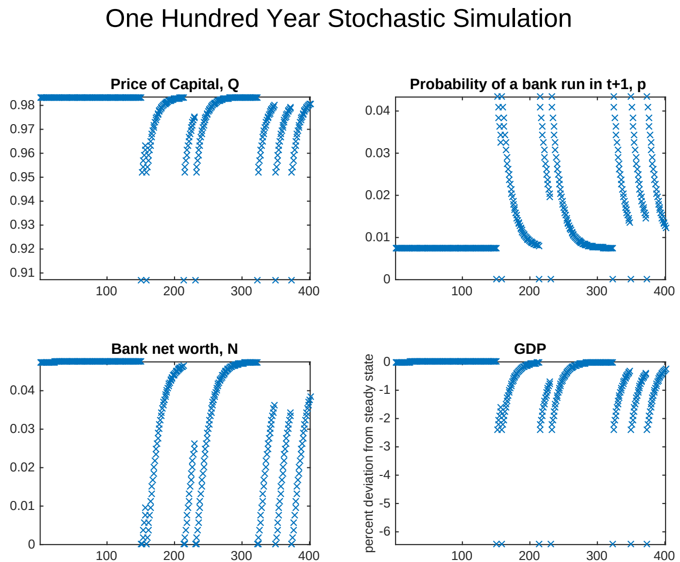
where s_t denotes a random realization of the state.

- All elements of π equal to zero except:

$$\begin{aligned}\pi_{1,2} &= 1,\ \pi_{i,i+1} = 1 - P_i \text{ for } i = 2, 3, ..., 121. \\ \pi_{i,1} &= P_i,\ i = 2, ..., 122\end{aligned}$$

Partial Realization of a Stochastic Equilibrium

- Following figure displays stochastic simulation of length 400 periods (i.e., 100 years).
- Note:
  - the periodic collapses in economic activity triggered by bank runs.
  - the probability of collapse is greater when the banking system has not yet fully restored its net worth.
- The first 150 periods (35 years) is a time of tranquility.
  - then, there is a bank run, soon followed by another. Economy has recovered after 10 years.
  - then, again a bank run followed by another.
  - finally, three bank runs in a row.
- Problem with bank run is that it leads to a period of mismanagement of capital, as less efficient institutions take over.


One Hundred Year Stochastic Simulation
Price of Capital, Q
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
100
200
300
400
Probability of a bank run in t+1, p
0.04
0.03
0.02
0.01
0
100
200
300
400
Bank net worth, N
0.04
0.03
0.02
0.01
0
100
200
300
400
GDP
percent deviation from steady state
0
-1
-2
-3
-4
-5
-6
100
200
300
400

Policy Implications

- In a banking crisis, it makes sense for the government to undertake ‘unconventional actions’.
  - could provide tax-financed deposits into banks, which it commits to roll over.
  - government makes use of fact that banks can’t run away from it to shore up intermediation system.
  - no doubt, there are inefficiencies associated with this type of action, but they may be smaller than the inefficiencies associated with a full-blown crisis.
- Question:
  - are restrictions on bank leverage, beyond what occurs in unregulated economy, desirable?
  - if yes, then what do these restrictions look like?