Estimating the Effects of Shocks to the Economy

- Vector Autoregression for a $N \times 1$ vector of observed variables:

$$Y_t = B_1 Y_{t-1} + ... + B_p Y_{t-p} + u_t,$$

$$E u_t u_t' = V$$

- $B\prime s$, $u's$ and V are Easily Obtained by OLS.
- Problem: $u's$ are statistical innovations.
  - We want impulse response functions to fundamental economic shocks, e_t.

$$u_t = C e_t,$$

$$E e_t e_t' = I,$$

$$CC' = V$$

Estimating the Effects of a Shock to the Economy ...

VAR: $Y_t = B_1 Y_{t-1} + ... + B_p Y_{t-p} + C e_t$

- Impulse Response to i^{th} Shock:

$$Y_t - E_{t-1} Y_t = C_i e_{it},$$

$$E_t Y_{t+1} - E_{t-1} Y_{t+1} = B_1 C_i e_{it}$$

$$\cdots$$

- To Compute Dynamic Response of Y_t to i^{th} Element of e_t We Need

$B_1, ..., B_p$ and C_i.

Identification Problem

$$Y_t = B_1 Y_{t-1} + ... + B_p Y_{t-p} + u_t$$

$$u_t = Ce_t, \; Eu_t u_t' = CC' = V$$

- We know $B's$ and V, we need C.
- Problem
  - N^2 Unknown Elements in C,
  - Only $N(N+1)/2$ Equations in

$$CC' = V$$

- Identification Problem: Not Enough Restrictions to Pin Down C
- Need More Identifying Restrictions!

Bivariate Blanchard and Quah Example

- Identification Assumption:

  Technology Shock is *Only* Shock that Has Long-Run Impact on (Forecast of) Level of Labor Productivity:

$$\text{(exclusion restriction)} \lim_{j \to \infty} [E_t y_{t+j} - E_{t-1} y_{t+j}] = f(\text{technology shock only})$$

$$\text{(sign restriction)} \; f' > 0$$

$$y_t = \frac{\text{output}}{\text{hour}}$$

- Blanchard-Quah/Jordi Gali:

  This Assumption Makes it Possible to Estimate Technology Shock, Even Without Direct Observations on Technology

Bivariate Blanchard and Quah Example ...

- Bivariate VAR:

$$Y_t = BY_{t-1} + u_t, \; Eu_t u_t' = V$$

$$u_t = Ce_t$$

$$Y_t = \begin{pmatrix} \Delta y_t \\ x_t \end{pmatrix}, \; C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \; e_t = \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}$$

e_{zt} ~ Technology Shock.

- From Applying OLS To Both Equations in VAR, We Know:

$$B, \; V$$

- Problem: $CC' = V$ Provides only **Three** Equations in **Four** Unknowns in C.
- Result: Assumption that e_{2t} Has No Long Run Impact on y_t Supplies the Extra Required Equation

Bivariate Blanchard and Quah Example ...

- Easy to Verify:

$$\underbrace{\overbrace{E_t[\Delta y_{t+1}+\Delta y_t]-E_{t-1}[\Delta y_{t+1}+\Delta y_t]}^{[E_t\Delta y_{t+1}-E_{t-1}\Delta y_{t+1}]+[E_t\Delta y_t-E_{t-1}\Delta y_t]}}_{E_t[y_{t+1}]-E_{t-1}[y_{t+1}]} = (1,0)\,[B+I]\,Ce_t$$

$$E_t[y_{t+2}]-E_{t-1}[y_{t+2}] = (1,0)\left[B^2+B+I\right]Ce_t$$

$$E_t[y_{t+j}]-E_{t-1}[y_{t+j}] = (1,0)\left[B^j+B^{j-1}+...+B^2+B+I\right]Ce_t$$

as $j\rightarrow\infty$:

$$\lim_{j\rightarrow\infty} E_t[y_{t+j}]-E_{t-1}[y_{t+j}] =$$

$$\lim_{j\rightarrow\infty}(1,0)\left[...+B^j+B^{j-1}+...+B^2+B+I\right]Ce_t$$

$$=(1,0)\,[I-B]^{-1}\,Ce_t$$

Bivariate Blanchard and Quah Example ...

- As $j \to \infty$:

$$\lim_{j \to \infty} E_t[y_{t+j}] - E_{t-1}[y_{t+j}] \;=\; (1,0)\,[I-B]^{-1}\,Ce_t$$

- Identification Assumption About Technology:

$$[I-B]^{-1}\,C \;=\; \begin{bmatrix} \text{number} & 0 \\ \text{number} & \text{number} \end{bmatrix}$$

- Final Result: Solve for C Using

$$\text{(exclusion restriction) } 1,2 \text{ element of } [I-B]^{-1}\,C \text{ is } zero$$

$$\text{(sign restriction) } 1,1 \text{ element of } [I-B]^{-1}\,C \text{ is positive}$$

$$CC' = V$$

- Conclude: Long-Run Restriction Supplies Extra Equation Needed to Achieve Identification.

Arbitrary Variables, Arbitrary Lags

- More General Case of Arbitrary Number (N) of Variables and Lags:

$$X_t = B_1 X_{t-1} + B_2 X_{t-2} + ... + B_p X_{t-p} + u_t$$

- To Compute Impulse Response to Technology Shock,
  - Require: $B_1, ..., B_p$ and C_1, First Column of C in $CC' = V$
  - Can Obtain by OLS: $B_1, ..., B_p$ and V
  - Identification Problem: Find C_1
- Solution: Use Restriction, as $j \to \infty$:

$$\lim_{j \to \infty} E_t[y_{t+j}] - E_{t-1}[y_{t+j}] = (1, 0, ..., 0) \left[I - B(1) \right]^{-1} C e_t$$

$$B(1) \equiv B_1 + B_2 + ... + B_p.$$

Arbitrary Variables, Arbitrary Lags ...

- VAR:

$$X_t = B_1 X_{t-1} + B_2 X_{t-2} + ... + B_p X_{t-p} + u_t$$

- Long-Run Restriction:

$$\text{(exclusion restriction) } [I - B(1)]^{-1} C = \begin{bmatrix} \text{number} & 0, ..., 0 \\ \text{numbers} & \text{numbers} \end{bmatrix}$$

$$\text{(sign restriction) } (1,1) \text{ element of } [I - B(1)]^{-1} C \text{ is } positive$$

$$CC' = V$$

- There Are Many C That Satisfy These Constraints. All Have the Same C_1.

Arbitrary Variables, Arbitrary Lags ...

- Using the Restrictions to Uniquely Pin Down C_1
- Let

$$D \equiv [I - B(1)]^{-1} C$$
$$\text{so, } DD' = [I - B(1)]^{-1} V [I - B(1)']^{-1} \equiv S_0 \text{ (Since } CC' = V)$$

- Exclusion Restriction Requires:

$$D = \begin{bmatrix} d_{11} & 0, ..., 0 \\ D_{21} & D_{22} \end{bmatrix}$$

- So

$$DD' = \begin{bmatrix} d_{11}^2 & d_{11} D_{21}' \\ D_{21} d_{11} & D_{21} D_{21}' + D_{22} D_{22}' \end{bmatrix} = \begin{bmatrix} S_0^{11} & S_0^{21\prime} \\ S_0^{21} & S_0^{22} \end{bmatrix}.$$

- Sign Restriction:

$$d_{11} > 0.$$

- Then, First Column of D Uniquely Pinned Down:

$$d_{11} = \sqrt{S_0^{11}},\ D_{21} = S_0^{21}/d_{11}$$

- First Column of C Uniquely Pinned Down:

$$C_1 = [I - B(1)]\, D_1.$$