Perturbation and Projection Methods for Solving DSGE Models

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Discussion of projections taken from Christiano-Fisher, 'Algorithms for Solving Dynamic Models with Occasionally Binding Constraints', 2000, *Journal of Economic Dynamics and Control.*
Discussion of perturbations taken from Judd's textbook.

Outline

- A Toy Example to Illustrate the basic ideas.
  - Functional form characterization of model solution.
  - Use of Projections and Perturbations.

- Neoclassical model.
  - Projection methods
  - Perturbation methods
    - Make sense of the proposition, 'to a first order approximation, can replace equilibrium conditions with linear expansion about nonstochastic steady state and solve the resulting system using certainty equivalence'

Simple Example

- Suppose that *x* is some exogenous variable and that the following equation implicitly defines *y*:

$$h(x, y) = 0, \text{ for all } x \in X$$

- Let the solution be defined by the 'policy rule', *g*:

$$y = g(x)$$

'Error function'

satisfying

$$R(x; g) \equiv h(x, g(x)) = 0$$

for all $x \in X$

The Need to Approximate

- Finding the policy rule, *g*, is a big problem outside special cases

  – 'Infinite number of unknowns (i.e., one value of *g* for each possible *x*) in an infinite number of equations (i.e., one equation for each possible *x*).'

- Two approaches:

  – projection and perturbation

Projection

- Find a parametric function, $\hat{g}(x;\gamma)$, where γ is a vector of parameters chosen so that it imitates the property of the exact solution, i.e., $R(x;g) = 0$ for all $x \in X$, as well as possible.

- Choose values for γ so that

$$\hat{R}(x;\gamma) = h(x,\hat{g}(x;\gamma))$$

is close to zero for $x \in X$.

- The method is defined by how 'close to zero' is defined and by the parametric function, $\hat{g}(x;\gamma)$, that is used.

Projection, continued

- Spectral and finite element approximations
  - **Spectral functions**: functions, $\hat{g}(x;\gamma)$, in which each parameter in γ influences $\hat{g}(x;\gamma)$ for all $x \in X$ example:

$$\hat{g}(x;\gamma) = \sum_{i=0}^{n} \gamma_i H_i(x), \ \gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$

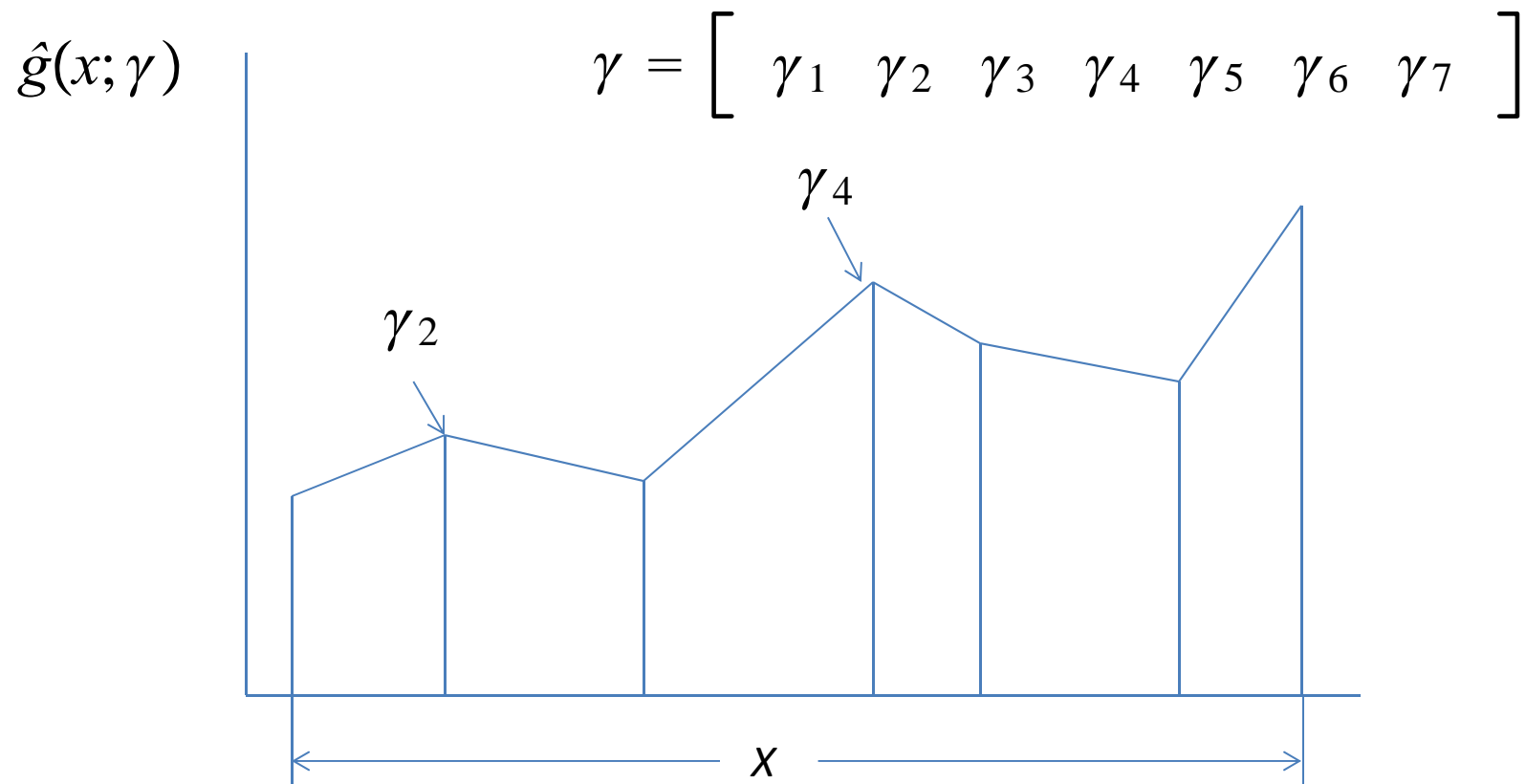$H_i(x) = x^i$ ~ordinary polynominal (not computationaly efficient)

$H_i(x) = T_i(\varphi(x)),$

$T_i(z) : [-1,1] \rightarrow [-1,1], \ i^{th}$ order Chebyshev polynomial

$\varphi : X \rightarrow [-1,1]$

Projection, continued

– Finite element approximations: functions, $\hat{g}(x;\gamma)$, in which each parameter in γ influences $\hat{g}(x;\gamma)$ over only a subinterval of $x \in X$

$\hat{g}(x;\gamma)$

$$\gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \end{bmatrix}$$

γ_4

γ_2

X

Projection, continued

- 'Close to zero': two methods

- **Collocation**, for n values of $x: x_1, x_2, \ldots, x_n \in X$ choose n elements of $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$ so that

$$\hat{R}(x_i; \gamma) = h(x_i, \hat{g}(x_i; \gamma)) = 0, \; i = 1, \ldots, n$$

  – how you choose the grid of x's matters...

Example of Importance of Grid Points

- Here is an example, taken from a related problem, the problem of *interpolation*.
  - You get to evaluate a function on a set of grid points that you select, and you must guess the shape of the function between the grid points.

- Consider the function,

$$f(k) = \frac{1}{1 + k^2}, \ k \in [-5, 5]$$

- Next slide shows what happens when you select 11 equally-spaced grid points and interpolate by fitting a 10th order polynomial.
  - As you increase the number of grid points on a fixed interval grid, oscillations in tails grow more and more violent.

Function Approximation with Fixed Interval Grid

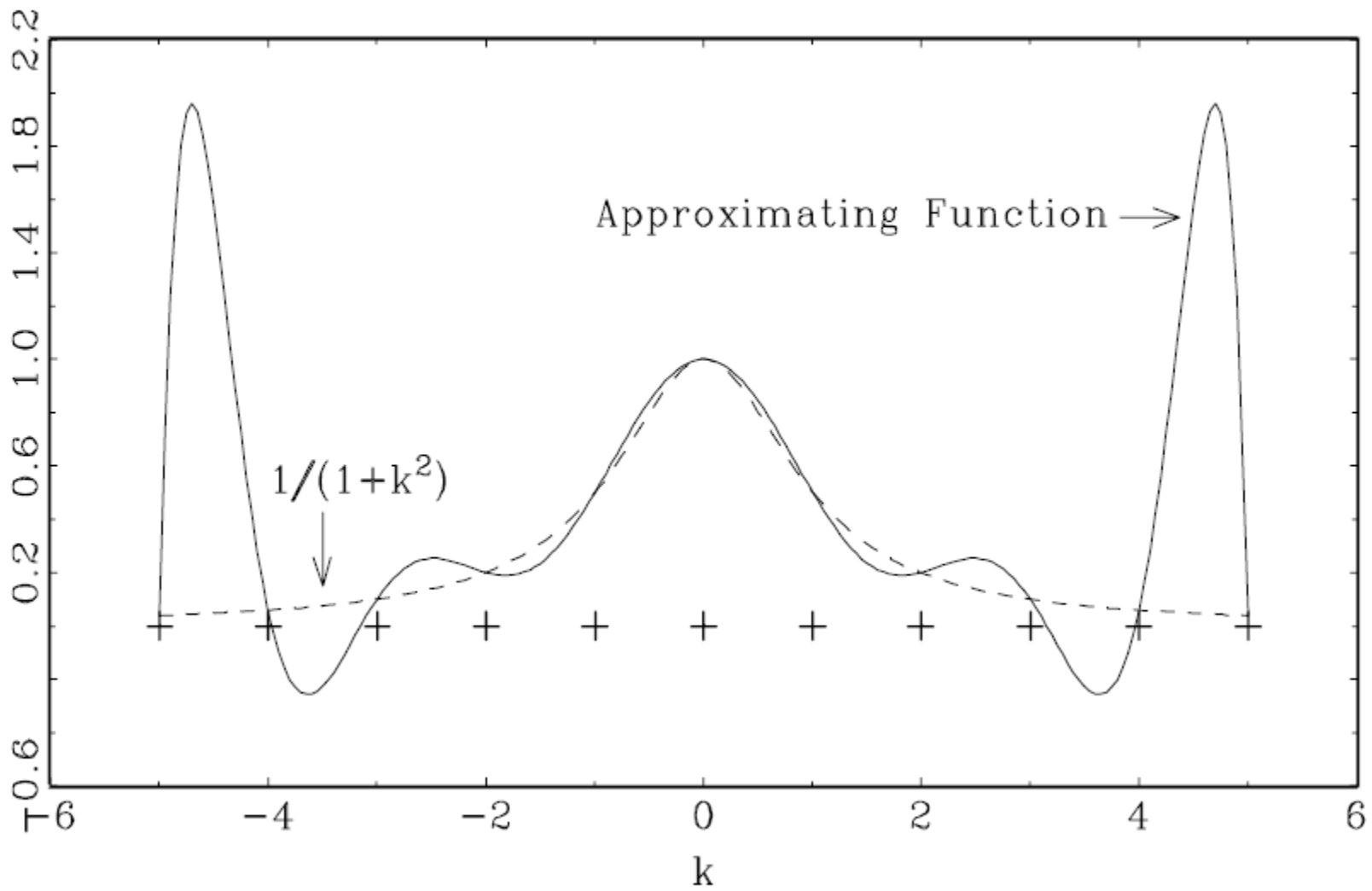Approximating Function $\longrightarrow$

$1/(1+k^2)$

Figure from Christiano-Fisher, JEDC, 1990

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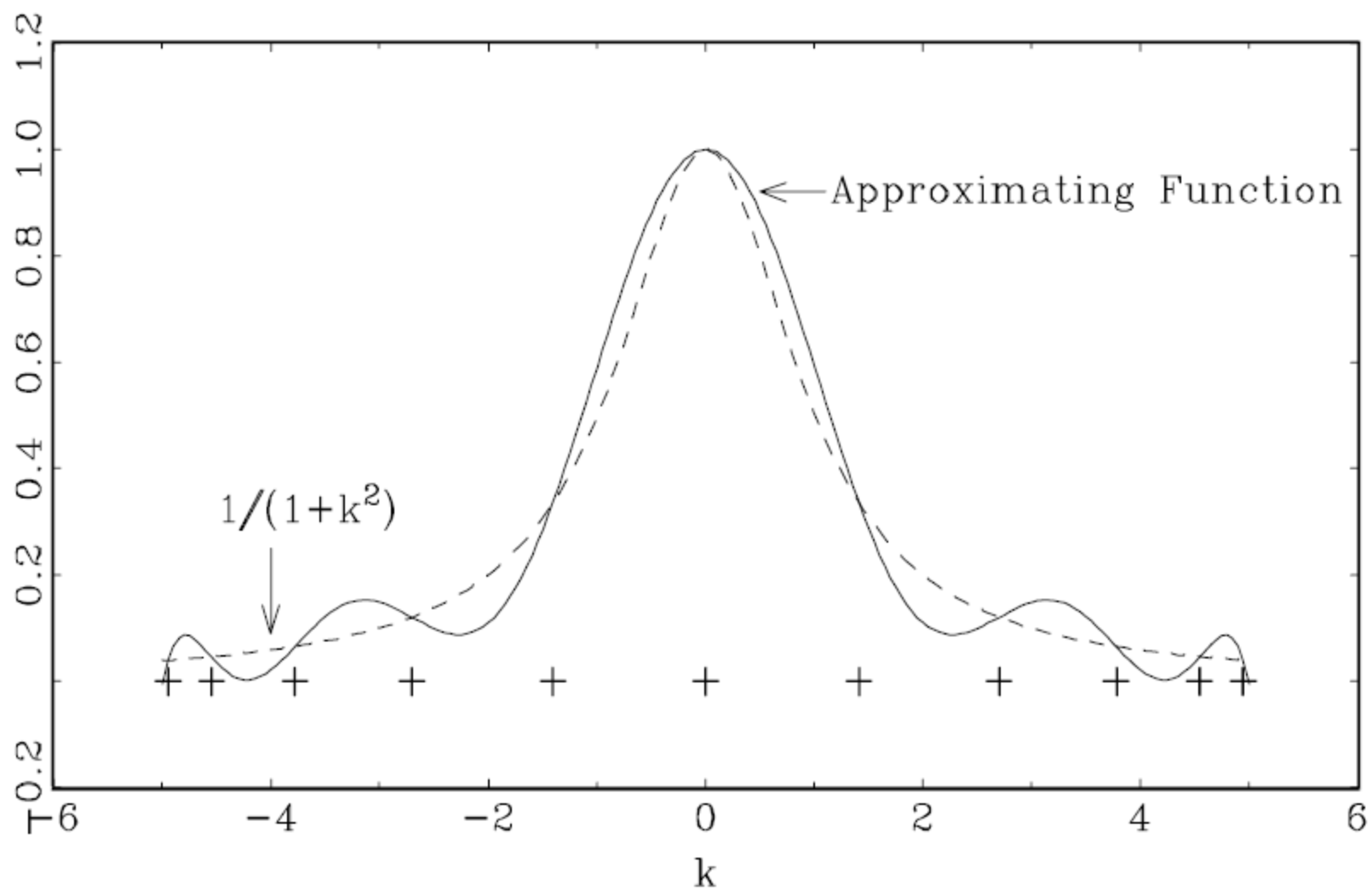
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  - As you increase the number of grid points on a fixed interval grid, oscillations in tails grow more and more violent.

- Chebyshev approximation theorem: distribute more points in the tails (by selecting zeros of Chebyshev polynomial) and get convergence in sup norm.

$$T_0(x) = 1 \quad T_2(x) = 2x^2 - 1$$
$$T_1(x) = x \quad T_3(x) = 4x^3 - 3x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Zeros: $\quad x_k = \cos\left(\dfrac{\pi}{2}\dfrac{2k-1}{n}\right), \quad k = 1, \ldots, n.$

Chebyshev polynomials

-0.71   0.71

Function Approximation with Chebychev Zeros

$1/(1+k^2)$

Approximating Function

Projection, continued

- 'Close to zero': two methods

- **Collocation**, for *n* values of $x : x_1, x_2, \ldots, x_n \in X$ choose *n* elements of $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$ so that

$$\hat{R}(x_i; \gamma) = h(x_i, \hat{g}(x_i; \gamma)) = 0, \ i = 1, \ldots, n$$

  – how you choose the grid of *x*'s matters...

- **Weighted Residual**, for *m>n* values of $x : x_1, x_2, \ldots, x_m \in X$ choose the *n* γ_i's

$$\sum_{j=1}^{m} w_j^i h(x_j, \hat{g}(x_j; \gamma)) = 0, \ i = 1, \ldots, n$$

Perturbation

- Projection uses the 'global' behavior of the functional equation to approximate solution.

  – Problem: requires finding zeros of non-linear equations. Iterative methods for doing this are a pain.

  – Advantage: can easily adapt to situations the policy rule is not continuous or simply non-differentiable (e.g., occasionally binding zero lower bound on interest rate).

- Perturbation method uses Taylor series expansion (computed using implicit function theorem) to approximate model solution.

  – Advantage: can implement procedure using non-iterative methods.

  – Possible disadvantages:
    - Global properties of Taylor series expansion not necessarily very good.
    - Does not work when there are important non-differentiabilities (e.g., occasionally binding zero lower bound on interest rate).

Taylor Series Expansion

- Let $f : R \rightarrow R$ be *k+1* differentiable on the open interval and continuous on the closed interval between *a* and *x*.

  - Then,
  $$f(x) = P_k(x) + R_k(x)$$

  - where

  Taylor series expansion about $x = a$:

  $$P_k(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2!}f^{(2)}(a)(x-a)^2 + \ldots + \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

  remainder:

  $$R_k(x) = \frac{1}{(k+1)!}f^{(k+1)}(\zeta)(x-a)^{k+1}, \text{ for some } \zeta \text{ between } x \text{ and } a$$
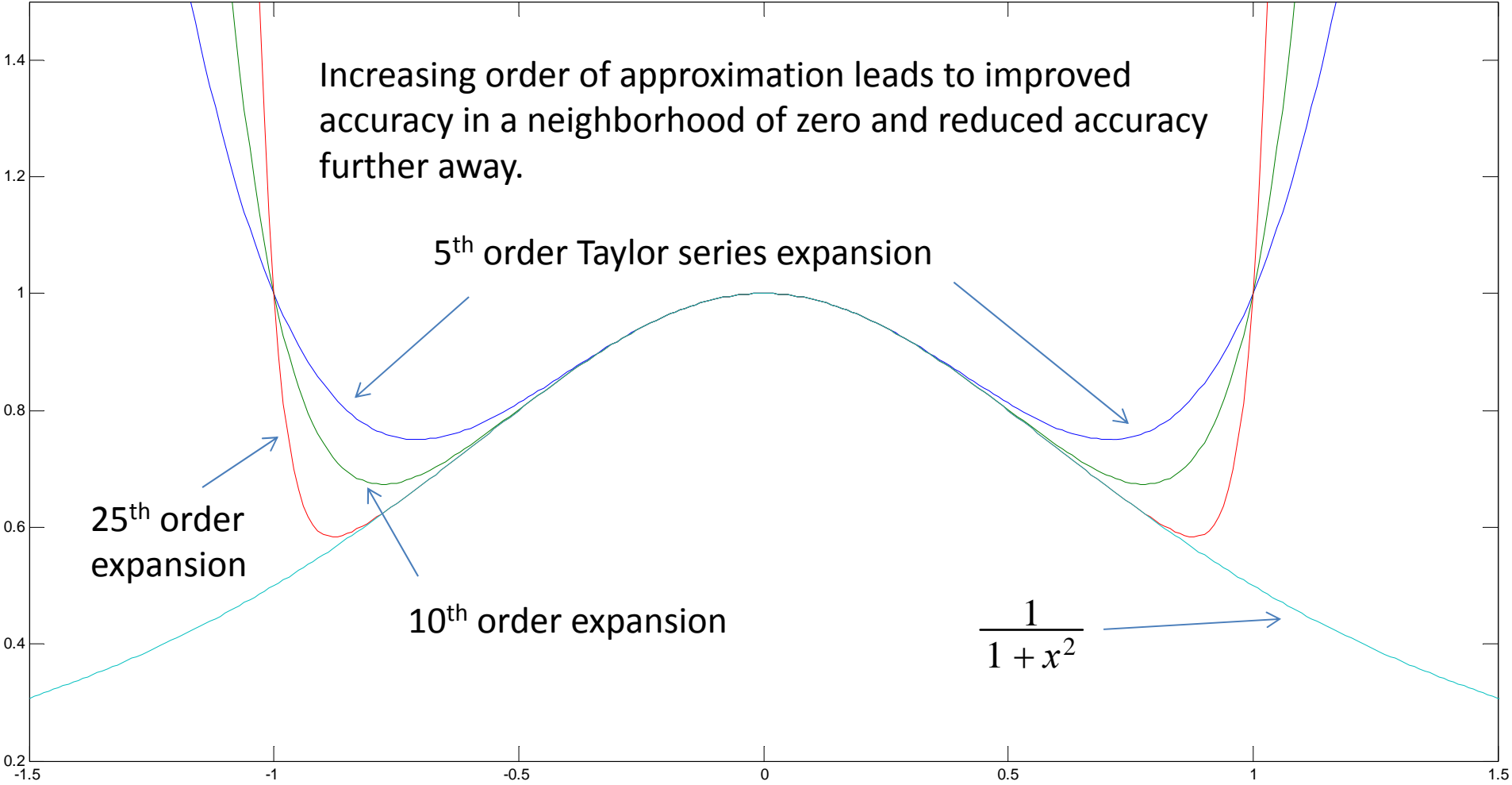
  - Question: is the Taylor series expansion a good approximation for *f*?

Taylor Series Expansion

- It's not as good as you might have thought.

- The next slide exhibits the accuracy of the Taylor series approximation to the Runge function.

  - In a small neighborhood of the point where the approximation is computed (i.e., *0*), higher order Taylor series approximations are increasingly accurate.

  - Outside that small neighborhood, the quality of the approximation deteriorates with higher order approximations.

Taylor Series Expansions about 0 of Runge Function

Increasing order of approximation leads to improved accuracy in a neighborhood of zero and reduced accuracy further away.

5th order Taylor series expansion

25th order expansion

10th order expansion

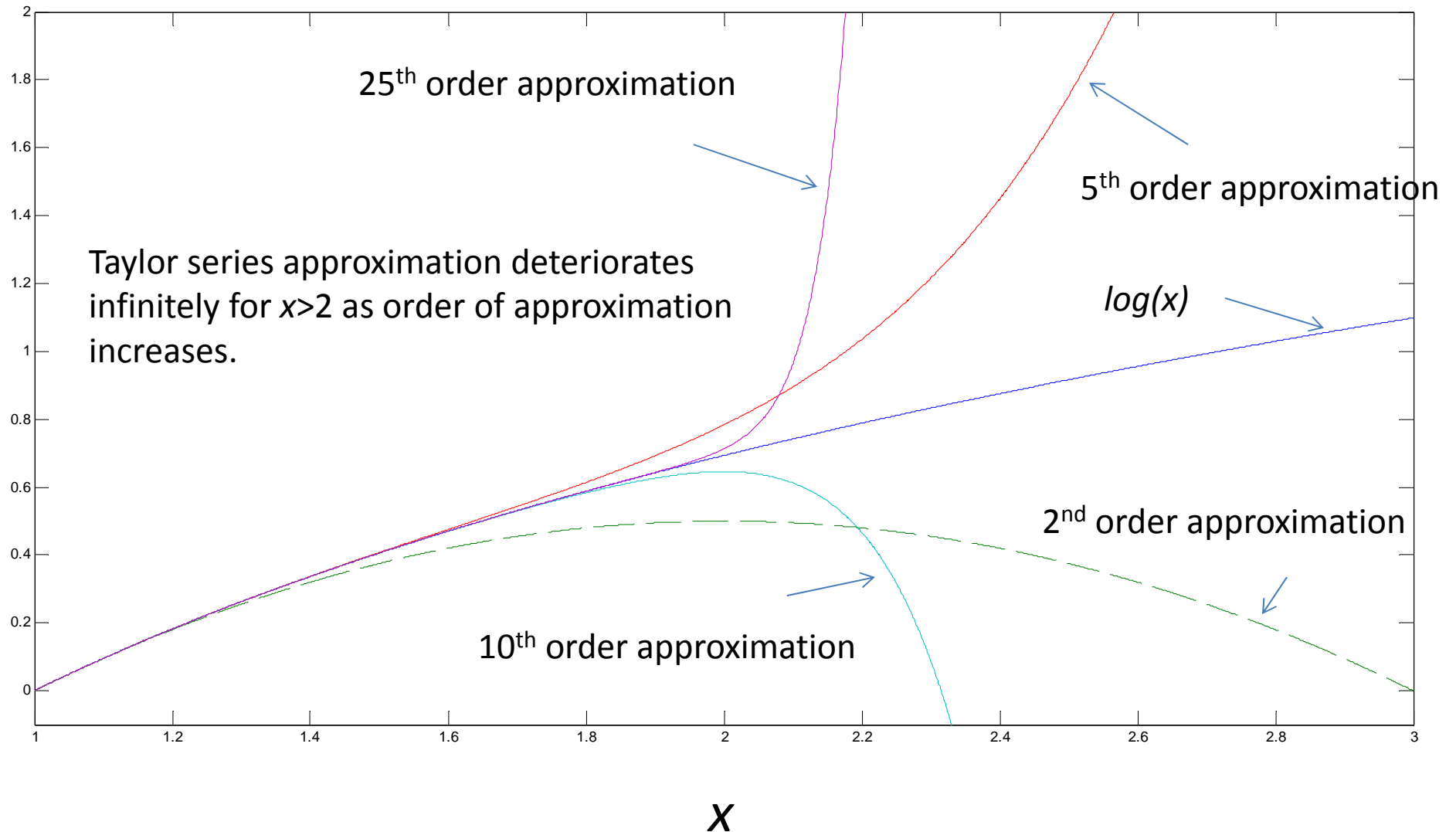$$\frac{1}{1+x^2}$$

x

Taylor Series Expansion

- Another example: the log function
  - It is often used in economics.
  - Surprisingly, Taylor series expansion does not provide a great global approximation.

- Approximate *log(x)* by its kth order Taylor series approximation at the point, *x=a*:

$$\log(x) = \log(a) + \sum_{i=1}^{k} (-1)^{i+1} \frac{1}{i} \left(\frac{x-a}{a} \right)^{i}$$

  - This expression diverges as $N \rightarrow \infty$ for

$$x \text{ such that } \left| \frac{x-a}{a} \right| \geq 1$$

Taylor Series Expansion of Log Function About *x=1*

25th order approximation

5th order approximation

Taylor series approximation deteriorates infinitely for *x*>2 as order of approximation increases.

log(x)

2nd order approximation

10th order approximation

x

Taylor Series Expansion

- In general, cannot expect Taylor series expansion to converge to actual function, globally.

    - There are some exceptions, e.g., Taylor's series expansion of $f(x) = e^x, \cos(x), \sin(x)$ about *x=0* converges to *f(x)* even for *x* far from *0*.

    - Problem: in general it is difficult to say for what values of *x* the Taylor series expansion gives a good approximation.

Taylor Versus Weierstrass

- Problems with Taylor series expansion does not represent a problem with polynomials *per se* as approximating functions.

- Weierstrass approximation theorem: for every continuous function, *f(x)*, defined on *[a,b]*, and for every $\varepsilon > 0$, there exists a finite-ordered polynomial, *p(x)*, on *[a,b]* such that

$$|f(x) - p(x)| < \varepsilon, \text{ for all } x \in [a,b]$$

- Weierstrass – polynomials may approximate well, even if sometimes the Taylor series expansion is not very good.

- Weierstrass theorem asserts ***that there exists <u>some</u> sequence of polynomials the limit of which approximates a function well***.
    - We used the Runge function to illustrate two sequences that do not approximate well: interpolation with fixed interval weights and Taylor series expansion about zero.
    - Interpolation with Chebyshev zeroes gives an excellent approximation.

- Ok, we're done with the digression on the Taylor series expansion.

- Now, back to the discussion of the perturbation method.
  - It approximates a solution using the Taylor series expansion.

Perturbation Method

- Suppose there is a point, $x^* \in X$, where we know the value taken on by the function, *g*, that we wish to approximate:

$$g(x^*) = g^*, \text{ some } x^*$$

- Use the implicit function theorem to approximate *g* in a neighborhood of x^*

- Note:

$$R(x;g) = 0 \text{ for all } x \in X$$

$$\rightarrow$$

$$R^{(j)}(x;g) \equiv \frac{d^j}{dx^j}R(x;g) = 0 \text{ for all } j, \text{ all } x \in X.$$

Perturbation, cnt'd

- Differentiate R with respect to x and evaluate the result at x^*:

$$R^{(1)}(x^*) = \frac{d}{dx}h(x,g(x))\big|_{x=x^*} = h_1(x^*,g^*) + h_2(x^*,g^*)g'(x^*) = 0$$

$$\rightarrow g'(x^*) = -\frac{h_1(x^*,g^*)}{h_2(x^*,g^*)}$$

- Do it again!

$$R^{(2)}(x^*) = \frac{d^2}{dx^2}h(x,g(x))\big|_{x=x^*} = h_{11}(x^*,g^*) + 2h_{12}(x^*,g^*)g'(x^*)$$

$$h_{22}(x^*,g^*)[g'(x^*)]^2 + h_2(x^*,g^*)g''(x^*).$$

$\rightarrow$ Solve this linearly for $g''(x^*)$.

Perturbation, cnt'd

- Preceding calculations deliver (assuming enough differentiability, appropriate invertibility, a high tolerance for painful notation!), recursively:

$$g'(x^*), g''(x^*), \ldots, g^{(n)}(x^*)$$

- Then, have the following Taylor's series approximation:

$$g(x) \approx \hat{g}(x)$$

$$\hat{g}(x) = g^* + g'(x^*) \times (x - x^*)$$

$$+ \frac{1}{2} g''(x^*) \times (x - x^*)^2 + \ldots + \frac{1}{n!} g^{(n)}(x^*) \times (x - x^*)^n$$

Perturbation, cnt'd

- Check….
- Study the graph of

$$R(x; \hat{g})$$

  – over $x \in X$ to verify that it is everywhere close to zero (or, at least in the region of interest).

Example: a Circle
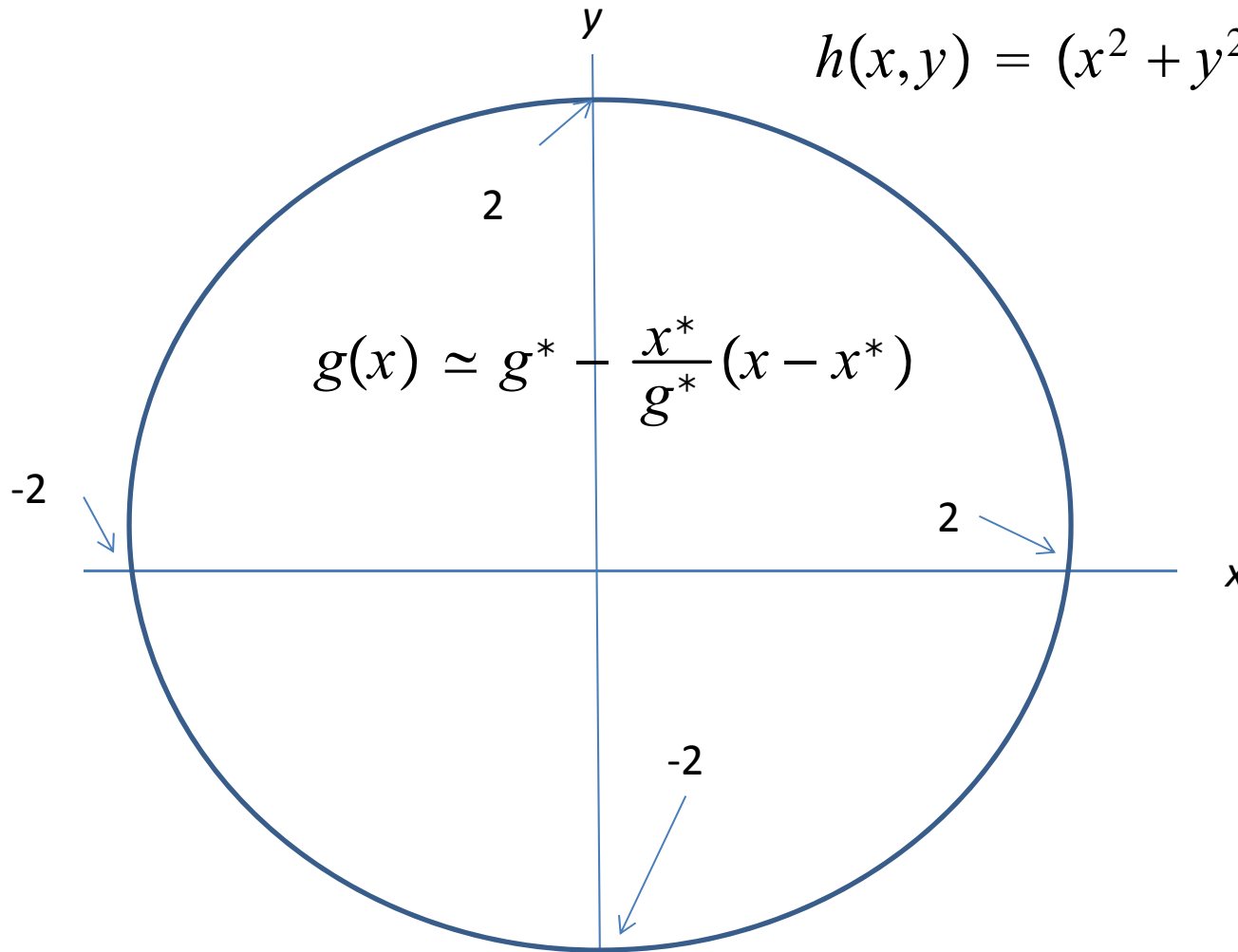
- Function:

$$h(x,y) = (x^2 + y^2) - 4 = 0.$$

- For each *x* except *x=-2, 2*, there are two distinct *y* that solve *h(x,y)=0*:

$$y = g_1(x) \equiv +\sqrt{4 - x^2} \,, \; y = g_2(x) \equiv -\sqrt{4 - x^2} \,.$$

- The perturbation method does not require that the function g that solves *h(x,g(x))=0* be unique.
  - When you specify the value of the function, *g*, at the point of the expansion, you select the function whose Taylor series expansion is delivered by the perturbation method.

Example of Implicit Function Theorem

$$h(x,y) = (x^2 + y^2) - 4 = 0.$$

$$g(x) \simeq g^* - \frac{x^*}{g^*}(x - x^*)$$

$$g'(x^*) = -\frac{h_1(x^*, g^*)}{h_2(x^*, g^*)} = -\frac{x^*}{g^*} \quad (h_2 \text{ had better not be zero!})$$

Outline

- A Toy Example to Illustrate the basic ideas.
  - Functional form characterization of model solution.
  - Projections and Perturbations.

- Neoclassical model.
  - Projection methods
  - Perturbation methods

Done!

- Stochastic Simulations and Impulse Responses
  - Focus on perturbation solutions of order two.
  - The need for pruning.

Neoclassical Growth Model

- Objective:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \ u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

- Constraints:

$$c_t + \exp(k_{t+1}) \leq f(k_t, a_t), \ t = 0, 1, 2, \dots.$$

$$a_t = \rho a_{t-1} + \varepsilon_t, \ \varepsilon_t \sim E\varepsilon_t = 0, \ E\varepsilon_t^2 = V_\varepsilon$$

$$f(k_t, a_t) = \exp(\alpha k_t)\exp(a_t) + (1-\delta)\exp(k_t)$$

Efficiency Condition

$$E_t[u'\left(\overbrace{f(k_t,a_t) - \exp(k_{t+1})}^{c_{t+1}}\right)$$

$$-\beta u'\left(\overbrace{f(k_{t+1},\rho a_t + \sigma \varepsilon_{t+1}) - \exp(k_{t+2})}^{c_{t+1}}\right)\overbrace{f_K(k_{t+1},\rho a_t + \sigma \varepsilon_{t+1})}^{\text{period } t+1 \text{ marginal product of capital}}] = 0.$$

$$k_t, a_t \sim \text{given numbers}$$

- Here,
$$\varepsilon_{t+1} \sim \text{random variable}$$

$$\text{time } t \text{ choice variable, } k_{t+1}$$

- Parameter, σ , indexes a set of models, with the model of interest corresponding to

$$\sigma = 1$$

Solution

- A policy rule,

$$k_{t+1} = g(k_t, a_t, \sigma).$$

- With the property:

$$R(k_t, a_t, \sigma; g) \equiv E_t \Big\{ u' \Big(\overbrace{f(k_t, a_t) - \exp[g(k_t, a_t, \sigma)]}^{c_t} \Big)$$

$$- \beta u' \Bigg(\overbrace{f \bigg(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma \varepsilon_{t+1}}^{a_{t+1}} \bigg) - \exp\bigg[g \bigg(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma \varepsilon_{t+1}}^{a_{t+1}}, \sigma \bigg) \bigg]}^{c_{t+1}} \Bigg)$$

$$\times f_K \bigg(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma \varepsilon_{t+1}}^{a_{t+1}} \bigg) \Big\} = 0,$$

- for all a_t, k_t, σ.

Projection Methods

- Let
$$\hat{g}(k_t, a_t, \sigma; \gamma)$$

  – be a function with finite parameters (could be either spectral or finite element, as before).

- Choose parameters, γ, to make
$$R(k_t, a_t, \sigma; \hat{g})$$

  – as close to zero as possible, over a range of values of the state.
  – use weighted residuals or Collocation.

Occasionally Binding Constraints

- Suppose we add the non-negativity constraint on investment:

$$\exp(g(k_t, a_t, \sigma)) - (1 - \delta)\exp(k_t) \geq 0$$

- Express problem in Lagrangian form and optimum is characterized in terms of equality conditions with a multiplier and with a complementary slackness condition associated with the constraint.

- Conceptually straightforward to apply preceding method. For details, see Christiano-Fisher, 'Algorithms for Solving Dynamic Models with Occasionally Binding Constraints', 2000, *Journal of Economic Dynamics and Control.*
  - This paper describes alternative strategies, based on parameterizing the expectation function, that may be easier, when constraints are occasionally binding constraints.

Perturbation Approach

- Straightforward application of the perturbation approach, as in the simple example, requires knowing the value taken on by the policy rule at a point.

- The overwhelming majority of models used in macro do have this property.

  - In these models, can compute non-stochastic steady state without any knowledge of the policy rule, g.
  - Non-stochastic steady state is k^* such that

$$k^* = g\left(\overbrace{k^*}, \quad \overbrace{0}^{a=0 \text{ (nonstochastic steady state in no uncertainty case)}}, \quad \overbrace{0}^{\sigma=0 \text{ (no uncertainty)}} \right)$$

  - and

$$k^* = \log\left\{ \left[\frac{\alpha\beta}{1-(1-\delta)\beta} \right]^{\frac{1}{1-\alpha}} \right\}.$$

Perturbation

- Error function:

$$R(k_t, a_t, \sigma; g) \equiv E_t\{u'\left(\overbrace{f(k_t, a_t) - \exp[g(k_t, a_t, \sigma)]}^{c_t}\right)$$

$$- \beta u'\left[\overbrace{f(g(k_t, a_t, \sigma), \rho a_t + \sigma \varepsilon_{t+1}) - \exp[g(g(k_t, a_t, \sigma), \rho a_t + \sigma \varepsilon_{t+1}, \sigma)]}^{c_{t+1}}\right]$$

$$\times f_K(g(k_t, a_t, \sigma), \rho a_t + \sigma \varepsilon_{t+1})\} = 0,$$

- for all values of k_t, a_t, σ.

- So, all order derivatives of R with respect to its arguments are zero (assuming they exist!).

Four (Easy to Show) Results About Perturbations

- Taylor series expansion of policy rule:

$$g(k_t, a_t, \sigma) \simeq \overbrace{k + g_k(k_t - k) + g_a a_t + g_\sigma \sigma}^{\text{linear component of policy rule}}$$

$$\overbrace{+\frac{1}{2}\left[g_{kk}(k_t - k)^2 + g_{aa}a_t^2 + g_{\sigma\sigma}\sigma^2\right] + g_{ka}(k_t - k)a_t + g_{k\sigma}(k_t - k)\sigma + g_{a\sigma}a_t\sigma + \ldots}^{\text{second and higher order terms}}$$

- $g_\sigma = 0$: to a first order approximation, 'certainty equivalence'

- All terms found by solving linear equations, except coefficient on past endogenous variable, g_k ,which requires solving for eigenvalues

- To second order approximation: slope terms certainty equivalent –
$$g_{k\sigma} = g_{a\sigma} = 0$$
- Quadratic, higher order terms computed recursively.

First Order Perturbation

- Working out the following derivatives and evaluating at $k_t = k^*, a_t = \sigma = 0$

$$R_k(k_t, a_t, \sigma; g) = R_a(k_t, a_t, \sigma; g) = R_\sigma(k_t, a_t, \sigma; g) = 0$$

'problematic term'

Source of certainty equivalence
In linear approximation

- Implies:

$$R_k = u''(f_k - e^g g_k) - \beta u' f_{Kk} g_k - \beta u''(f_k g_k - e^g g_k^2) f_K = 0$$

$$R_a = u''(f_a - e^g g_a) - \beta u'[f_{Kk} g_a + f_{Ka} \rho] - \beta u''(f_k g_a + f_a \rho - e^g[g_k g_a + g_a \rho]) f_K = 0$$

$$R_\sigma = -[u' e^g + \beta u''(f_k - e^g g_k) f_K] g_\sigma = 0$$

Absence of arguments in these functions reflects they are evaluated in $k_t = k^*, a_t = \sigma = 0$

Technical notes for following slide

$$u''(f_k - e^g g_k) - \beta u' f_{Kk} g_k - \beta u''(f_k g_k - e^g g_k^2)f_K = 0$$

$$\frac{1}{\beta}(f_k - e^g g_k) - u'\frac{f_{Kk}}{u''}g_k - (f_k g_k - e^g g_k^2)f_K = 0$$

$$\frac{1}{\beta}f_k - \left[\frac{1}{\beta}e^g + u'\frac{f_{Kk}}{u''} + f_k f_K\right]g_k + e^g g_k^2 f_K = 0$$

$$\frac{1}{\beta}\frac{f_k}{e^g f_K} - \left[\frac{1}{\beta f_K} + \frac{u'}{u''}\frac{f_{Kk}}{e^g f_K} + \frac{f_k}{e^g}\right]g_k + g_k^2 = 0$$

$$\frac{1}{\beta} - \left[1 + \frac{1}{\beta} + \frac{u'}{u''}\frac{f_{Kk}}{e^g f_K}\right]g_k + g_k^2 = 0$$

- Simplify this further using:

$$\beta f_K \sim \text{steady state equation}$$

$$f_K = \alpha K^{\alpha-1}\exp(a) + (1 - \delta),\ K \equiv \exp(k)$$

$$\quad = \alpha\exp[(\alpha - 1)k + a] + (1 - \delta)$$

$$f_k = \alpha\exp[\alpha k + a] + (1 - \delta)\exp(k) = f_K e^g$$

$$f_{Kk} = \alpha(\alpha - 1)\exp[(\alpha - 1)k + a]$$

$$f_{KK} = \alpha(\alpha - 1)K^{\alpha-2}\exp(a) = \alpha(\alpha - 1)\exp[(\alpha - 2)k + a] = f_{Kk}e^{-g}$$

- to obtain polynomial on next slide.

First Order, cont'd

- Rewriting $R_k = 0$ term:

$$\frac{1}{\beta} - \left[1 + \frac{1}{\beta} + \frac{u'}{u''} \frac{f_{KK}}{f_K} \right] g_k + g_k^2 = 0$$

- There are two solutions, $0 < g_k < 1, \; g_k > \frac{1}{\beta}$
  - Theory (see Stokey-Lucas) tells us to pick the smaller one.
  - In general, could be more than one eigenvalue less than unity: multiple solutions.
- Conditional on solution to $g_k, \; g_a$ solved for linearly using $R_a = 0$ equation.
- These results all generalize to multidimensional case

Numerical Example

- Parameters taken from Prescott (1986):

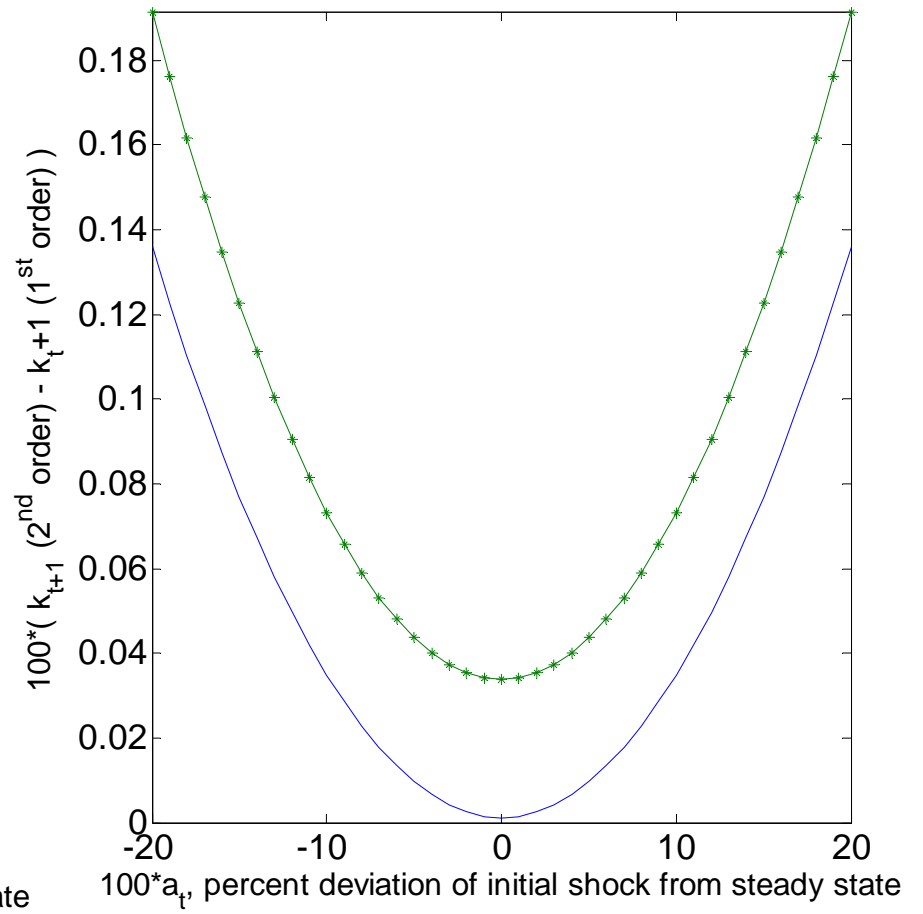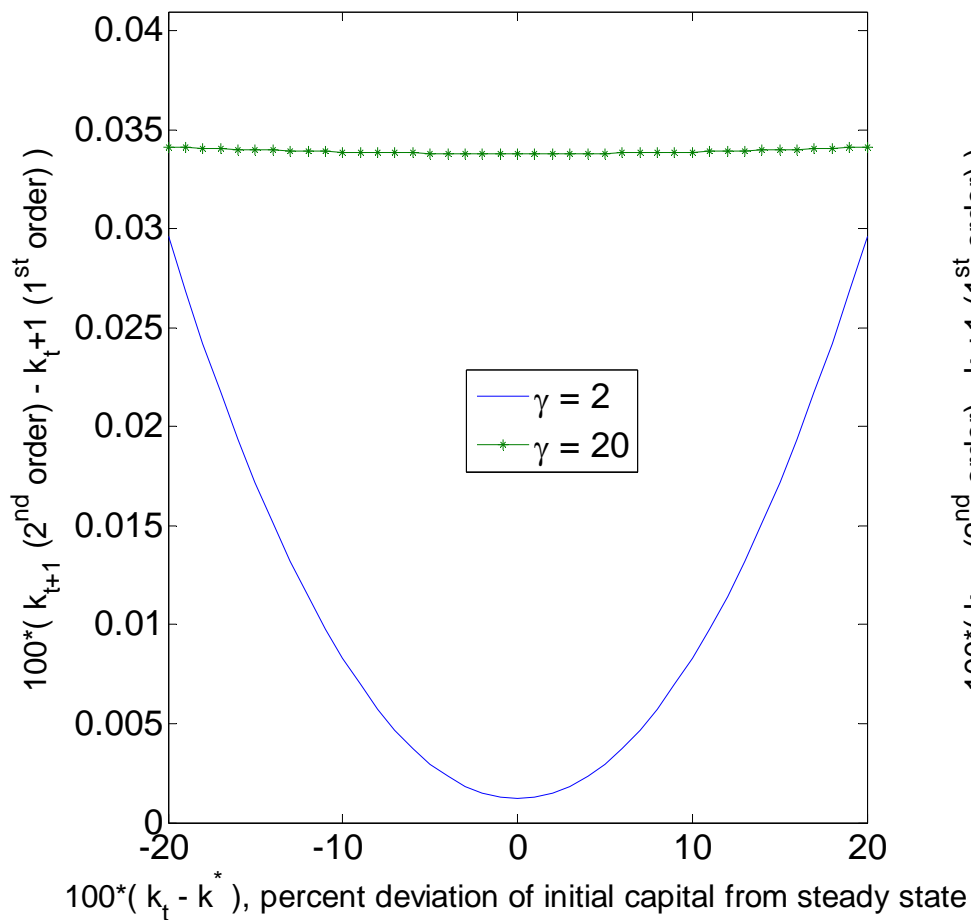$$\beta = 0.99, \ \gamma = 2(20), \ \alpha = 0.36, \ \delta = 0.02, \ \rho = 0.95, \ V_\varepsilon = 0.01^2$$

- Second order approximation:

$$\hat{g}(k_t, a_{t-1}, \varepsilon_t, \sigma) = \overbrace{k^*}^{3.88} + \overbrace{g_k}^{0.98\,(0.996)}(k_t - k^*) + \overbrace{g_a}^{0.06\,(0.07)} a_t + \overbrace{g_\sigma}^{0} \sigma$$

$$+ \frac{1}{2}[\ \overbrace{g_{kk}}^{0.014\,(0.00017)}(k_t - k)^2 + \overbrace{g_{aa}}^{0.067\,(0.079)} a_t^2 + \overbrace{g_{\sigma\sigma}}^{0.000024\,(0.00068)} \overbrace{\sigma^2}^{1}\ ]$$

$$+ \overbrace{g_{ka}}^{-0.035\,(-0.028)}(k_t - k)a_t + \overbrace{g_{k\sigma}}^{0}(k_t - k)\sigma + \overbrace{g_{a\sigma}}^{0} a_t\sigma$$

- Following is a graph that compares the policy rules implied by the first and second order perturbation.

- The graph itself corresponds to the baseline parameterization, and results are reported in parentheses for risk aversion equal to 20.

'If initial capital is 20 percent away from steady state, then capital choice differs by 0.03 (0.035) percent between the two approximations.'

'If shock is 6 standard deviations away from its mean, then capital choice differs by 0.14 (0.18) percent between the two approximations'

Left plot:
- y-axis: $100*(k_{t+1}$ (2nd order) - k_t+1 (1st order))
- x-axis: $100*(k_t - k^*)$, percent deviation of initial capital from steady state
- legend: $\gamma = 2$, $\gamma = 20$

Right plot:
- y-axis: $100*(k_{t+1}$ (2nd order) - k_t+1 (1st order))
- x-axis: $100*a_t$, percent deviation of initial shock from steady state

Number in parentheses at top correspond to $\gamma = 20$.

Conclusion

- For modest US-sized fluctuations and for aggregate quantities, it is reasonable to work with first order perturbations.

- First order perturbation: linearize (or, log-linearize) equilibrium conditions around non-stochastic steady state and solve the resulting system.

  – This approach assumes 'certainty equivalence'. Ok, as a first order approximation.

Solution by Linearization

- (log) Linearized Equilibrium Conditions:

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

- Posit Linear Solution:

$$z_t = A z_{t-1} + B s_t$$

$$s_t - P s_{t-1} - \epsilon_t = 0.$$

Exogenous shocks

- To satisfy equil conditions, *A* and *B* must:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \ \ F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

- If there is exactly one *A* with eigenvalues less than unity in absolute value, that's the solution. Otherwise, multiple solutions.

- Conditional on *A*, solve linear system for *B*.