Equilibrium Conditions for the Simple New Keynesian Model

Lawrence J. Christiano

August 24, 2014

- Baseline NK model with no capital and with a competitive labor market.
  - private sector equilibrium conditions
  - Details: http://faculty.wcas.northwestern.edu/~lchrist/course/Korea_2012/intro_NK.pdf
- Use the equilibrium conditions of this model:
  - as a base to introduce financial frictions
  - to illustrate the application of solution methods.

Households

- Problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}$$

$$\text{s.t. } P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + \text{Profits net of taxes}_t$$

- First order conditions:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

$$\exp(\tau_t) C_t N_t^{\varphi} = \frac{W_t}{P_t}.$$

Goods Production

- A homogeneous final good is produced using the following (Dixit-Stiglitz) production function:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

- Each intermediate good, $Y_{i,t}$, is produced as follows:

$$Y_{i,t} = \overbrace{A_t}^{=\exp(a_t)} N_{i,t},\ a_t = \rho a_{t-1} + \varepsilon_t^a$$

- Before discussing the firms that operate these production functions, we briefly investigate the socially efficient ('First Best') allocation of labor across i, for given N_t :

$$N_t = \int_0^1 N_{it} di$$

Efficient Sectoral Allocation of Labor

- With Dixit-Stiglitz final good production function, there is a socially optimal allocation of resources to all the intermediate activities, $Y_{i,t}$
  - It is optimal to run them all at the same rate, *i.e.*, $Y_{i,t} = Y_{j,t}$ for all $i, j \in [0, 1]$.
- For given N_t, it is optimal to set $N_{i,t} = N_{j,t}$ for all $i, j \in [0, 1]$
- In this case, final output is given by

$$Y_t = e^{a_t} N_t.$$

- Best way to see this is to suppose that labor is *not* allocated equally to all activities.
  - But, this can happen in a million different ways when there is a continuum of inputs!
  - Explore one simple deviation from $N_{i,t} = N_{j,t}$ for all $i, j \in [0, 1]$.

Suppose Labor *Not* Allocated Equally

- Example:

$$N_{it} = \begin{cases} 2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\ 2(1-\alpha)N_t & i \in \left[\frac{1}{2}, 1\right] \end{cases}, \; 0 \leq \alpha \leq 1.$$

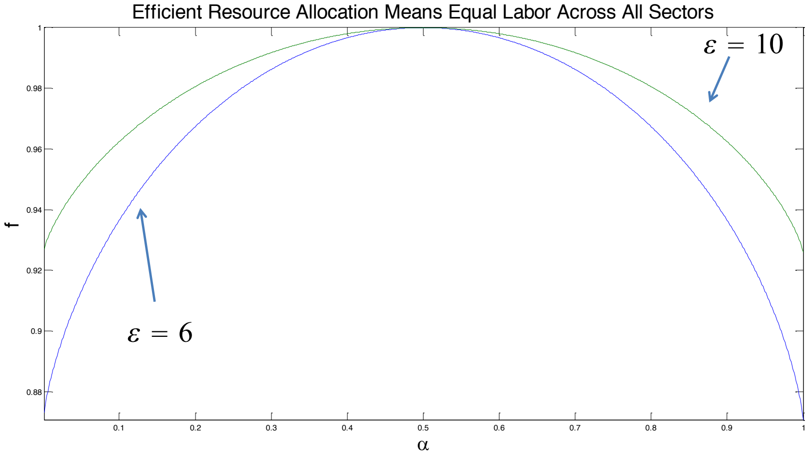
- Note that this is a particular distribution of labor across activities:

$$\int_0^1 N_{it} di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2(1-\alpha) N_t = N_t$$

Labor *Not* Allocated Equally, cnt'd

$$
\begin{aligned}
Y_t &= \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\int_0^{\frac{1}{2}} Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} \left[\int_0^{\frac{1}{2}} N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} \left[\int_0^{\frac{1}{2}} (2\alpha N_t)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 (2(1-\alpha)N_t)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} N_t \left[\int_0^{\frac{1}{2}} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} N_t \left[\frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t} N_t f(\alpha)
\end{aligned}
$$

$$f(\alpha) = \left[\frac{1}{2}(2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2}(2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$



Final Goods Production

- Final good firms:
  - maximize profits:

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} dj,$$

subject to:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

  - Foncs:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon} \rightarrow \overbrace{P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}}^{\text{"cross price restrictions"}}$$

Intermediate Goods Production

- Demand curve for i^{th} monopolist:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon}.$$

- Production function:

$$Y_{i,t} = \exp(a_t) N_{i,t},\ a_t = \rho a_{t-1} + \varepsilon_t^a$$

- Calvo Price-Setting Friction:

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1-\theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases}.$$

- Real marginal cost:

$$s_t = \frac{\frac{d\text{Cost}}{d\text{worker}}}{\frac{d\text{output}}{d\text{worker}}} = \frac{\overbrace{(1-\nu)}^{\text{minimize monopoly distortion by setting } = \frac{\varepsilon-1}{\varepsilon}} \frac{W_t}{P_t}}{\exp(a_t)}$$

Optimal Price Setting by Intermediate Goods Producers

- Let

$$\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t},\ \bar{\pi}_t \equiv \frac{P_t}{P_{t-1}}.$$

- Optimal price setting:

$$\tilde{p}_t = \frac{K_t}{F_t},$$

where

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} (1)$$

$$F_t = 1 + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1}. (2)$$

- Note:

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1} + (\beta\theta)^2 E_t \bar{\pi}_{t+2}^{\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+2} + \dots$$

Goods and Price Equilibrium Conditions

- Cross-price restrictions imply, given the Calvo price-stickiness:

$$P_t = \left[(1-\theta) \tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

- Dividing latter by P_t and solving:

$$\tilde{p}_t = \left[\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} \rightarrow \frac{K_t}{F_t} = \left[\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} \quad (3)$$

- Relationship between aggregate output and aggregate inputs:

$$C_t = p_t^* A_t N_t, \text{ (6)}$$

$$\text{where } p_t^* = \left[(1-\theta) \left(\frac{1-\theta \bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta \frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \quad (4)$$

Tak Yun Distortion: a Closer Look

- Tak Yun showed (JME):

$$p_t^* = p^*\left(\bar{\pi}_t, p_{t-1}^*\right) \equiv \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1}$$

  - Distortion, p_t^*, increasing function of lagged distortion, p_{t-1}^*.
  - Current shocks affect current distortion via $\bar{\pi}_t$ only.

- Derivatives:

$$p_1^*\left(\bar{\pi}_t, p_{t-1}^*\right) = -\left(p_t^*\right)^2 \varepsilon\theta\bar{\pi}_t^{\varepsilon-2}\left[\frac{\bar{\pi}_t}{p_{t-1}^*} - \left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}}\right]$$

$$p_2^*\left(\bar{\pi}_t, p_{t-1}^*\right) = \left(\frac{p_t^*}{p_{t-1}^*}\right)^2 \theta\bar{\pi}_t^{\varepsilon}.$$

Linear Expansion of Tak Yun Distortion in Undistorted Steady State

- Linearizing about $\bar{\pi}_t = \bar{\pi}$, $p^*_{t-1} = p^*$:

$$dp^*_t = p^*_1 (\bar{\pi}, p^*) \, d\bar{\pi}_t + p^*_2 (\bar{\pi}, p^*) \, dp^*_{t-1},$$

where $dx_t \equiv x_t - x$, for $x_t = p^*_t, p^*_{t-1}, \bar{\pi}_t$.

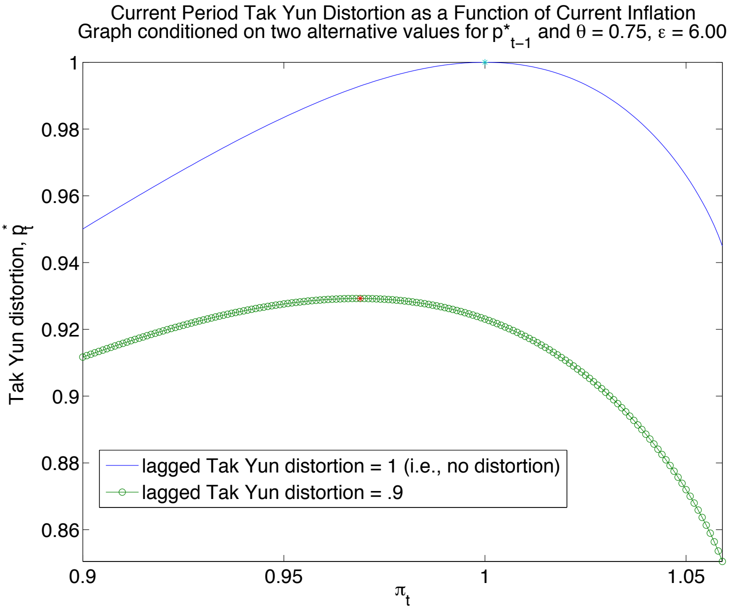
- In an undistorted steady state (i.e., $\bar{\pi}_t = p^*_t = p^*_{t-1} = 1$) :

$$p^*_1 (1,1) = 0, \; p^*_1 (1,1) = \theta.$$

so that

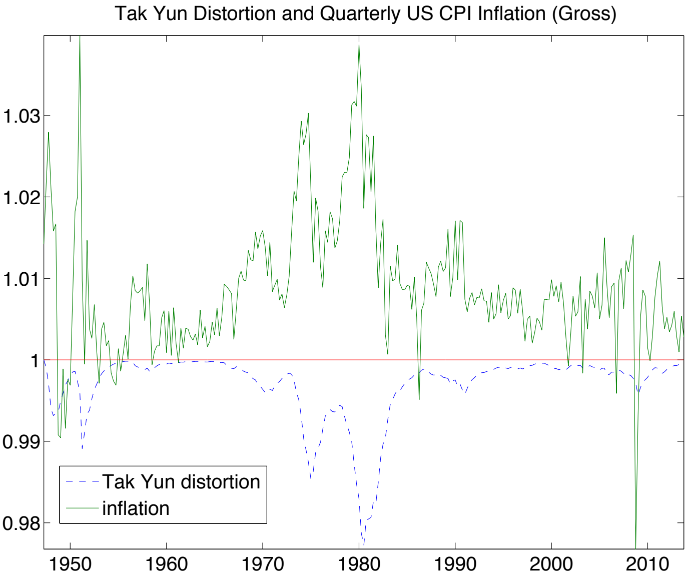
$$\begin{aligned} dp^*_t &= 0 \times d\bar{\pi}_t + \theta dp^*_{t-1} \\ &\rightarrow p^*_t = 1 - \theta + \theta p^*_{t-1} \end{aligned}$$

- Often, people that linearize NK model ignore p^*_t.
  - Reflects that they linearize the model around a price-undistorted steady state.



Current Period Tak Yun Distortion as a Function of Current Inflation
Graph conditioned on two alternative values for p*_{t−1} and θ = 0.75, ε = 6.00
Tak Yun distortion, p*_t
π_t
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.9
0.95
1
1.05
lagged Tak Yun distortion = 1 (i.e., no distortion)
lagged Tak Yun distortion = .9

Ignoring Tak Yun Distortion, a Mistake?



Linearizing around Efficient Steady State

- In steady state (assuming $\bar{\pi} = 1$, $1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$)

$$p^* = 1,\ K = F = \frac{1}{1 - \beta\theta},\ s = \frac{\varepsilon - 1}{\varepsilon},\ \Delta a = \tau = 0,\ N = 1$$

- Linearizing the Tack Yun distortion, (4):

$$\boxed{p_t^* = 1,\ t \text{ large enough}}$$

- Denote the *output gap* in ratio form by X_t:

$$X_t \equiv \frac{C_t}{A_t \exp\left(-\frac{\tau_t}{1+\varphi}\right)} = p_t^* N_t \exp\left(\frac{\tau_t}{1+\varphi}\right),$$

where the denominator is the socially efficient ('First Best') level of consumption.

Output Gap and First Best Consumption, Employment

- Explained above that with socially efficient sectoral allocation of labor,

$$Y_t = \exp(a_t) N_t.$$

- First best level of employment and consumption is solution to

$$N_t^{\text{best}} = \arg\max_N \left\{ \log\left[\exp(a_t) N\right] - \exp(\tau_t) \frac{N^{1+\varphi}}{1+\varphi} \right\}$$

so,

$$N_t^{\text{best}} = \exp\left(-\frac{\tau_t}{1+\varphi}\right),\ C_t^{\text{best}} = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)$$

- Then, using expression on previous slide for X_t (and, $x_t \equiv \hat{X}_t$):

$$\boxed{x_t = \hat{N}_t + \frac{d\tau_t}{1+\varphi}}$$

NK IS Curve, Baseline Model

- The intertemporal Euler equation, (5), after substituting for C_t in terms of X_t:

$$\frac{1}{X_t A_t \exp\left(-\frac{\tau_t}{1+\varphi}\right)} = \beta E_t \frac{1}{X_{t+1} A_{t+1} \exp\left(-\frac{\tau_{t+1}}{1+\varphi}\right)} \frac{R_t}{\bar{\pi}_{t+1}}$$

$$\frac{1}{X_t} = E_t \frac{1}{X_{t+1} R^*_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}},$$

where

$$R^*_{t+1} \equiv \frac{1}{\beta} \exp\left(a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right)$$

then, use

$$\widehat{z_t u_t} = \hat{z}_t + \hat{u}_t, \ \widehat{\left(\frac{u_t}{z_t}\right)} = \hat{u}_t - \hat{z}_t$$

to obtain:

$$\hat{X}_t = E_t \left[\hat{X}_{t+1} - \left(\hat{R}_t - \widehat{\bar{\pi}}_{t+1} - \hat{R}^*_{t+1}\right)\right]$$

NK IS Curve, Baseline Model

- We now want to establish (when the steady state is efficient) the following result:

$$\begin{aligned} & E_t\left(\hat{R}_t - \widehat{\bar{\pi}}_{t+1} - \hat{R}^*_{t+1}\right) \\ = \; & r_t - E_t \pi_{t+1} - r^*_t, \end{aligned}$$

  where

$$r_t \equiv \log R_t,\ r^*_t \equiv E_t \log R^*_{t+1},\ \pi_{t+1} \equiv \log \bar{\pi}_{t+1}.$$

- Note:

$$\begin{aligned} Z_t &= \exp(z_t)\,, \text{ where } z_t \equiv \log Z_t \\ \hat{Z}_t &\equiv \frac{dZ_t}{Z} = \frac{d\exp(z_t)}{Z} = \frac{Zdz_t}{Z} = dz_t. \end{aligned}$$

- The result follows easily from the following facts in efficient steady state:

$$\log R^* = \log R,\ \log \bar{\pi} = 0.$$

NK IS Curve, Baseline Model

- Substituting

$$\hat{X}_t = E_t\left[\hat{X}_{t+1} - \left(\hat{R}_t - \widehat{\bar{\pi}}_{t+1} - \hat{R}^*_{t+1}\right)\right],\ x_t \equiv \hat{X}_t,$$

we obtain NK IS curve:

$$\boxed{x_t = E_t x_{t+1} - E_t\left[r_t - \pi_{t+1} - r^*_t\right]}$$

- Also,

$$r^*_t = -\log\left(\beta\right) + E_t\left[a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right].$$

Linearized Marginal Cost in Baseline Model

- Marginal cost (using $da_t = a_t$, $d\tau_t = \tau_t$ because $a = \tau = 0$):

$$\begin{aligned} s_t &= (1-\nu)\frac{\bar{w}_t}{A_t},\ \bar{w}_t = \exp(\tau_t) N_t^{\varphi} C_t \\ &\rightarrow \widehat{\bar{w}}_t = \tau_t + a_t + (1+\varphi)\hat{N}_t \end{aligned}$$

- Then,

$$\boxed{\hat{s}_t = \widehat{\bar{w}}_t - a_t = (\varphi+1)\left[\tfrac{\tau_t}{\varphi+1} + \hat{N}_t\right] = (\varphi+1)\,x_t}$$

Linearized Phillips Curve in Baseline Model

- Log-linearize equilibrium conditions, (1)-(3), around steady state:

$\hat{K}_t = (1-\beta\theta)\,\hat{s}_t + \beta\theta\left(\varepsilon\hat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right)$ (1)
$\hat{F}_t = \beta\theta\,(\varepsilon - 1)\,\hat{\bar{\pi}}_{t+1} + \beta\theta\hat{F}_{t+1}$ (2)
$\hat{K}_t = \hat{F}_t + \frac{\theta}{1-\theta}\hat{\bar{\pi}}_t$ (3)

- Substitute (3) into (1)

$$\hat{F}_t + \frac{\theta}{1-\theta}\hat{\pi}_t = (1-\beta\theta)\,\hat{s}_t + \beta\theta\left(\varepsilon\hat{\bar{\pi}}_{t+1} + \hat{F}_{t+1} + \frac{\theta}{1-\theta}\hat{\pi}_{t+1}\right)$$

- Simplify the latter using (2), to obtain the NK Phillips curve:

$$\boxed{\pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{s}_t + \beta\pi_{t+1}}$$

The Linearized Private Sector Equilibrium Conditions

$x_t = x_{t+1} - \left[r_t - \pi_{t+1} - r_t^*\right]$
$\pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{s}_t + \beta\pi_{t+1}$
$\hat{s}_t = (\varphi + 1)\, x_t$
$r_t^* = -\log(\beta) + E_t\left[a_{t+1} - a_t - \frac{\tau_{t+1}-\tau_t}{1+\varphi}\right]$

Nonlinear Private Sector Equilibrium Conditions

$$
\begin{aligned}
K_t &= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} \text{(1)} \\
F_t &= 1 + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1}. \text{(2)} \\
\frac{K_t}{F_t} &= \left[\frac{1-\theta\bar{\pi}_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \quad \text{(3)} \\
p_t^* &= \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1} \quad \text{(4)} \\
\frac{1}{C_t} &= \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \text{ (5)} \\
C_t &= p_t^* A_t N_t. \text{ (6)}
\end{aligned}
$$