The Extended Path Method

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Model Solution

- *Model solution*: a procedure for computing the response of the $N \times 1$ vector of endogenous variables, z_t, of a model to a sequence of values of (potentially stochastic) exogenous variables, y_t.
- *Model* is summarized by a set of equilibrium conditions:

$$E_t v\left(z_{t-1}, z_t, z_{t+1}, y_t, y_{t+1}\right) = 0,\ t \geq 1.$$

and a specification of a stochastic process (possibly deterministic) for the exogenous shocks.

Standard Model Solution Method: Policy Rule

- Standard strategies to model solution -
  - policy rule approach:
    - find $z_t = g\left(z_{t-1}, y_t\right)$ such that

$$E_t v\left(z_{t-1}, g\left(z_{t-1}, y_t\right), g\left(g\left(z_{t-1}, y_t\right), y_{t+1}\right), y_t, y_{t+1}\right) = 0,\ t \geq 1.$$

  - methods for finding policy rule
    - perturbation (with pruning) and projection.
  - Given a realization, $y_1, y_2, ..., y_T$, compute a sequence, $z_1, ..., z_T$:

$$\begin{aligned} z_1 &= g\left(z_0, y_1\right) \\ z_2 &= g\left(z_1, y_2\right) \\ &\dots \end{aligned}$$

Problems with Standard Method

- Has difficulties with certain exotic situations that have become of interest recently.
- Example: forward guidance monetary policy ('Evans rule')
  - keep interest rate at zero until either the unemployment rate hits 6.5 percent or the inflation rate rises to 2.5 percent.
  - as soon as one of these thresholds is achieved, revert to Taylor rule:

$$R_t = \max\left\{1, \rho R_{t-1} + (1-\rho)\left[\phi_\pi \pi_t + \phi_y y_t\right]\right\}$$

  - problem: equilibrium conditions, v, now include 'if, then' statements and max operator. Rules out perturbation (not differentiable) and makes projection difficult (though not impossible..see Christiano-Fisher, JEDC, 2001).
  - see Christiano, Eichenbaum and Trabandt, "Understanding the Great Recession," (forthcoming, AEJ-M) for application of extended path.

Extended Path Model Solution Method

- The *extended path method* can be used for deterministic simulations and for stochastic simulations.
  - A simple stochastic version of the simulation procedure imposes the following condition wherever the expectation operator appears:
  $$Ef(x) = f(Ex).$$
  - We apply this (certainty equivalence) condition, even though it is at best only approximately correct in the non-linear equilibrium conditions that we consider.
- The extended path method was originally proposed in Fair and Taylor, 1983, "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models," Econometrica, vol. 51, no. 4, July, pp. 1169-1185.
- Recent work towards dropping the certainty equivalence assumption is described in Stephane Adjemian and Michel Juillard, "Stochastic Extended Path Approach" (it is not discussed here).

Extended Path: Deterministic Case

- Suppose we have a sequence of values for the exogenous variables, $y_1, y_2, ..., y_T$, with $\lim_{j\to\infty} y_{T+j} = y$
- Compute z, *nonstochastic steady state* value of z_t:

$$v(z, z, z, y, y) = 0.$$

Extended Path: Deterministic Case, cnt'd

- For given z_0, solve for $z_1, ..., z_{T^*-1}$ in the following system of $T^* - 1$ equations:

$$\begin{array}{rcl} v\left(z_0, z_1, z_2, y_1, y_2\right) & = & 0 \\ v\left(z_1, z_2, z_3, y_2, y_3\right) & = & 0 \\ & & \vdots \\ v\left(z_{T^*-3}, z_{T^*-2}, z_{T^*-1}, y_{T^*-2}, y_{T^*-1}\right) & = & 0 \\ v\left(z_{T^*-2}, z_{T^*-1}, z, y_{T^*-1}, y\right) & = & 0. \end{array}$$

- Note: if T^* is too small, then it won't be possible to drive all these equations to zero. In that case, increase T^*. But, don't make T^* unecessarily large!

Extended Path: Deterministic Case, cnt'd

- To solve the problem, consider

$$V(\gamma) = \begin{pmatrix} v(z_0, z_1, z_2, y_1, y_2) \\ \vdots \\ v(z_{T^*-2}, z_{T^*-1}, z, y_{T^*-1}, y) \end{pmatrix},$$

where

$$\gamma = \begin{pmatrix} z_1 \\ \vdots \\ z_{T^*-1} \end{pmatrix},$$

and z_0, z, and the y's are taken as given. We seek γ^* such that $V(\gamma^*) = 0$.

Extended Path: Deterministic Case, cnt'd

- Gradient method for finding γ^*:
  - Compute a sequence, $\gamma_1, \gamma_2, ...$ that (hopefully!) converges to γ^* starting from an initial guess, γ_0.
  - Suppose $\gamma_0, ..., \gamma_{r-1}$ are given and we wish to approximate γ_r. Let

$$V(\gamma) \simeq \hat{V}_r(\gamma) = V(\gamma_{r-1}) + V'(\gamma_{r-1})(\gamma - \gamma_{r-1}),$$

where

$$V'(\gamma_{r-1}) = \frac{dV(\gamma_{r-1})}{d\gamma'_{r-1}},$$

so that V' is a square, $T^* - 1 \times T^* - 1$, matrix with a block-Toeplitz pattern, mostly zeros.

  - The value of γ_r is the value of γ such that $\hat{V} = 0.$,That is

$$\gamma = \gamma_{r-1} - \left[V'(\gamma_{r-1})\right]^{-1} V(\gamma_{r-1})$$

to zero.

Extended Path: Stochastic Case

- Compute a stochastic realization, $y_1, ..., y_T$, from a time series representation for $\{y_t\}$.
- This time series representation can be virtually anything.
  - A linear time series model:

$$y_t = P_0 + P_1 y_{t-1} + \varepsilon_t,$$

  - A mixture of stochastic and deterministic terms.
  - A model with regime switching in response to exogenous shocks or to the values of endogenous variables.
  - Need only to know how to compute $E\left[y_{t+j}|\Omega_t\right], j \geq 1$, where

$$\Omega_t = \{z_0, z_1, ..., z_{t-1}, y_1, ..., y_t\}.$$

Extended Path: Stochastic Case, cnt'd

- Objective:
  - Given realization, $y_1, ..., y_T$, compute a sequence, $z_1, ..., z_T$, that satisfies equilibrium conditions.
  - For each t, want z_t to be a function of Ω_t only.
- At each date, t, agents observe Ω_t and they compute forecasts

$$y_{t+j}^t \equiv E\left[y_{t+j}|\Omega_t\right], \ \lim_{j\rightarrow\infty} y_{t+j}^t = y, \ y_{T^*}^t \simeq y.$$

- Agents proceed as though they have certainty equivalence, acting as though they believe forecasts are certain to occur.
- In this respect, extended path resembles first order perturbation, but extended path otherwise works with the exact non-linear equations.

Stochastic Extended Path: Date 1

- Need to find z_1 as a function of Ω_1
- Equilibrium Conditions

$$E_1 v\left(z_0, z_1, z_2, y_1, y_2\right) \overbrace{\approx}^{\text{certainty equivalence}} v\left(z_0, z_1, z_2^1, y_1, y_2^1\right),$$

where

$$z_{t+j}^t \equiv E_t z_{t+j}$$

- To compute z_1, require z_2^1 which satisfies

$$E_1 v\left(z_1, z_2, z_3, y_2, y_3\right) \overbrace{\approx}^{\text{certainty equivalence, again}} v\left(z_1, z_2^1, z_3^1, y_2^1, y_3^1\right).$$

- Similarly, z_1 requires solving for z_{1+j}^t for $j > 1$:

$$\begin{aligned} & E_1 v\left(z_{1+j-1}, z_{1+j}, z_{1+j+1}, y_{1+j}, y_{1+j+1}\right) \\ \approx \;& v\left(z_{1+j-1}^1, z_{1+j}^1, z_{1+j+1}^1, y_{1+j}^1, y_{1+j+1}^1\right). \end{aligned}$$

Stochastic Extended Path: Date 1

- We have,

$$\lim_{t\to\infty} z_t^1 = z.$$

  - Select the smallest $T^* > T$ such that $z_{T^*}^1 \simeq z$.

- Solve for z_1 by solving for $z_1, z_2^1, z_3^1, ..., z_{T^*-1}^1$ in

$$\begin{aligned}
v\left(z_0, z_1, z_2^1, y_1, y_2^1\right) &= 0 \\
v\left(z_1, z_2^1, z_3^1, y_2^1, y_3^1\right) &= 0 \\
&\vdots \\
v\left(z_{T^*-3}^1, z_{T^*-2}^1, z_{T^*-1}^1, y_{T^*-2}^1, y_{T^*-1}^1\right) &= 0 \\
v\left(z_{T^*-2}^1, z_{T^*-1}^1, z, y_{T^*-1}^1, y\right) &= 0.
\end{aligned}$$

- Same as deterministic problem!

Stochastic Extended Path: Date > 1

- Given $\Omega_t = \{z_0, z_1, ..., z_{t-1}, y_1, ..., y_t\}$, find the value of z_t that solves the following $T^* - 1$ equations for

$$z_t, z_{t+1}^t, ..., z_{T^*}^t,$$

imposing $z_{T^*}^t = z$:

$$\begin{aligned} v\left(z_{t-1}, z_t, z_{t+1}^t, y_t, y_{t+1}^t\right) &= 0 \\ &\vdots \\ v\left(z_{T^*-2}^t, z_{T^*-1}^t, z, y_{T^*-1}^t, y\right) &= 0. \end{aligned}$$

- Do this for $z_1, z_2, ..., z_T$.

Application to Zero Lower Bound Situation

- NK model with competitive labor markets and without capital.
  - Monetary policy:

$$R_t = \max\left\{1, \frac{1}{\beta}\left(\bar{\pi}_t\right)^{1.5}\right\}.$$

- Eggertsson-Woodford (2003) scenario: economy is in nonstochastic steady state up to and including date 0.
  - In date 1 agents' discount rate unexpectedly falls to $r^l < r$, where r is normal value:

$$\begin{aligned} \beta &= \frac{1}{1+r} = 0.99, \\ r &= 0.0101,\ r^l = -0.01. \end{aligned}$$

  - With constant probability, $p = 0.8$, the discount rate remains at r^l and with probability $1-p$ it jumps back up to r, which is its absorbing state.
  - ZLB on interest rate remains binding while $r_t = r^l$.

Widespread View About Properties of E-W Model in ZLB

- Expansionary Regress Hypothesis:
  - "Technological regress produces economic expansion if it occurs while ZLB binds."
  - Interpretation: model implies that a natural (or, other) disaster leads to expansion.
  - It has been argued that this supposed implication of E-W framework is so ridiculous that the whole framework and its implications should be rejected.
- Stakes high:
  - E-W analysis is basis for forward guidance monetary policy and conventional aggregate demand view for current poor US economic performance.

Expansionary Regress Hypothesis

- Basic idea:
  - A bad technology shock raises marginal cost and, other things the same, inflation.
  - With ZLB sufficiently binding, nominal rate will not rise, so real rate must fall.
  - Consumption and output are stimulated ('substitution effect').
- Problem
  - Response of consumption to technology shock also depends on wealth effect.
  - If the technology shock is sufficiently persistent, then wealth effect dominates substitution effect.
  - Does not have to be hugely persistent.

Model of Fall Into ZLB

- Standard Simple New Keynesian model.
  - Equilibrium conditions correspond to six pse conditions, plus
$$R_t = \max\left\{1, \frac{1}{\beta}(\bar{\pi}_t)^{1.5}\right\},$$
where R_t is the gross nominal rate of interest and $\bar{\pi}_t$ is the net inflation rate.
- Adopt Eggertsson-Woodford scenario.
  - Economy is in steady state and in date 1 agents' discount rate unexpectedly is low, falling from r (its normal value) to $r^l < r$.
  - Here,
$$\begin{aligned}\beta &= \frac{1}{1+r} = 0.99, \\ r &= 0.0101,\ r^l = -0.02/4.\end{aligned}$$
  - With constant probability, $p = 0.8$, the discount rate remains at r^l and with probability $1-p$ it jumps back up to r, which is an absorbing state.

Private Sector Equilibrium Conditions

$$
\begin{aligned}
K_t &= \frac{\varepsilon}{\varepsilon-1}\left[(1-\nu)\frac{N_t^{\varphi}C_t}{e^{a_t}}\right] + \frac{1}{1+r_t}\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1}, \\
F_t &= 1 + \frac{1}{1+r_t}\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1},\ \frac{K_t}{F_t} = \left[\frac{1-\theta\bar{\pi}_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}, \\
p_t^* &= \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1}, \\
\frac{1}{C_t} &= \frac{1}{1+r_t}E_t\frac{1}{C_{t+1}}\frac{R_t}{\bar{\pi}_{t+1}},\ C_t = p_t^* e^{a_t} N_t.
\end{aligned}
$$

- $r_1 = r^l$, $P\left[r_{t+1} = r^l | r_t = r^l\right] = p$, $P\left[r_{t+1} = r^h | r_t = r^h\right] = 1.$

Parameter Values and Experiment

- Parameter values:

$$\theta = 0.75,\ \varepsilon = 6,\ 1 - \nu = \frac{\varepsilon - 1}{\varepsilon},\ \varphi = 1.$$

  and

$$r = \frac{1}{\beta} - 1 = 0.01,\ r^l = -0.01$$

- Technology shock

$$a_t = \rho_1 a_{t-1} + \rho_2 a_{t-2} + v_t$$

- Experiment:
  - $r_t = r^l$ for $t = 1, ..., 16$, $r_t = r$ for $t > 16$, $T^* = 116$
  - Consider $v_1 = -0.10$, $v_t = 0$ for $t > 1$. Compute the solution.
  - Consider $v_t = 0$ for $t \geq 0$. Compute the solution.

Findings

- Figures:
  - Figure 1: $\rho_1 = 0.95, \rho_2 = 0$ ('baseline'),
  - Figure 2: $\rho_1 = 0.5$, $\rho_2 = 0$ ('transitory'),
  - Figure 3: $\rho_1 = 0.95$, $\rho_2 = 0.2$ ('persistent').
- When technology shock is persistent, output and employment drop in response to negative technology shock.
  - Drop is proportionally smaller in the ZLB than out of ZLB.
  - Required persistence in line with degree of persistence commonly assumed in RBC literature, and with findings in empirical literature more generally.
- When technology shock lacks persistence then 'Expansionary Regress' is true.

Figure 1: ZLB Episode With and Without Negative Technology Shock, AR(1) coefficient on technology, 0.95

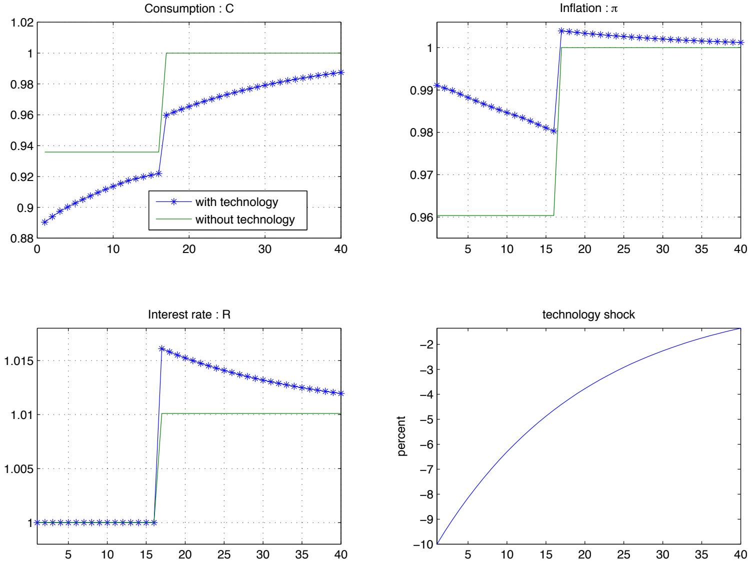

Figure 2: ZLB Episode With and Without Negative Technology Shock, AR(1) coefficient on technology, 0.50

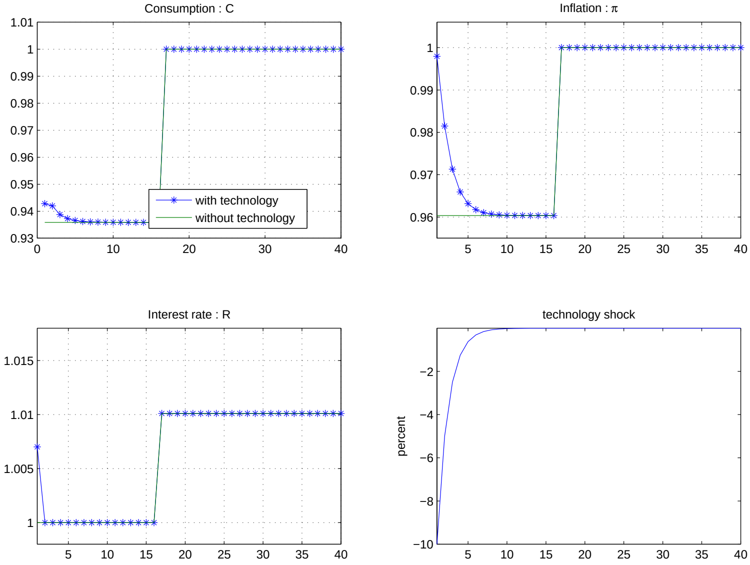

Figure 3: ZLB Episode With and Without Negative Technology Shock
AR(2) roots, 0.95 and 0.2