Foundations for the New Keynesian Model

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Objective

- Describe a very simple model economy with no monetary frictions.
  - Describe its properties.
  - 'markets work well'

- Modify the model to include price setting frictions.
  - Now markets won't necessarily work so well, unless monetary policy is good.

Model

- Household preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\},$$

$$\tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \ \varepsilon_t^\tau \sim iidN(0, \sigma_\varepsilon^2)$$

Production

- Final output requires lots of intermediate inputs:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \ \varepsilon > 1$$

- Production of intermediate inputs:

$$Y_{i,t} = e^{a_t} N_{i,t}, \ \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a, \ \varepsilon_t^a \sim iidN(0, \sigma_a^2)$$

- Constraint on allocation of labor:

$$\int_0^1 N_{it} di = N_t$$

Efficient Allocation of Total Labor

- Suppose total labor, N_t, is fixed.

- What is the best way to allocate N_t among the various activities, $0 \leq i \leq 1$?

- Answer:
  - allocate labor equally across all the activities

$$N_{it} = N_t, \text{ all } i$$

Suppose Labor *Not* Allocated Equally

- Example:

$$N_{it} = \begin{cases} 2\alpha N_t & i \in \left[0, \frac{1}{2} \right] \\ 2(1-\alpha)N_t & i \in \left[\frac{1}{2}, 1 \right] \end{cases}, \ 0 \leq \alpha \leq 1.$$

- Note that this is a particular distribution of labor across activities:
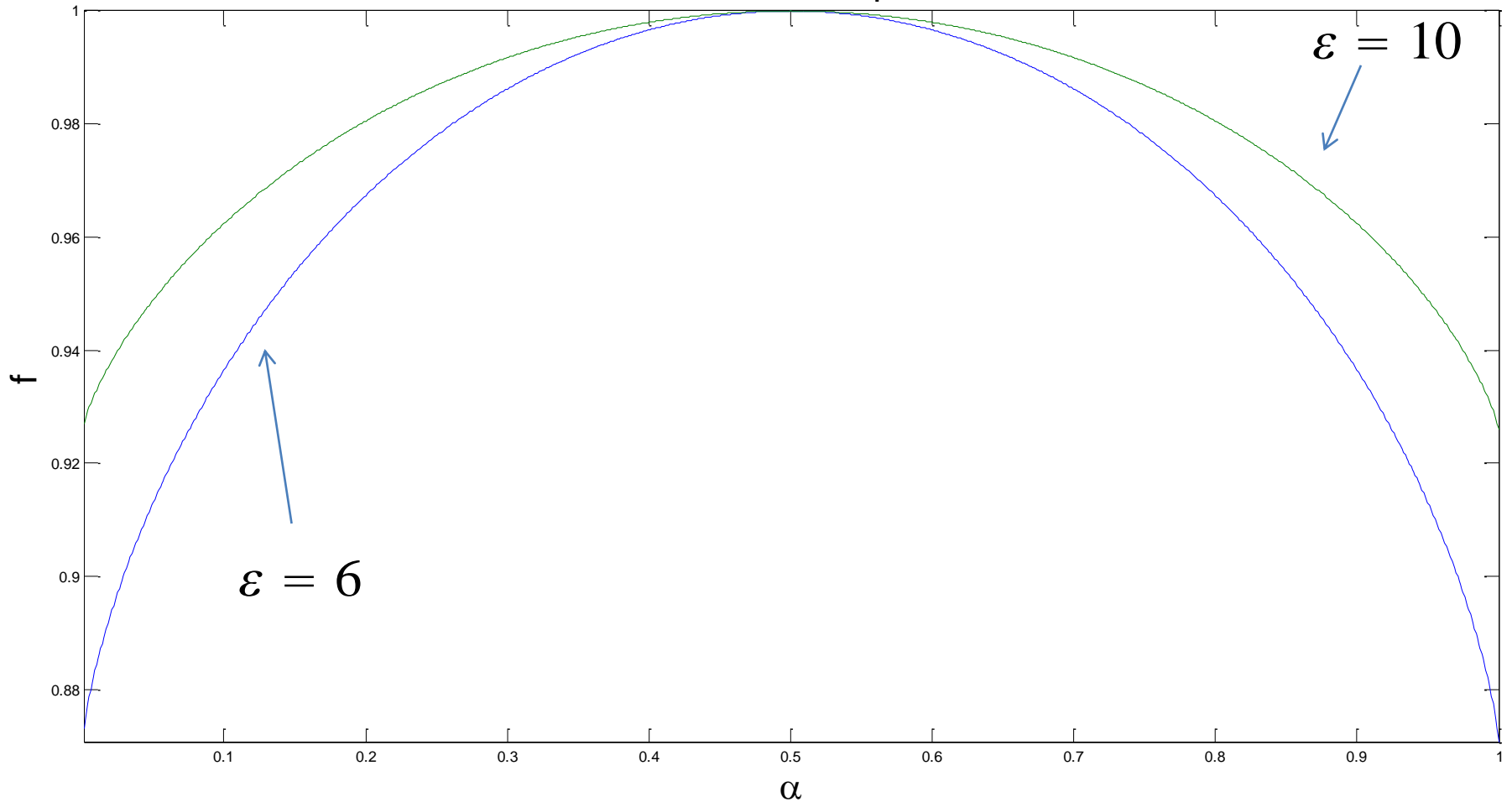
$$\int_0^1 N_{it}di = \frac{1}{2}2\alpha N_t + \frac{1}{2}2(1-\alpha)N_t = N_t$$

Labor *Not* Allocated Equally, cnt'd

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= \left[\int_0^{\frac{1}{2}} Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= e^{a_t} \left[\int_0^{\frac{1}{2}} N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= e^{a_t} \left[\int_0^{\frac{1}{2}} (2\alpha N_t)^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 (2(1-\alpha)N_t)^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= e^{a_t} N_t \left[\int_0^{\frac{1}{2}} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= e^{a_t} N_t \left[\frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= e^{a_t} N_t f(\alpha)$$

$$f(\alpha) = \left[\ \frac{1}{2}(2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2}(2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}}\ \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

Efficient Resource Allocation Means Equal Labor Across All Sectors

Economy with Efficient *N* Allocation

- Efficiency dictates

$$N_{it} = N_t \text{ all } i$$

- So, with efficient production:

$$Y_t = e^{a_t} N_t$$

- Resource constraint:

$$C_t \leq Y_t$$

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \ \varepsilon_t^\tau \sim iid,$$

Efficient Determination of Labor

- Lagrangian:

$$\max_{C_t,N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \overbrace{u(C_t,N_t,\tau_t)}^{=\log C_t - \exp(\tau_t)\frac{N_t^{1+\varphi}}{1+\varphi}} + \lambda_t[e^{a_t}N_t - C_t] \right\}$$

- First order conditions:

$$u_c(C_t,N_t,\tau_t) = \lambda_t, \; u_n(C_t,N_t,\tau_t) + \lambda_t e^{a_t} = 0$$

- or:

$$u_{n,t} + u_{c,t} e^{a_t} = 0$$

marginal cost of labor in consumption units $= \dfrac{-\frac{du}{dN_t}}{\frac{du}{dC_t}} = \dfrac{dC_t}{dN_t}$

marginal product of labor

$$\overbrace{\frac{-u_{n,t}}{u_{c,t}}} \qquad = \qquad \overbrace{e^{a_t}}$$

Efficient Determination of Labor, cont'd

- Solving the fonc's:

$$\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}$$

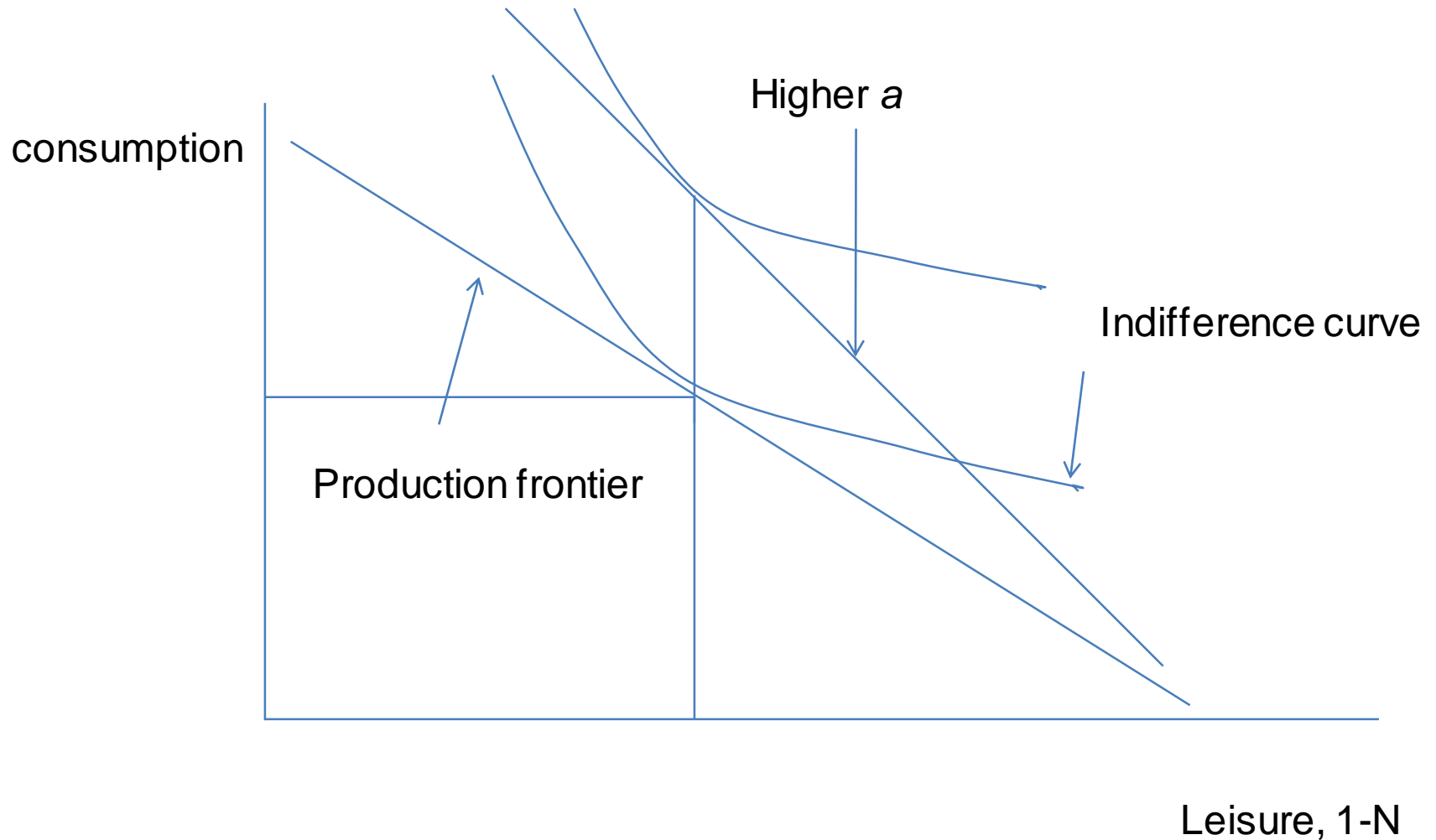$$C_t \exp(\tau_t) N_t^{\varphi} = e^{a_t}$$

$$e^{a_t} N_t \exp(\tau_t) N_t^{\varphi} = e^{a_t}$$

$$\rightarrow N_t = \exp\left(\frac{-\tau_t}{1+\varphi}\right)$$

$$\rightarrow C_t = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)$$

- Note:
  - Labor responds to preference shock, *not* to tech shock

Response to a Jump in *a*

consumption

Higher *a*

Indifference curve

Production frontier

Leisure, 1-N

Decentralizing the Model

- Give households budget constraints and place them in markets.

- Give the production functions to firms and suppose that they seek to maximize profits.

Households

- Solve:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\},$$

- Subject to:

$$C_t + \overbrace{B_{t+1}}^{\text{bonds purchases in } t} \leq \overbrace{w_t}^{\text{wage rate}} N_t + \overbrace{\pi_t}^{\text{profits}} + \overbrace{r_{t-1}}^{\text{(real) interest on bonds}} B_t$$

- First order conditions:

$$\frac{-u_{n,t}}{u_{c,t}} = C_t \exp(\tau_t) N_t^{\varphi} = w_t \quad \text{'marginal cost of working equals marginal benefit'}$$

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \quad \text{'marginal cost of saving equals marginal benefit'}$$

Final Good Firms

- Final good firms buy $Y_{i,t}$, $i \in (0,1)$, at given prices, $P_{i,t}$, to maximize profits:

$$Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Subject to

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

- Fonc's:

$$P_{i,t} = \left(\frac{Y_t}{Y_{i,t}} \right)^{\frac{1}{\varepsilon}}$$

$$\rightarrow Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t, \ 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$

Intermediate Good Firms

- For each $Y_{i,t}$ there is a single producer who is a monopolist in the product market and hires labor, $N_{i,t}$ in competitive labor markets.
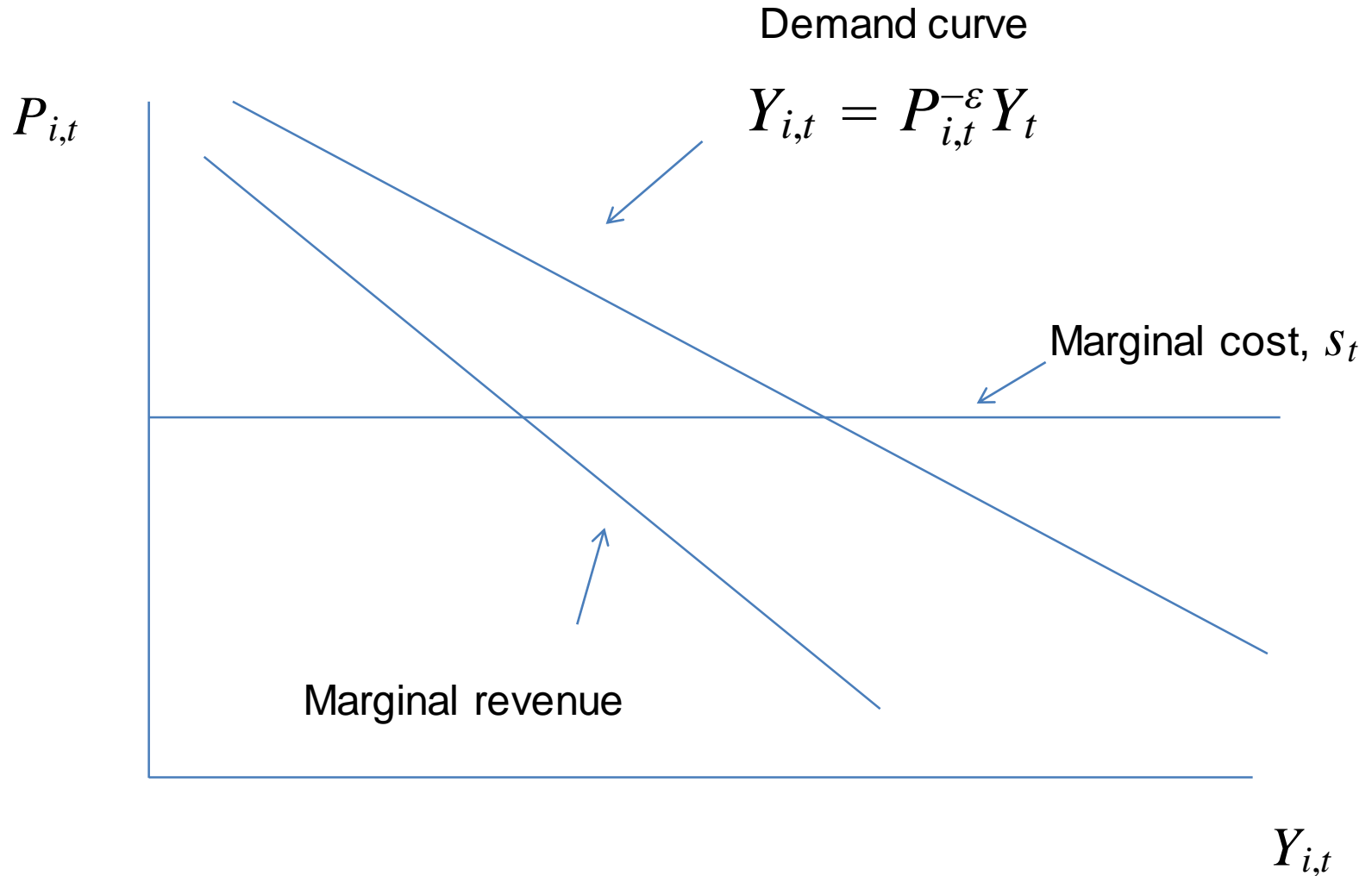
- Marginal cost of production:

(real) marginal cost $= s_t = \dfrac{\frac{dCost}{dwor\ker}}{\frac{doutput}{dwor\ker}} = \dfrac{\left(1 - \overbrace{v}^{\text{subsidy payment to firm}}\right) w_t}{\exp(a_t)}$

- Subsidy will be required to ensure markets work efficiently.

Intermediate Good Firms

Demand curve

$$Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t$$

Marginal cost, s_t

Marginal revenue

$P_{i,t}$

$Y_{i,t}$

i*th* Intermediate Good Firm

- Problem: $\max_{N_{it}} P_{it} Y_{it} - s_t Y_{it}$

- Subject to demand for $Y_{i,t}$: $\quad Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t$

- Problem:

$$\max_{N_{it}} P_{it} P_{i,t}^{-\varepsilon} Y_t - s_t P_{i,t}^{-\varepsilon} Y_t$$

fonc : $(1 - \varepsilon) P_{it}^{-\varepsilon} Y_t + \varepsilon s_t P_{i,t}^{-\varepsilon-1} Y_t = 0$

$$P_{it} = \frac{\varepsilon}{\varepsilon - 1} s_t \text{ 'price is markup over marginal cost'}$$

- Note: all prices are the same, so resources allocated efficiently across intermediate good firms.

$$P_{i,t} = P_{j,t} = 1, \text{ because } 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$

Equilibrium

- Pulling things together:

$$1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1 - v)w_t}{\exp(a_t)}$$

$$\underset{\substack{\text{household fonc}\\ =}}{} \frac{\varepsilon(1 - v)}{\varepsilon - 1} \frac{\frac{-u_{n,t}}{u_{c,t}}}{\exp(a_t)}$$

$$\underset{\substack{\text{if } \frac{\varepsilon(1-v)}{\varepsilon-1} = 1\\ =}}{} \frac{\frac{-u_{n,t}}{u_{c,t}}}{\exp(a_t)}.$$

If proper subsidy is provided to monopolists, employment is efficient:

$$\text{if } 1 - v = \frac{\varepsilon - 1}{\varepsilon}, \text{ then } \frac{-u_{n,t}}{u_{c,t}} = \exp(a_t)$$

Equilibrium Allocations

- With efficient subsidy,

$$\underbrace{\frac{-u_{n,t}}{u_{c,t}}}_{} \overset{\text{functional form}}{\underbrace{=}} C_t \exp(\tau_t)N_t^{\varphi} \overset{\text{resource constraint}}{\underbrace{=}} \exp(a_t)\exp(\tau_t)N_t^{1+\varphi} = \exp(a_t)$$

$$\rightarrow N_t = \exp\left(\frac{-\tau_t}{1+\varphi}\right)$$

$$C_t = e^{a_t}N_t = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)$$

- Bond market clearing implies:

$$B_t = 0 \text{ always}$$

Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t$$

$$\rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp\left[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right]}$$

$$= \frac{1}{\beta \exp\left[E_t\left(-\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right) + \frac{1}{2}V\right]}, \quad V = \sigma_a^2 + \left(\frac{1}{1+\varphi}\right)^2 \sigma_\lambda^2$$

$$\log r_t = -\log \beta + E_t\left(\overbrace{\Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1+\varphi}}^{c_{t+1} - c_t}\right) + \frac{1}{2}V$$

Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t$$

$$\rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp\left[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right]}$$

$$= \frac{1}{\beta \exp\left[E_t\left(-\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right) + \frac{1}{2}V\right]}, \quad V = \sigma_a^2 + \left(\frac{1}{1+\varphi}\right)^2 \sigma_\lambda^2$$
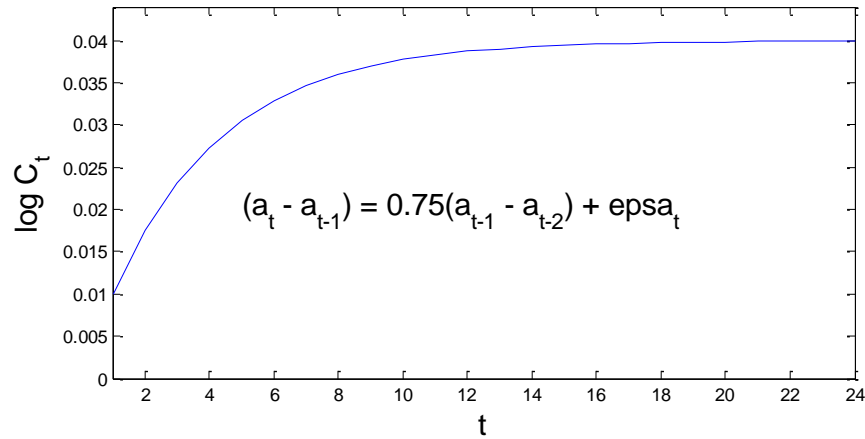
$$\log r_t = -\log\beta + E_t\left(\overbrace{\Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1+\varphi}}^{c_{t+1} - c_t}\right) + \frac{1}{2}V$$
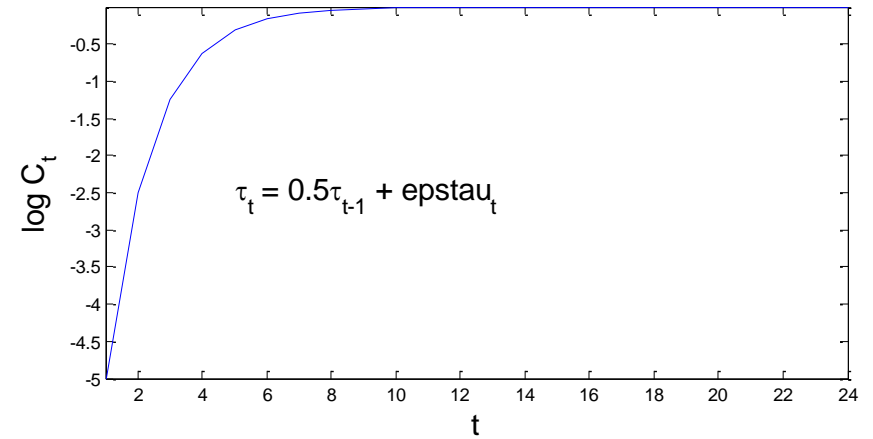
using assumptions about Δa_t and τ_t

$$\overset{=}{} \qquad -\log\beta + \rho\Delta a_t - \frac{(\lambda-1)\tau_t}{1+\varphi} + \frac{1}{2}V$$
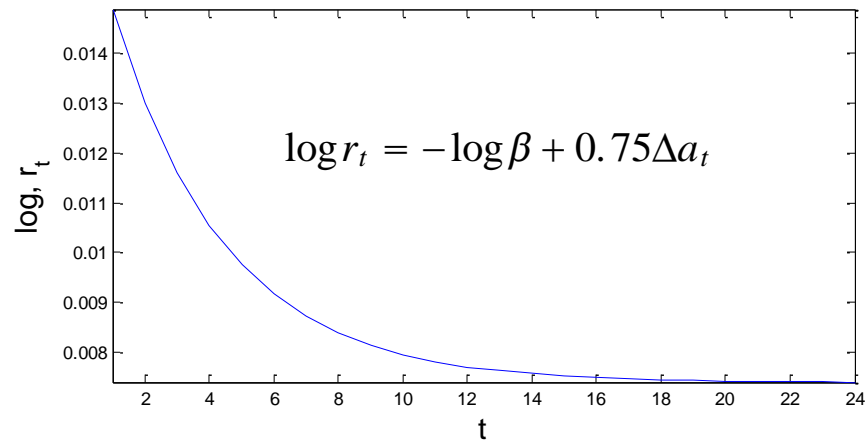
Dynamic Properties of the Model

Response to .01 Technology Shock in Period 1

$$(a_t - a_{t-1}) = 0.75(a_{t-1} - a_{t-2}) + epsa_t$$

Response to .01 Preference Shock in Period 1

$$\tau_t = 0.5\tau_{t-1} + epstau_t$$

interest rate

$$\log r_t = -\log \beta + 0.75\Delta a_t$$

interest rate

$$\log r_t = -\log \beta - (0.5 - 1)\tau_t/(1 + 1)$$
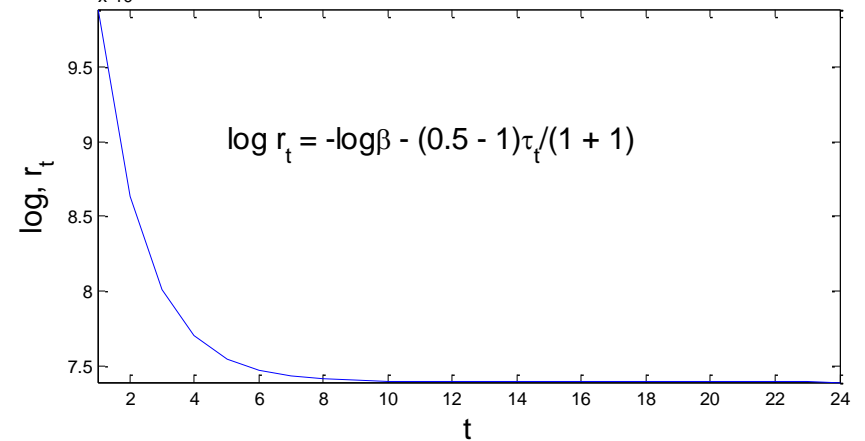
Key Features of Equilibrium Allocations

- Allocations *efficient* (as long as monopoly power neutralized)
- Employment does not respond to technology
  - Improvement in technology raises marginal product of labor and marginal cost of labor by same amount.
- First best consumption not a function of intertemporal considerations
  - Discount rate irrelevant.
  - Anticipated future values of shocks irrelevant.
- Natural rate of interest steers consumption and employment towards their natural levels.

Introducing Price Setting Frictions (Clarida-Gali-Gertler Model)

- Households maximize:

$$E_0 \sum_{t=0}^{\infty} \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \ \varepsilon_t^\tau \sim iid,$$

Profits, net of taxes raised by Government to finance subsidies.

- Subject to:

$$P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + T_t$$

- Intratemporal first order condition:

$$C_t \exp(\tau_t) N_t^\varphi = \frac{W_t}{P_t}$$

Household Intertemporal FONC

- Condition:

$$1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \frac{R_t}{1 + \pi_{t+1}}$$

  – or

$$1 = \beta E_t \frac{C_t}{C_{t+1}} \frac{R_t}{1 + \pi_{t+1}}$$
$$= \beta E_t \exp[\log(R_t) - \log(1 + \pi_{t+1}) - \Delta c_{t+1}]$$
$$\simeq \beta \exp[\log(R_t) - E_t \pi_{t+1} - E_t \Delta c_{t+1}], \ c_t \equiv \log(C_t)$$

  – take log of both sides:

$$0 = \log(\beta) + r_t - E_t \pi_{t+1} - E_t \Delta c_{t+1}, \ r_t = \log(R_t)$$

  – or

$$c_t = -\log(\beta) - [r_t - E_t \pi_{t+1}] + c_{t+1}$$

Final Good Firms

- Buy $Y_{i,t}$, $i \in [0, 1]$ at prices $P_{i,t}$ and sell Y_t for P_t

- Take all prices as given (competitive)

- Profits:

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Production function:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} di, \ \varepsilon > 1,$$

- First order condition:

$$Y_{i,t} = Y_t \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \qquad \rightarrow P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}$$

Intermediate Good Firms

- Each ith good produced by a single monopoly producer.

- Demand curve:

$$Y_{i,t} = Y_t \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon}$$

- Technology:

$$Y_{i,t} = \exp(a_t) N_{i,t}, \ \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a,$$

- Calvo Price-setting Friction

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1 - \theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases}$$

Marginal Cost

real marginal cost $=$ $\quad s_t \; = \; \dfrac{\dfrac{dCost}{dwor\ker}}{\dfrac{dOutput}{dwor\ker}} \; = \; \dfrac{(1-v)W_t/P_t}{\exp(a_t)}$

$$= \; \dfrac{\overbrace{(1-v)}^{=\frac{\varepsilon-1}{\varepsilon}\;\text{in efficient setting}} \; C_t \exp(\tau_t) N_t^{\varphi}}{\exp(a_t)}$$

The Intermediate Firm's Decisions

- i*th* firm is required to satisfy whatever demand shows up at its posted price.

- Its only real decision is to adjust price whenever the opportunity arises.

Intermediate Good Firm

- Present discounted value of ith firm profits:

period $t+j$ profits sent to household

marginal value of dividends to household=$u_{c,t+j}/P_{t+j}$

revenues   total cost

$$E_t^i \sum_{j=0}^{\infty} \beta^j \overbrace{\upsilon_{t+j}} \left[\overbrace{P_{i,t+j}Y_{i,t+j}} - \overbrace{P_{t+j}s_{t+j}Y_{i,t+j}} \right]$$

Conditional expectation over price and aggregate uncertainty (e.g., Δa_t and τ_t).

- Each of the $1-\theta$ firms that can optimize price choose $\tilde{P}_t$ to optimize

Conditional expectation over aggregate uncertainty alone.

in selecting price, firm only cares about
future states in which it can't reoptimize

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\theta^j} \upsilon_{t+j}[\tilde{P}_t Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j}].$$

Intermediate Good Firm Problem

- Substitute out the demand curve:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}]$$

$$= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon}].$$

- Differentiate with respect to $\tilde{P}_t$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [(1-\varepsilon)(\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1}] = 0,$$

- or

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0.$$

Intermediate Good Firm Problem

- Objective:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0.$$

$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j P_{t+j}^{\varepsilon} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_t}, \; X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}}, \; j \geq 1 \\ 1, \; j = 0. \end{cases}, \; X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}}, \; j > 0$$

Intermediate Good Firm Problem

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0$$

- Solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

- But, still need expressions for $K_t, F_t.$

$$K_t = E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}$$

$$= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left(\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}$$

$$= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j}$$

$$= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta \overbrace{E_t E_{t+1}}^{=E_t \text{ by LIME}} \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j}$$

$$= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \overbrace{E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j}}^{\text{exactly } K_{t+1}!}$$

$$= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}$$

- From previous slide:

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

- Substituting out for marginal cost:

$$\frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} (1 - v) \frac{\overbrace{W_t/P_t}^{d\text{Cost}/d\text{labor}}}{\underbrace{\exp(a_t)}_{d\text{Output}/d\text{labor}}}$$

$$= \frac{\varepsilon}{\varepsilon - 1} (1 - v) \frac{\overbrace{\exp(\tau_t)N_t^{\varphi}C_t}^{=\frac{W_t}{P_t} \text{ by household optimization}}}{\exp(a_t)}.$$

In Sum

- solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},$$

- Where:

$$K_t = (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t) N_t^{\varphi} C_t}{\exp(a_t)} + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

$$F_t \equiv E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{1-\varepsilon} F_{t+1}$$

To Characterize Equilibrium

- Have equations characterizing optimization by firms and households.

- Still need:
  - Expression for all the prices. Prices, $P_{i,t}$, $0 \leq i \leq 1$, will all be different because of the price setting frictions.
  - Relationship between aggregate employment and aggregate output not simple because of price distortions:

$$Y_t \neq e^{a_t} N_t, \text{ in general}$$

- This part of the analysis is the reason why it made Calvo famous – it's not easy.

Going for Prices

- Aggregate price relationship

$$P_t = \left[\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

$$= \left[\int_{\text{firms that reoptimize price}} P_{i,t}^{(1-\varepsilon)} di + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

$$\overset{\text{all reoptimizers choose same price}}{=} \left[(1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

- In principle, to solve the model need all the prices, $P_t, P_{i,t}, 0 \leq i \leq 1$
  - Fortunately, that won't be necessary.

Key insight

$$\int_{\text{firms that don't reoptimize price in } t} P_{i,t}^{(1-\varepsilon)} \, di$$

add over prices, weighted by # of firms posting that price

$$=$$

$$\int \left[\overbrace{f_{t-1,t}(\omega)}^{\text{'number' of firms that had price, } P(\omega), \text{ in } t-1 \text{ and were not able to reoptimize in } t} P(\omega)^{(1-\varepsilon)} \right] d\omega$$

Applying the Insight

- By Calvo randomization assumption

total 'number' of firms with price $P(\omega)$ in $t-1$

$$f_{t-1,t}(\omega) = \theta \times \overbrace{f_{t-1}(\omega)} \quad \text{, for all } \omega$$

- Substituting:

$$\int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di = \int f_{t-1,t}(\omega) P(\omega)^{(1-\varepsilon)} d\omega$$

$$= \theta \int f_{t-1}(\omega) P(\omega)^{(1-\varepsilon)} d\omega$$

$$= \theta P_{t-1}^{(1-\varepsilon)}$$

Expression for $\tilde{p}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[(1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

  – Only two variables here!

- Divide by P_t:

$$1 = \left[(1-\theta)\tilde{p}_t^{(1-\varepsilon)} + \theta \left(\frac{1}{\bar{\pi}_t} \right)^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{p}_t = \left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}$$

Relation Between Aggregate Output and Aggregate Inputs

- Technically, there is no 'aggregate production function' in this model

  - If you know how many people are working, N, and the state of technology, a, you don't have enough information to know what Y is.

  - Price frictions imply that resources will not be efficiently allocated among different inputs.

    - Implies Y low for given a and N. How low?
    - Tak Yun (JME) gave a simple answer.

Tak Yun Algebra

$$Y_t^* = \int_0^1 Y_{i,t}\,di \left(= \int_0^1 A_t N_{i,t}\,di \overset{\text{labor market clearing}}{=} A_t N_t \right)$$

$$\overset{\text{demand curve}}{=} Y_t \int_0^1 \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} \int_0^1 (P_{i,t})^{-\varepsilon}\,di$$

Calvo insight

$$= Y_t P_t^{\varepsilon} (P_t^*)^{-\varepsilon}$$

- Where: $\quad P_t^* \equiv \left[\int_0^1 P_{i,t}^{-\varepsilon}\,di \right]^{\frac{-1}{\varepsilon}} = \left[(1-\theta)\tilde{P}_t^{-\varepsilon} + \theta (P_{t-1}^*)^{-\varepsilon} \right]^{\frac{-1}{\varepsilon}}$

Relationship Between Agg Inputs and Agg Output

- Rewriting previous equation:

$$Y_t = \left(\frac{P_t^*}{P_t} \right)^\varepsilon Y_t^*$$

$$= p_t^* e^{a_t} N_t,$$

- 'efficiency distortion':

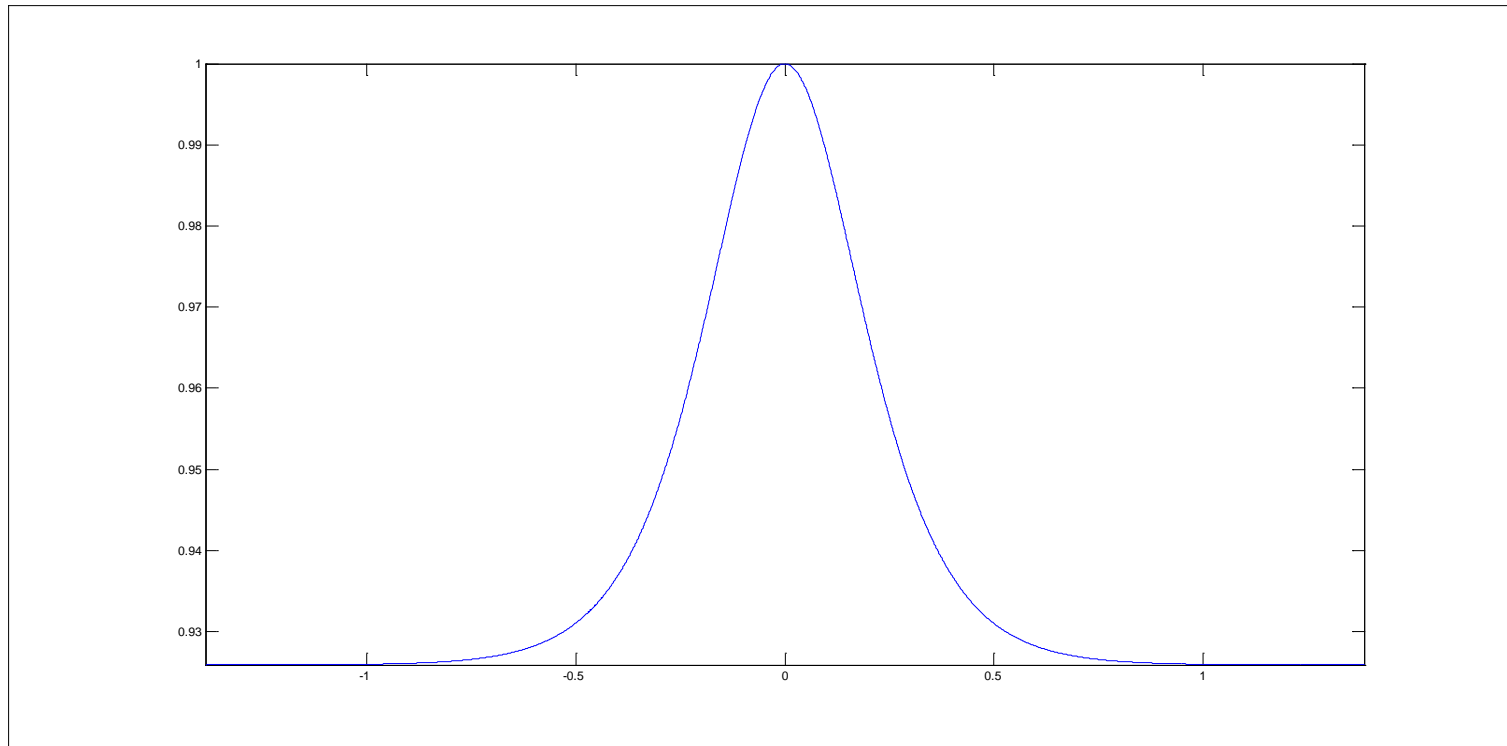$$p_t^* : \begin{cases} \leq 1 \\ = 1 \quad P_{i,t} = P_{j,t}, \text{ all } i,j \end{cases}$$

Example of Efficiency Distortion

$$P_{j,t} = \begin{cases} P^1 & 0 \leq j \leq \alpha \\ P^2 & \alpha \leq j \leq 1 \end{cases} \quad . \quad p_t^* = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} = \left(\frac{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}}{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}} \right)^{\varepsilon}$$

Example of Efficiency Distortion

$$P_{j,t} = \begin{cases} P^1 & 0 \leq j \leq \alpha \\ P^2 & \alpha \leq j \leq 1 \end{cases} \quad \cdot \quad p_t^* = \left(\frac{P_t^*}{P_t}\right)^\varepsilon = \left(\frac{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}}{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}\right)^\varepsilon$$

$\alpha = 0.5, \varepsilon = 10$

$\log P^1/P^2$

Collecting Equilibrium Conditions

- Price setting:

$$K_t = (1-v)\frac{\varepsilon}{\varepsilon - 1}\frac{\exp(\tau_t)N_t^{\varphi}C_t}{A_t} + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon}K_{t+1} \ (1)$$

$$F_t = 1 + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1}F_{t+1} \ (2)$$

- Intermediate good firm optimality and restriction across prices:

$$\underbrace{\frac{K_t}{F_t}}_{=\tilde{p}_t \text{ by firm optimality}} = \overbrace{\left[\frac{1 - \theta\bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta}\right]^{\frac{1}{1-\varepsilon}}}^{=\tilde{p}_t \text{ by restriction across prices}} \ (3)$$

Equilibrium Conditions

- Law of motion of (Tak Yun) distortion:

$$p_t^* = \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right]^{-1} \quad (4)$$

- Household Intertemporal Condition:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

- Aggregate inputs and output:

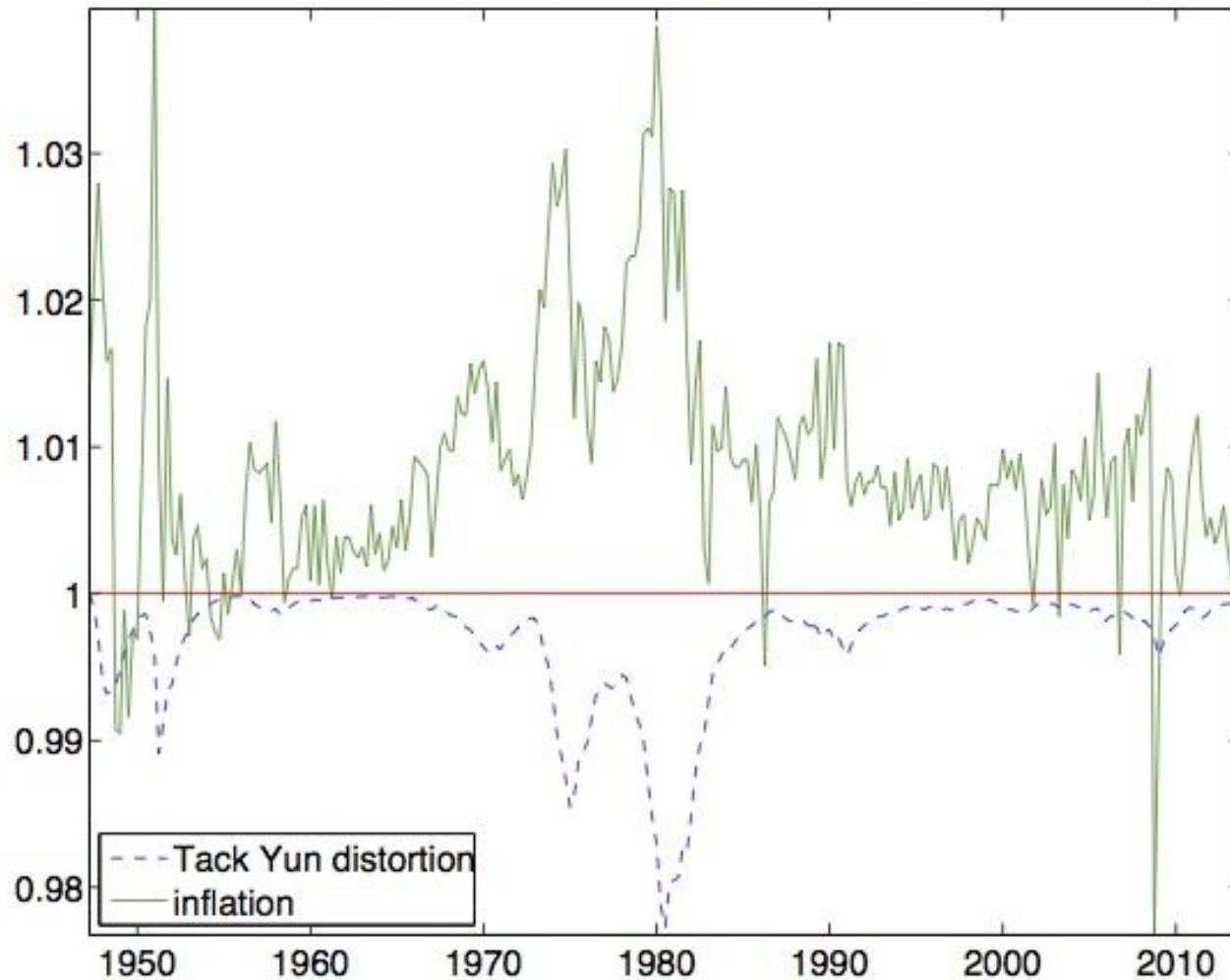$$C_t = p_t^* e^{a_t} N_t \quad (6)$$

- 6 equations, 8 unknowns:

$$v, \; C_t, \; p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t$$

- System under determined!

Ignoring Tack Yun Distortion, a Mistake?

Tack Yun Distortion and Quarterly US CPI Inflation (Gross)

Underdetermined System

- Not surprising: we added a variable, the nominal rate of interest.

- Also, we're counting subsidy as among the unknowns.

- Have two extra policy variables.

- One way to pin them down: compute optimal policy.

Ramsey-Optimal Policy

- 6 equations in 8 unknowns…..
  - Many configurations of the 8 unknowns that satisfy the 6 equations.
  - Look for the best configurations (Ramsey optimal)
    - Value of tax subsidy and of R represent optimal policy

- Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.

Ramsey Problem

$$\max_{v,p_t^*,C_t,N_t,R_t,\bar{\pi}_t,F_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right.$$

$$+ \lambda_{1t} \left[\frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} \left[1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right]$$

$$+ \lambda_{4t} \left[(1-v) \frac{\varepsilon}{\varepsilon-1} \frac{C_t \exp(\tau_t) N_t^{\varphi}}{e^{a_t}} + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[F_t \left(\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right]$$

$$\left. + \lambda_{6t} \left[C_t - p_t^* e^{a_t} N_t \right] \right\}$$

Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\max_{v,p_t^*,N_t,R_t,\bar{\pi}_t,F_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right.$$

$$+ \lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{e^{a_t}\beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1 - \theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} \left[1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta\theta F_{t+1} - F_t \right]$$

$$+ \lambda_{4t} \left[(1-v)\frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta\theta\bar{\pi}_{t+1}^\varepsilon K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[F_t \left(\frac{1 - \theta\bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \right\}$$

Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\max_{v,p_t^*,N_t,R_t,\bar{\pi}_t,F_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

defines R
$$+ \lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{e^{a_t}\beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right) \right]$$

defines F
$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta\theta F_{t+1} - F_t]$$

defines tax
$$+ \lambda_{4t} \left[(1-v)\frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta\theta\bar{\pi}_{t+1}^\varepsilon K_{t+1} - K_t \right]$$

defines K
$$+ \lambda_{5t} \left[F_t \left(\frac{1-\theta\bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \}$$

Solving the Ramsey Problem, cnt'd

- Simplified problem:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right) \right] \}$$

- First order conditions with respect to $p_t^*, \bar{\pi}_t, N_t$

$$p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^\varepsilon = \lambda_{2t}, \ \bar{\pi}_t = \left[\frac{(p_{t-1}^*)^{\varepsilon-1}}{1-\theta+\theta(p_{t-1}^*)^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}}, \ N_t = \exp\left(-\frac{\tau_t}{\varphi+1} \right)$$

- Substituting the solution for inflation into law of motion for price distortion:

$$p_t^* = \left[(1-\theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}.$$

Solution to Ramsey Problem

Eventually, price distortions
eliminated, regardless of shocks

$$p_t^* = \left[(1-\theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}$$

When price distortions
gone, so is inflation.

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

$$N_t = \exp\left(-\frac{\tau_t}{1+\varphi} \right)$$

Efficient ('first best')
allocations in real
economy

$$1 - v = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = p_t^* e^{a_t} N_t.$$

Consumption corresponds to efficient
allocations in real economy, eventually when price distortions gone

Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide

Convergence of price distortion

$$\theta = 0.75, \ \varepsilon = 10$$

$$p_t^* = \left[(1 - \theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}$$

p-star
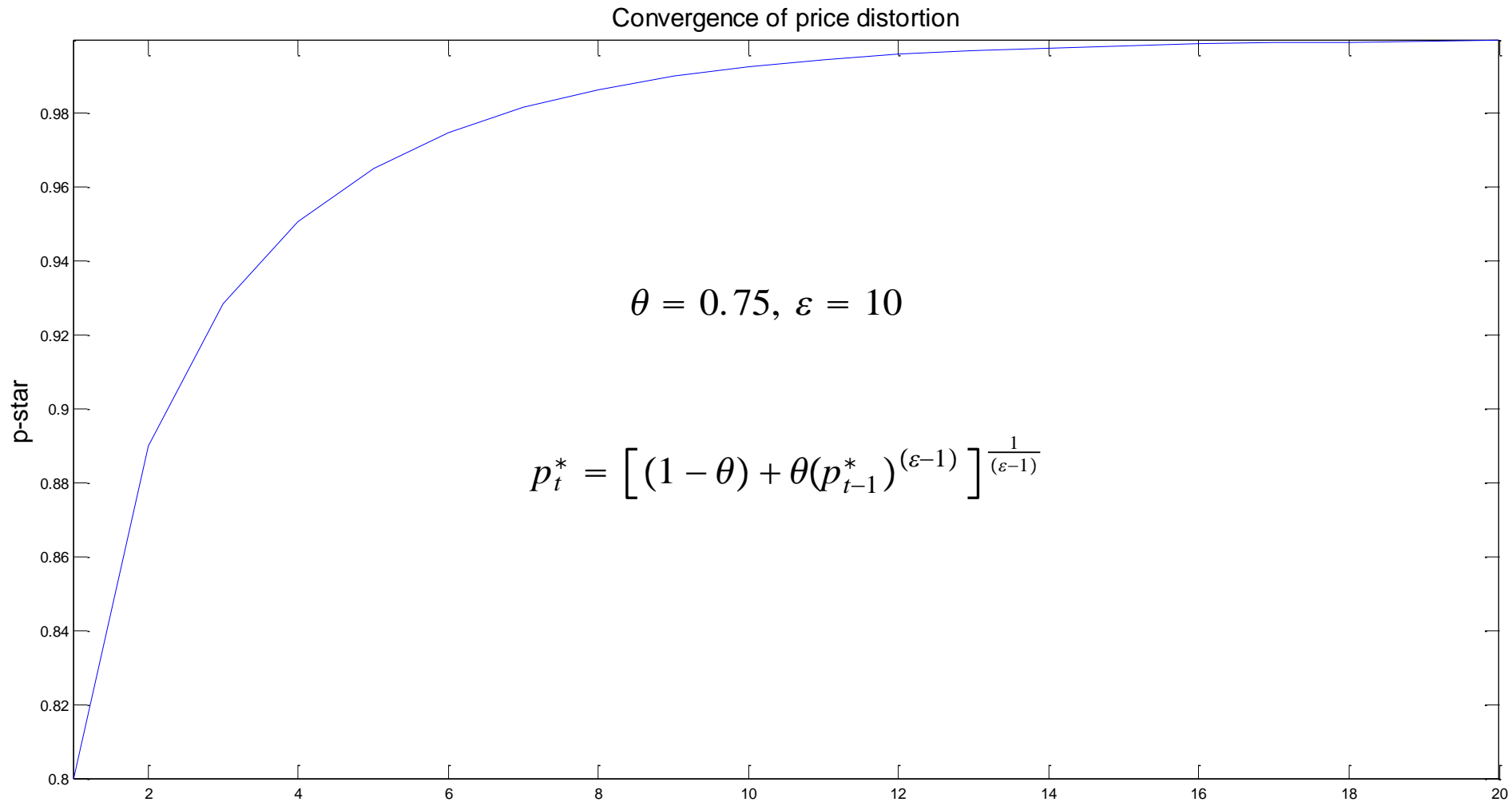
- The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., 'first best allocations')

- Refer to the Ramsey allocations as the 'natural allocations'….
  - Natural consumption, natural rate of interest, etc.