The Zero Bound

Based on work by:

Eggertsson and Woodford, 2003, 'The Zero Interest-Rate Bound and Optimal Monetary Policy,' Brookings Panel on Economic Activity.

Christiano, Eichenbaum, Rebelo, 'When is the Government Spending Multiplier Big?' (JPE, 2011)

Nonlinearity

- The New Keynesian model suggests that an economy may be vulnerable to welfare-reducing volatility when the zero lower bound on the nominal interest rate is binding.

- Exploring this idea and what to do about it requires solving a non-linear model.
  - Will apply the Eggertsson-Woodford approach
  - Deterministic Shooting Algorithm.

The ZLB Analysis (Over) Simplified

- Identity:

$$\text{expenditures} = \text{GDP}$$

- If one group reduces spending, then GDP must fall unless another group increases.

- Another group increases if real rate drops:

$$\frac{R}{\pi^e}$$

- If *R* is at lower bound and π^e cannot rise, have a problem.

The ZLB Analysis, cnt'd

- Several reasons π^e may not rise....all presume a lack of commitment in monetary policy

  – Ex post, monetary authority would not deliver high inflation (Eggertsson).

  – Real-world monetary authorities spent years persuading people they would not use inflation to stabilize economy. Fears consequences of loss of credibility in case they now raise π^e for stabilization purposes.

- In the presence of commitment, ZLB not a big problem.

The ZLB Analysis (Over) Simplified

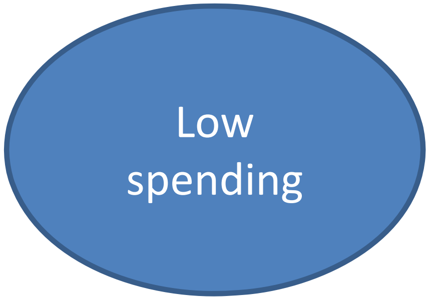
- Recession likely to follow, as real rate fails to drop.

- The recession could be very severe if a deflation spiral occurs.

$$\frac{R}{\pi^e}$$

  - The decrease in spending leads to a fall in marginal cost, which makes firms cut prices.
  - When there are price frictions, downward pressure on prices is manifest as a reduction in inflation.

Deflation Cycle in Zero Bound



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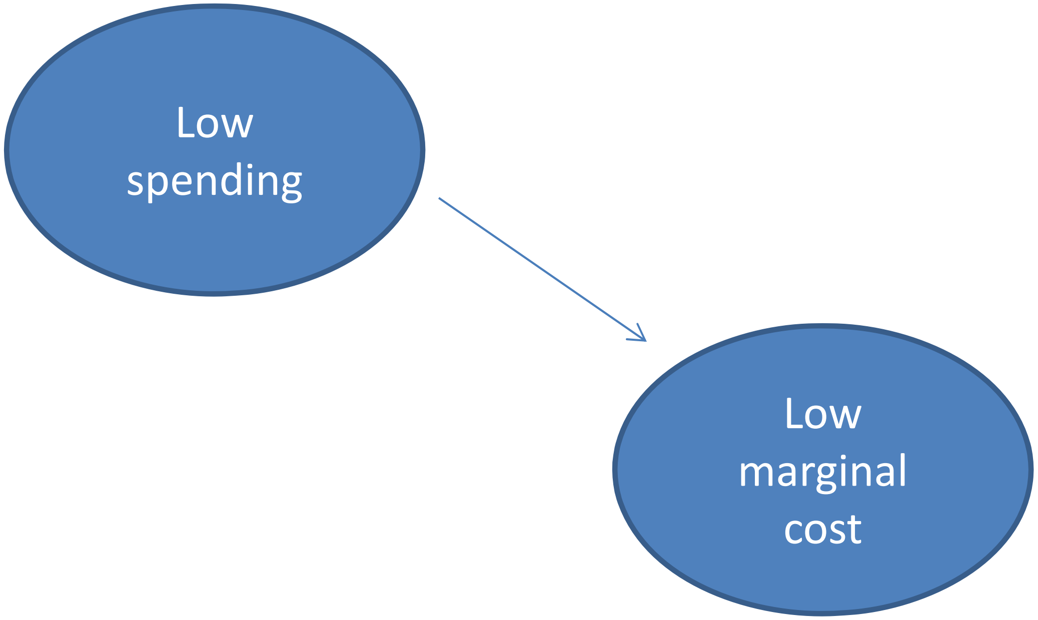

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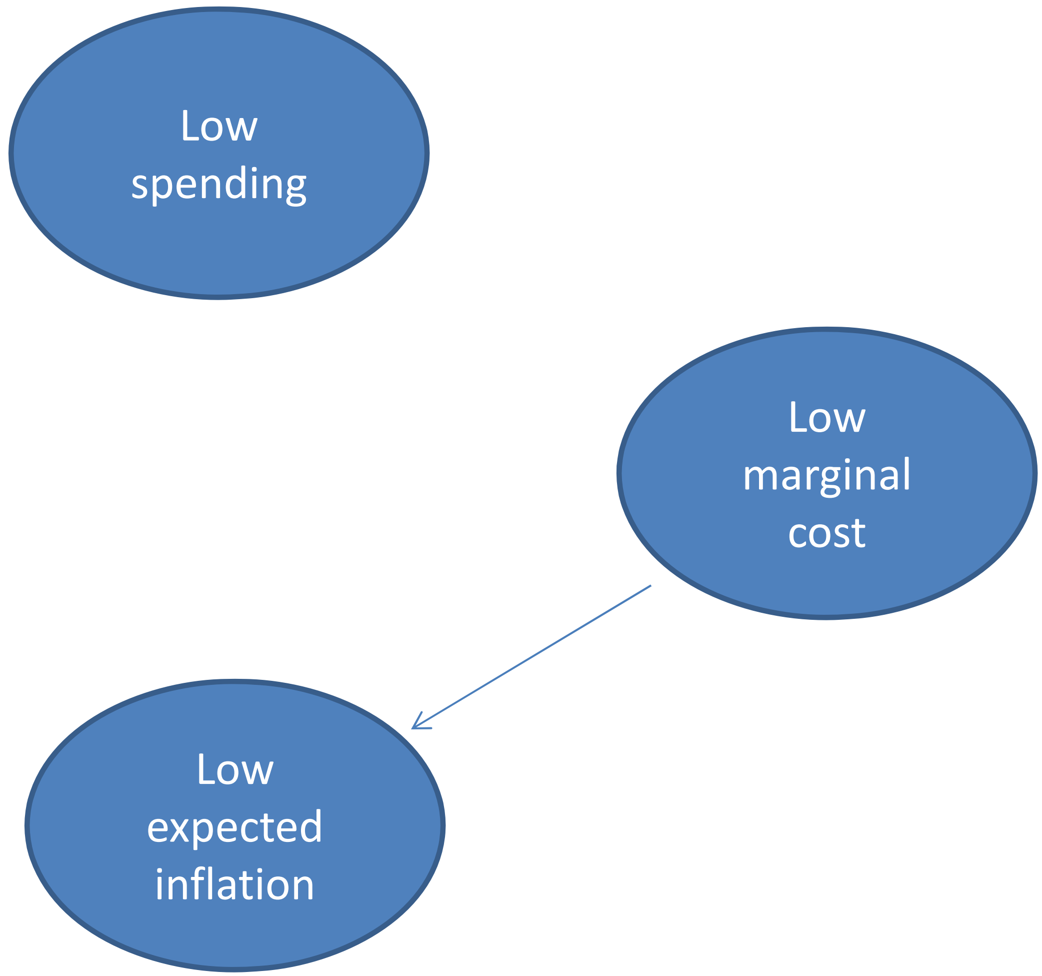

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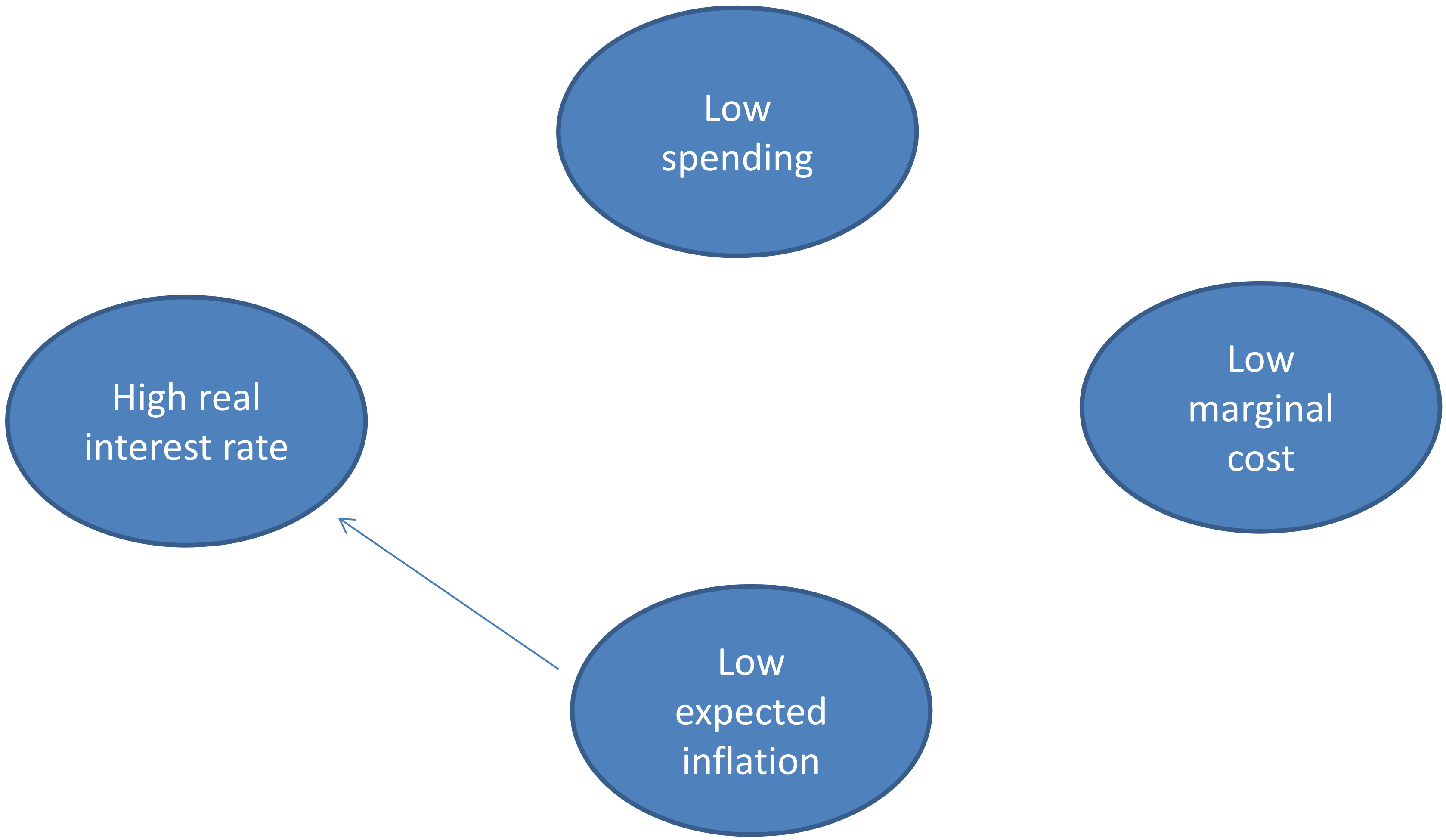

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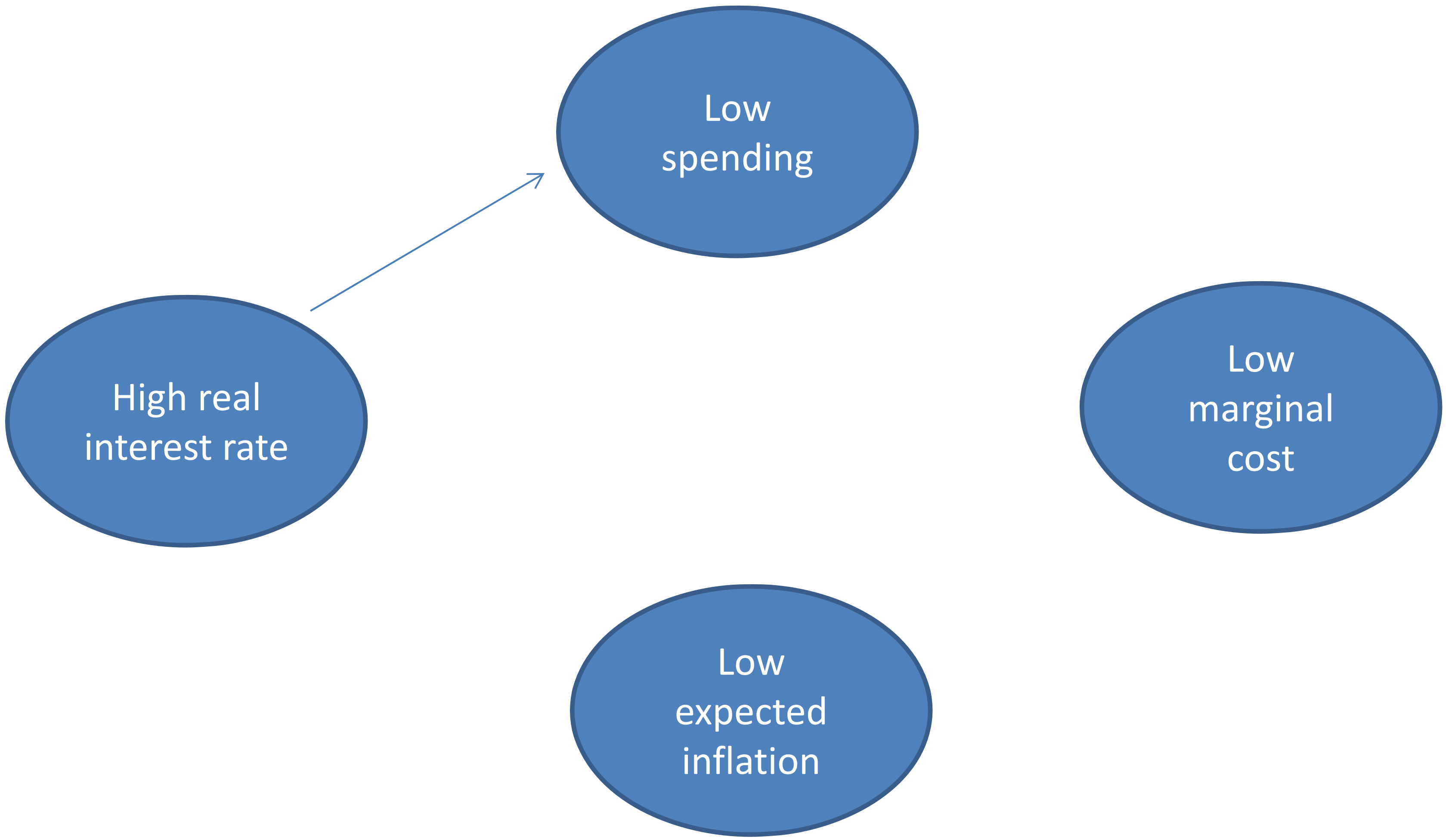

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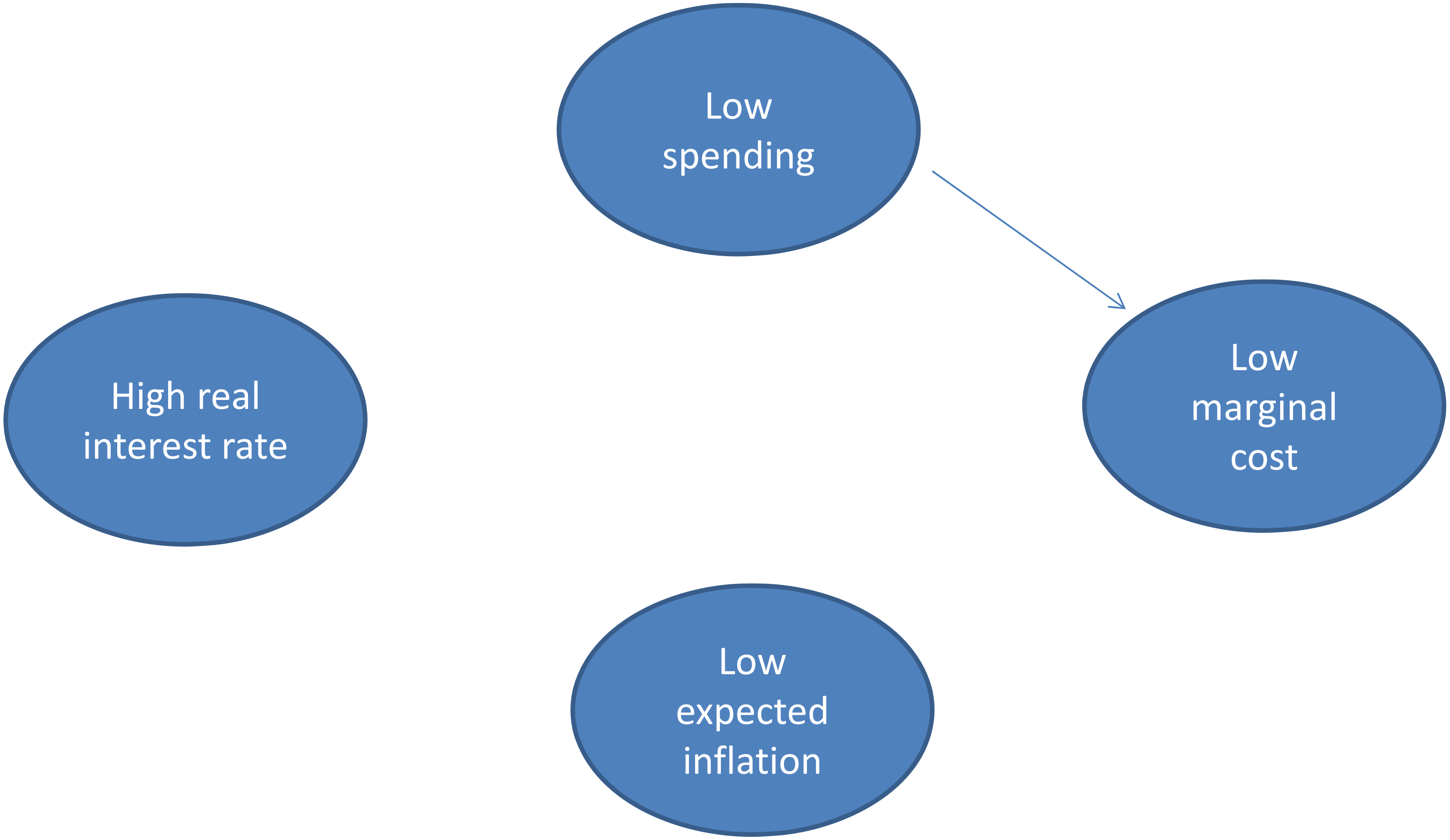

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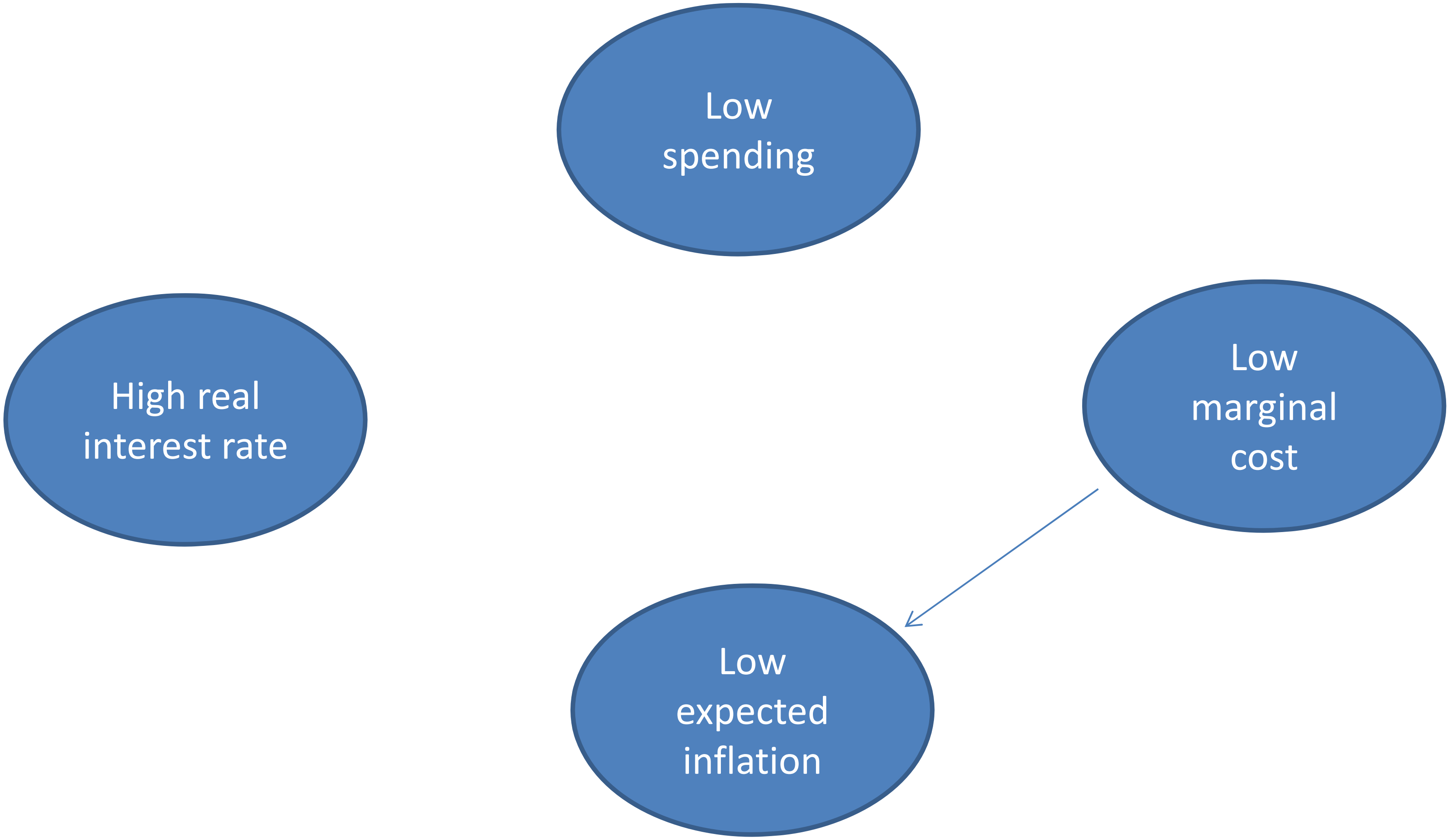

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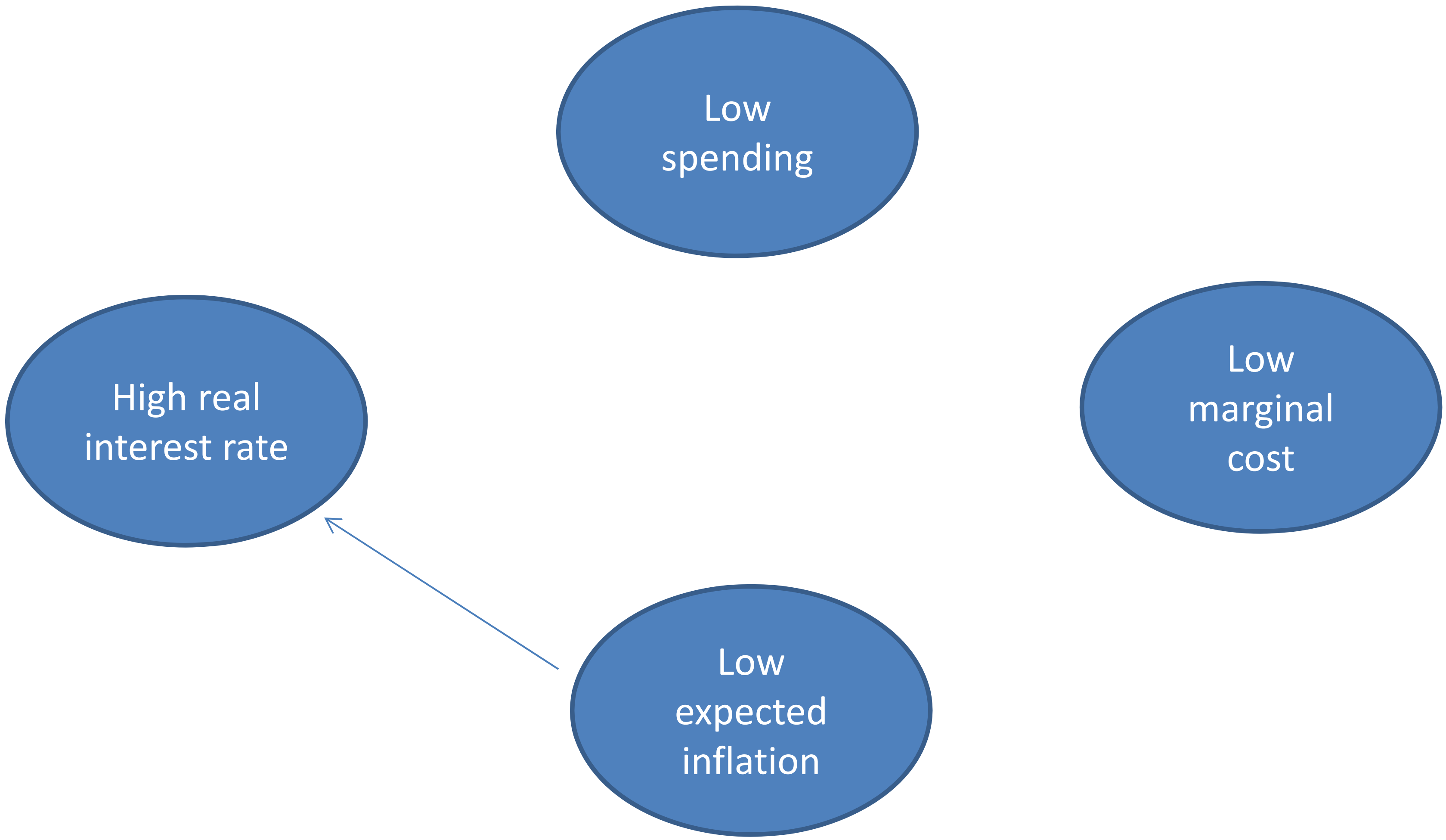

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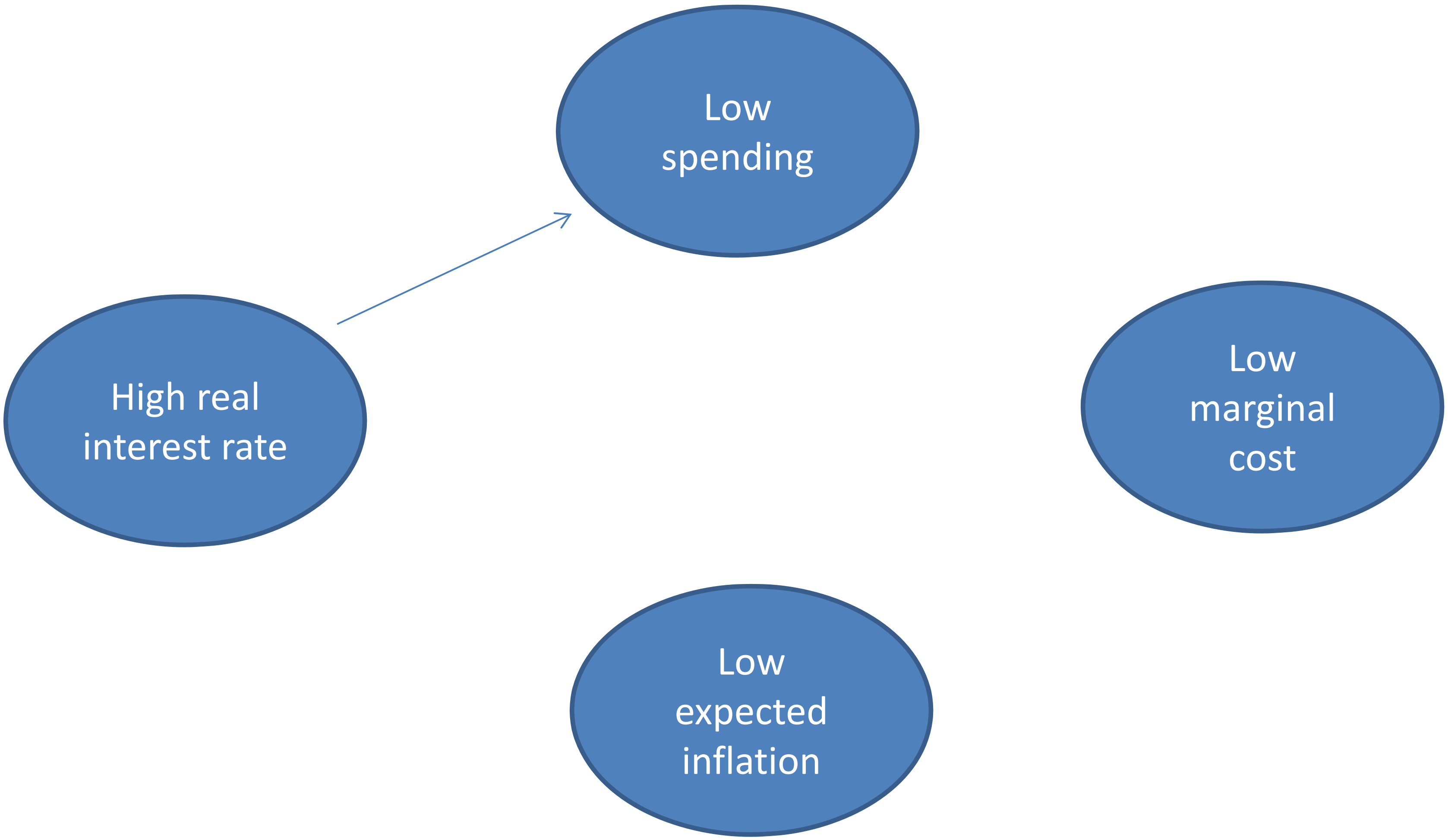

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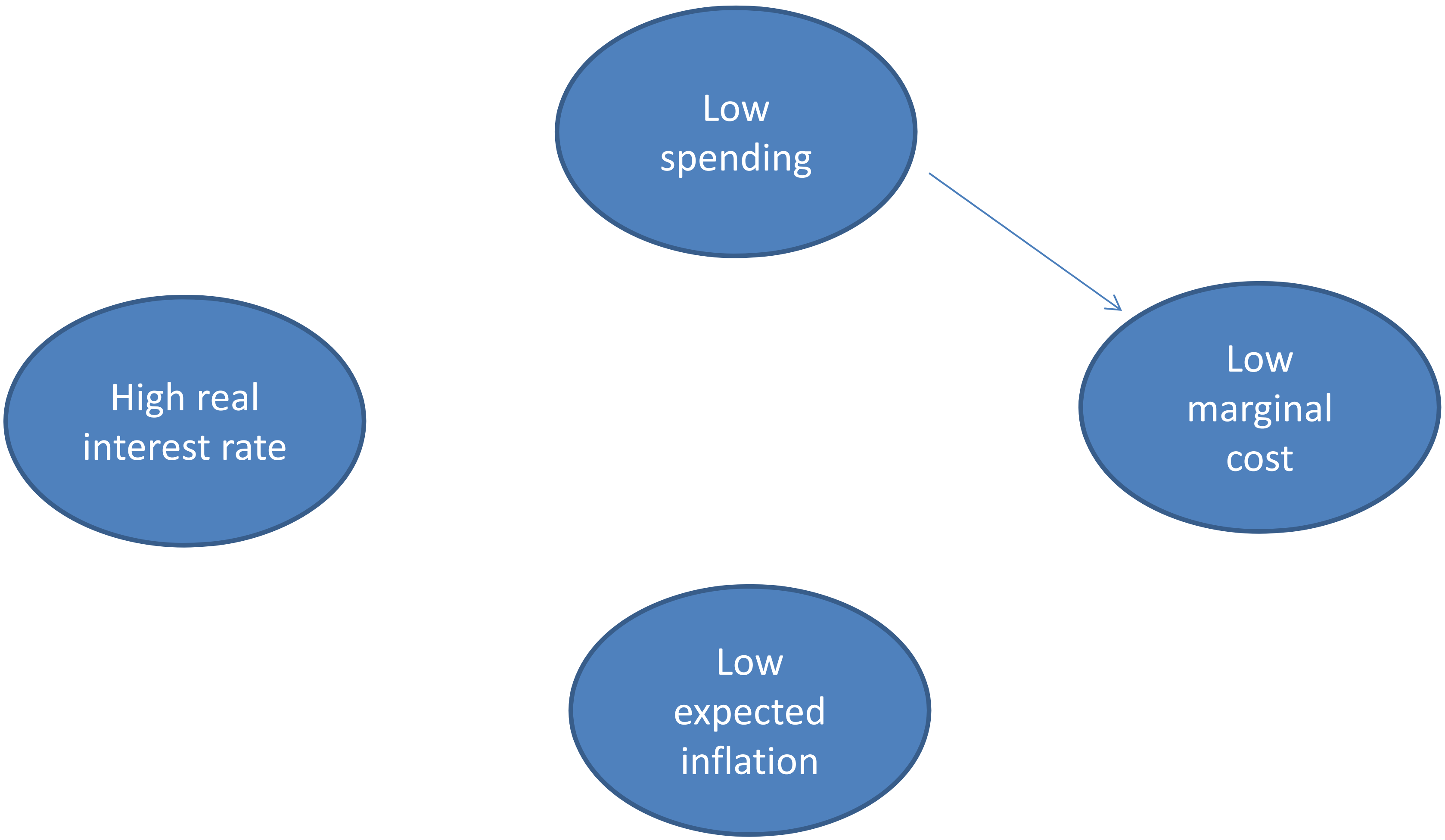

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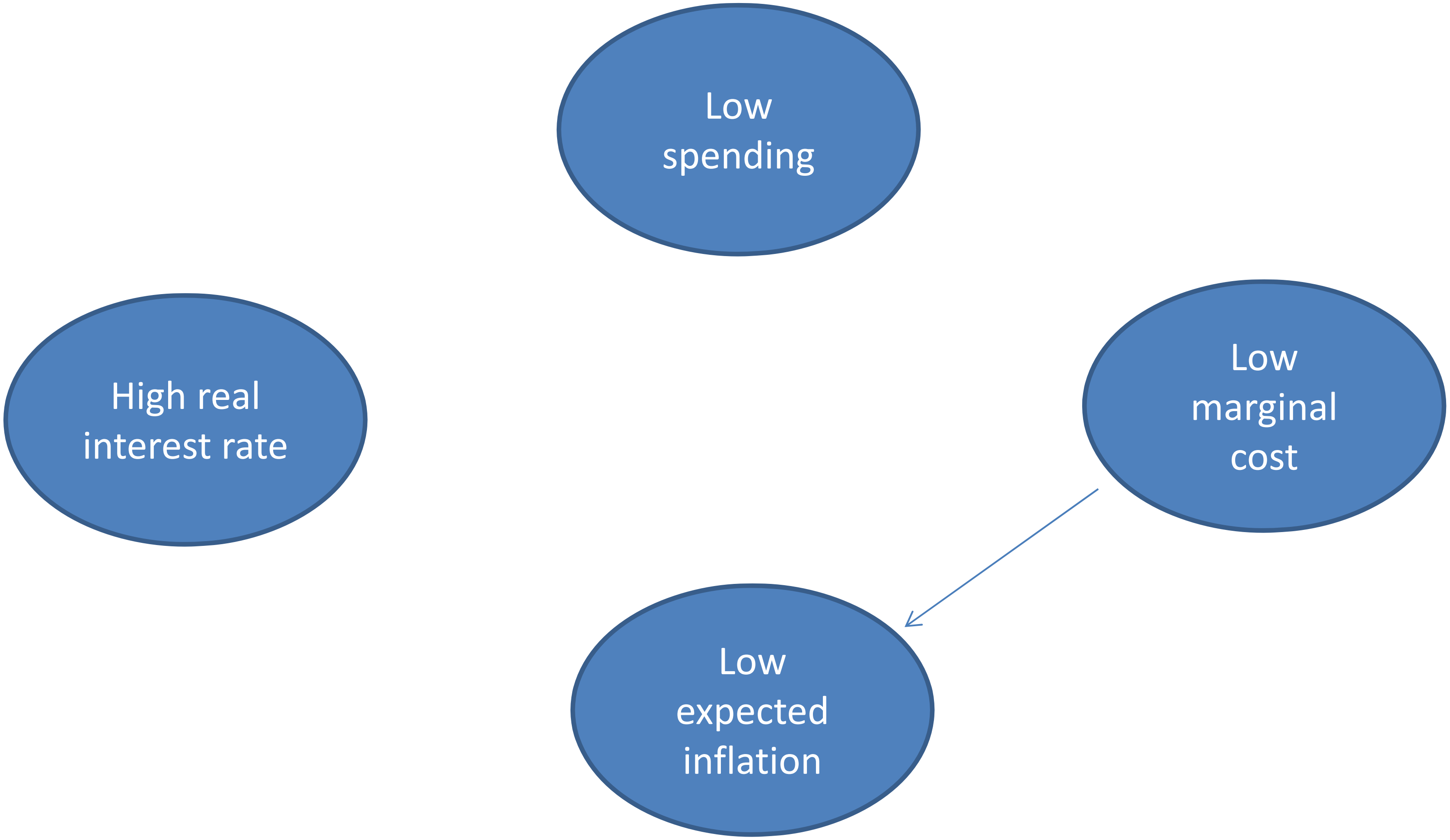

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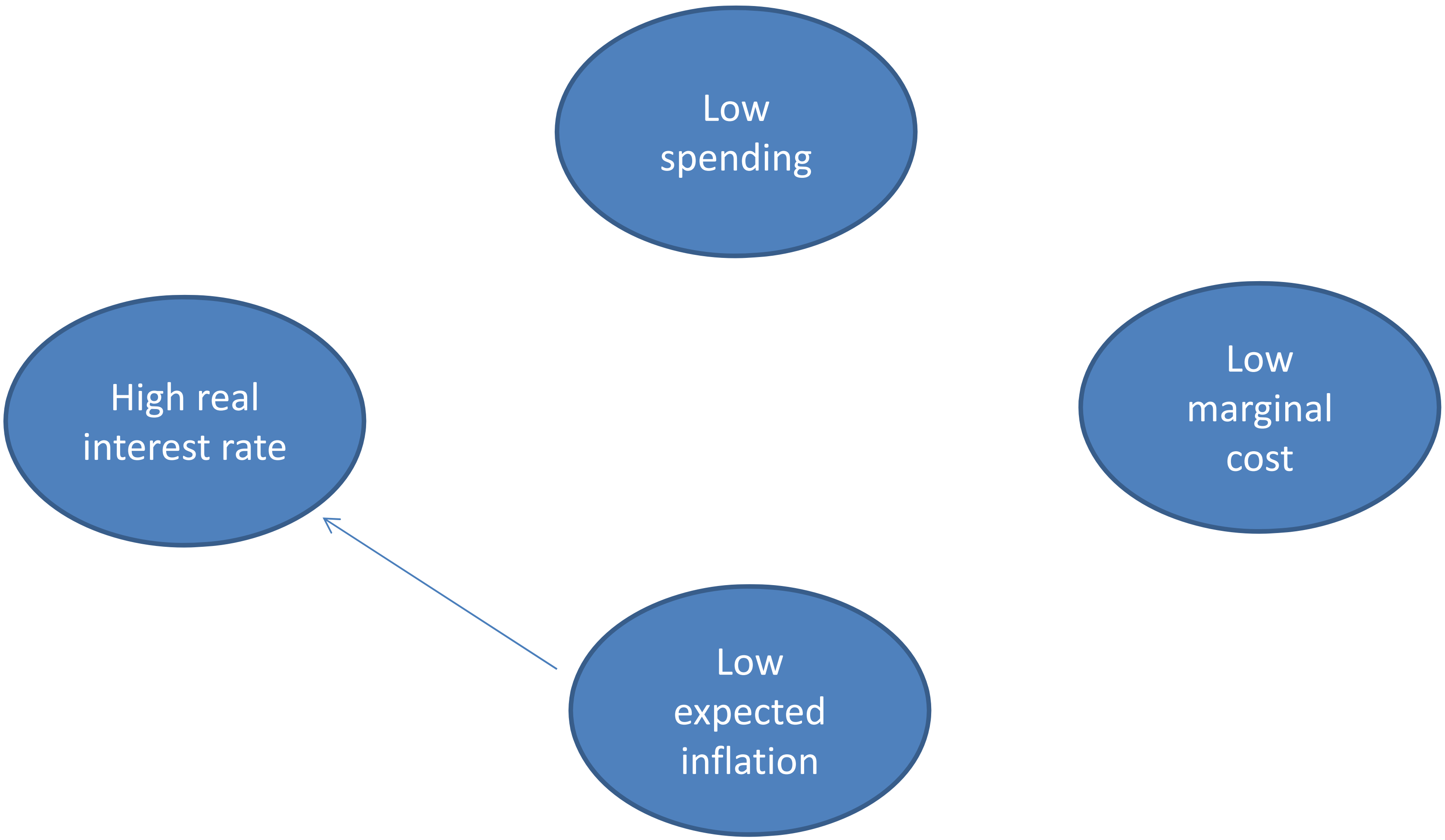

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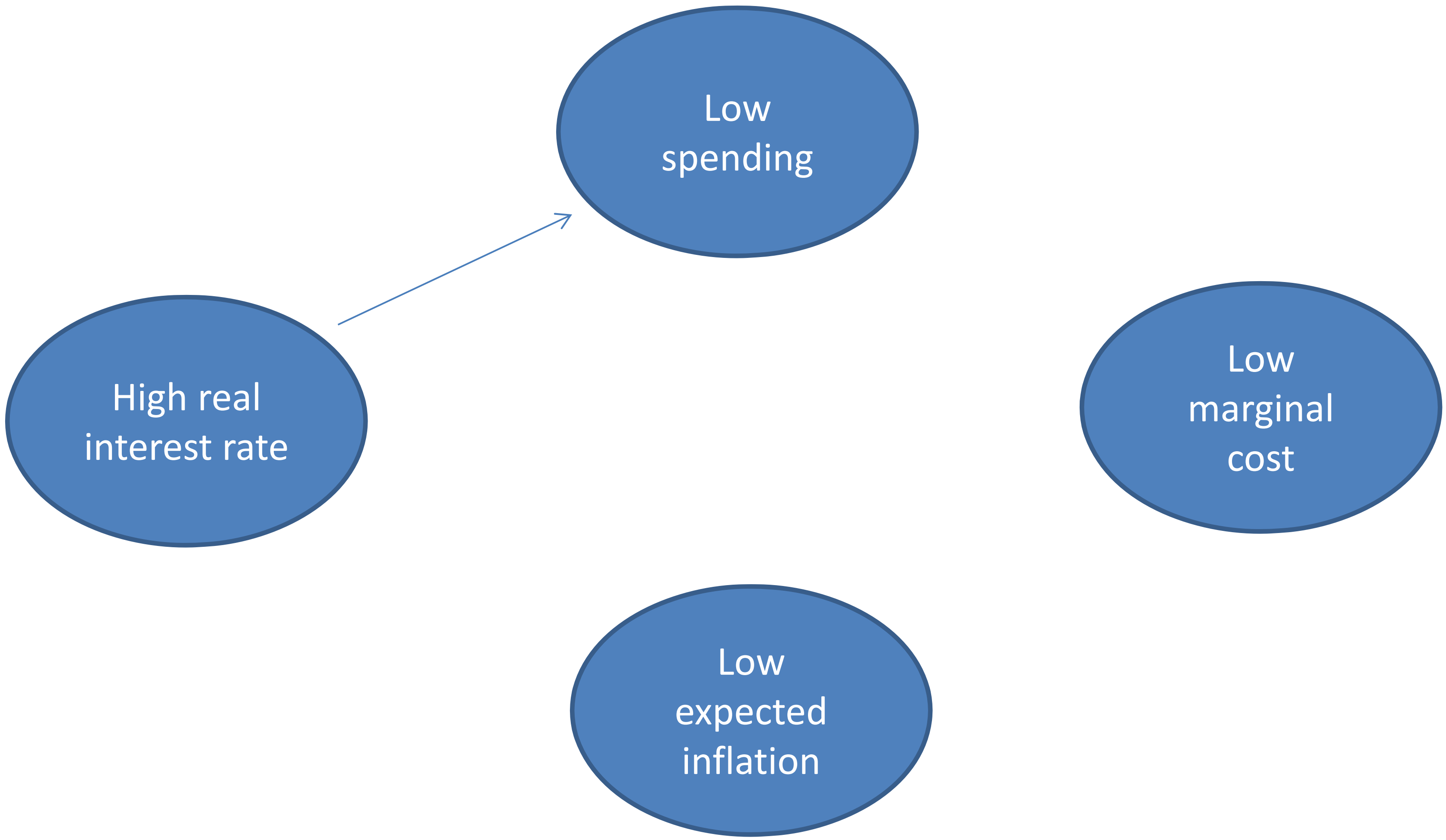

The Whole Analysis, cnt'd

- The preceding indicates that the drop in output might be substantial.

- Options for solving zlb problem

  – Direct: by interrupting destructive deflation spiral, increase government spending may have a very large effect on output.

  – Tax credits
    – Investment tax credit
    – 'cash for clunkers'

  – Increase anticipated inflation
    - Convert to a VAT tax in the future (Feldstein, Correia-Fahri-Nicolini-Teles).

  – Don't: cut labor tax rate or subsidize employment (Eggertsson)

Outline

- Analysis in ‘normal times’ when zlb constraint on interest rate can be ignored.
  – Show that the government spending multiplier is fairly small.

- Analysis when zlb is binding.
  – Government spending can have a big, welfare-improving impact on output.

Derivation of Model Equilibrium Conditions

- Households
  - First order conditions
- Firms:
  - final goods and intermediate goods
  - marginal cost of intermediate good firms
- Aggregate resources
- Monetary policy
- Three linearized equilibrium conditions:
  - Intertemporal, Pricing, Monetary policy
- Results

Model

- Household preferences and constraints:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left[C_t^{\gamma}(1-N_t)^{1-\gamma}\right]^{1-\sigma}-1}{1-\sigma} + v(G_t) \right]$$

$$P_t C_t + B_{t+1} \leq W_t N_t + (1+R_t)B_t + T_t,\ T_t \text{ ~lump sum taxes and profits}$$

- Optimality conditions

$$\overbrace{u_{c,t}}^{\text{marginal cost of giving up one unit of consumption to save}} = E_t\ \overbrace{\beta u_{c,t+1} \overbrace{\frac{1+R_{t+1}}{1+\pi_{t+1}}}^{\text{extra goods tomorrow from saving more today}}}^{\text{marginal benefit tomorrow from saving more today}},$$

Model

King-Plosser-Rebelo (KPR) preferences.

- Household preferences and constraints:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left[C_t^{\gamma} (1-N_t)^{1-\gamma} \right]^{1-\sigma} - 1}{1-\sigma} + v(G_t) \right]$$

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$$\overbrace{\frac{-u_{N,t}}{u_{c,t}}}^{\text{marginal cost (in units of goods) of labor effort}} = \overbrace{\frac{W_t}{P_t}}^{\text{marginal benefit of labor effort}}$$

Firms

- Final, homogeneous good

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \ \varepsilon > 1$$

  - Efficiency condition:

$$P_t(i) = P_t \left(\frac{Y_t}{Y_t(i)} \right)^{\frac{1}{\varepsilon}}$$

- i-th intermediate good

$$Y_t(i) = N_t(i)$$

  - Optimize price with probability *1*-θ, otherwise

$$P_t(i) = P_{t-1}(i)$$

Monetary Policy

- Monetary policy rule (after linearization)

$$dR_{t+1} = \rho_R dR_t + (1 - \rho_R)\left[\frac{\phi_1}{\beta}\pi_{t+k} + \frac{\phi_2}{\beta}\hat{Y}_{t+l} \right]$$

$$dR_{t+1} \equiv R_{t+1} - R,\ R = \frac{1}{\beta} - 1$$

$$\hat{Y}_t \equiv \frac{Y_t - Y}{Y}$$

$$k, l = 0, 1.$$

Pulling All the Equations Together

- IS equation:

$$\hat{Y}_t + [\gamma(1-\sigma)-1]g\hat{G}_t$$
$$= \hat{Y}_{t+1} + [\gamma(1-\sigma)-1]g\hat{G}_{t+1} - (1-g)[\beta dR_{t+1} - d\pi_{t+1}]$$

- Phillips curve:

$$\pi_t = \beta\pi_{t+1} + \kappa\Big[\Big(\tfrac{1}{1-g} + \tfrac{N}{1-N}\Big)\hat{Y}_t - \tfrac{g}{1-g}\hat{G}_t\Big]$$

- Monetary policy rule:

$$dR_{t+1} = \rho_R dR_t + (1-\rho_R)\Big[\tfrac{\phi_1}{\beta}\pi_{t+k} + \tfrac{\phi_2}{\beta}\hat{Y}_{t+l}\Big]$$

The Equations in Matrix Form

$$\begin{bmatrix} -\frac{1}{1-g} & -1 & 0 \\ 0 & \beta & 0 \\ l(1-\rho_R)\frac{\phi_2}{\beta} & k(1-\rho_R)\frac{\phi_1}{\beta} & 0 \end{bmatrix} \begin{pmatrix} \hat{Y}_{t+1} \\ \pi_{t+1} \\ dR_{t+2} \end{pmatrix}$$

$$+ \begin{bmatrix} \frac{1}{1-g} & 0 & \beta \\ \kappa\left(\frac{1}{1-g} + \frac{N}{1-N}\right) & -1 & 0 \\ (1-l)(1-\rho_R)\frac{\phi_2}{\beta} & (1-k)(1-\rho_R)\frac{\phi_1}{\beta} & -1 \end{bmatrix} \begin{pmatrix} \hat{Y}_t \\ \pi_t \\ dR_{t+1} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_R \end{bmatrix} \begin{pmatrix} \hat{Y}_{t-1} \\ \pi_{t-1} \\ dR_t \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{g[\gamma(\sigma-1)+1]}{1-g} \\ 0 \\ 0 \end{pmatrix} \hat{G}_{t+1} + \begin{pmatrix} -\frac{g[\gamma(\sigma-1)+1]}{1-g} \\ -\frac{\kappa g}{1-g} \\ 0 \end{pmatrix} \hat{G}_t,$$

- or, $\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t = 0.$

$$s_t = P s_{t-1} + \varepsilon_t,\; s_t \equiv \hat{G}_t,\; P = \rho$$

Solution:

- Undetermined coefficients, *A* and *B*:

$$z_t = Az_{t-1} + Bs_t$$

- *A* and *B* must satisfy:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0$$
$$\alpha_0(AB + BP) + \alpha_1 B + \beta_0 P + \beta_1 = 0.$$

- When $\rho_R = 0,\ \alpha_2 = 0 \rightarrow A = 0$ works .

Results

- Fiscal spending multiplier small, but can easily be bigger than unity (i.e., *C* rises in response to *G* shock)

- Contrasts with standard results in which multiplier is less than unity
  - Typical preferences in estimated models:

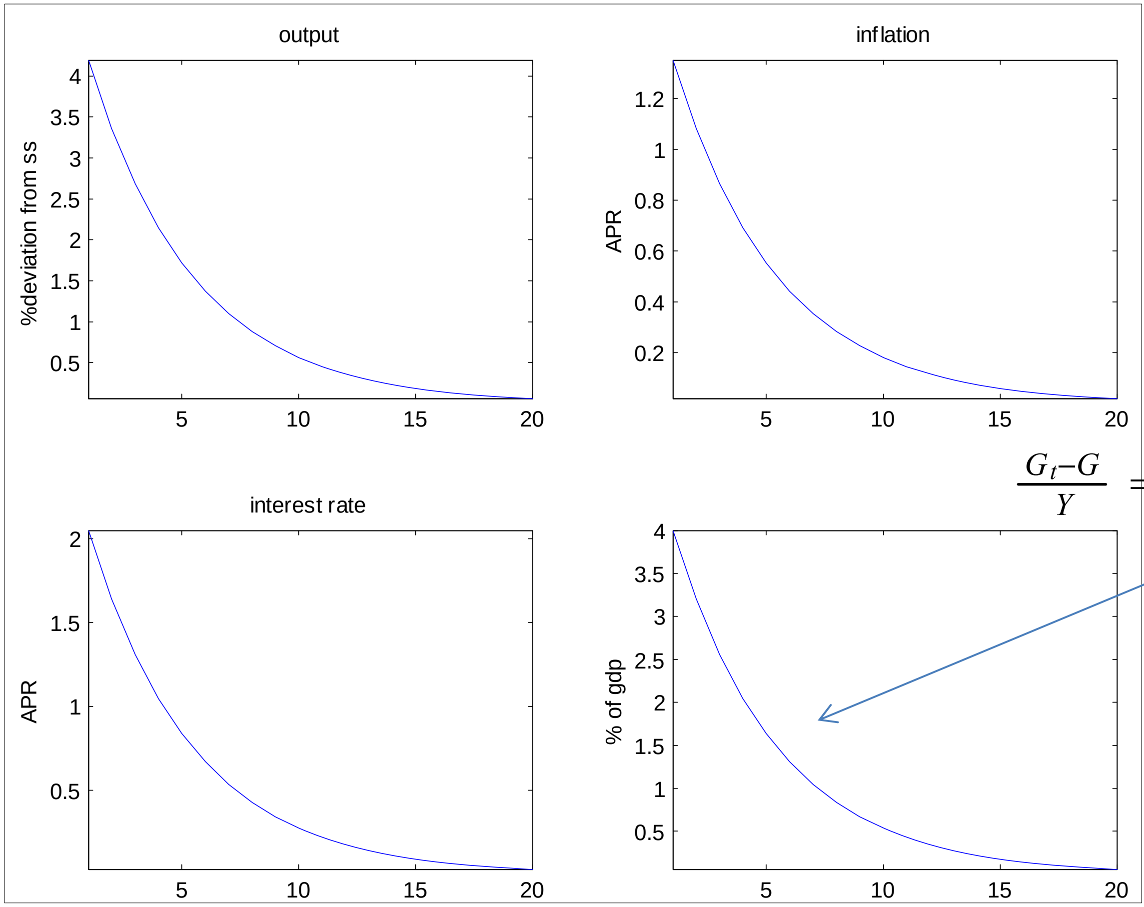
$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\gamma}}{1+\gamma} + v(G_t) \right], \; \psi, \gamma, \sigma > 0.$$

  - Marginal utility of *C* independent of *N* for CGG
  - Marginal utility of *C* increases in *N* for KPR.

Simulation Results

- Benchmark parameter values:

$$\kappa = 0.035,\ \beta = 0.99,\ \phi_1 = 1.5,\ \phi_2 = 0,\ N = 0.23,\ g = 0.2,\ \sigma = 2,\ \rho = 0.8,\ \rho_R = 0$$



$$\frac{G_t - G}{Y} = \frac{G_t - G}{G}\frac{G}{Y} = \hat{G}_t g$$

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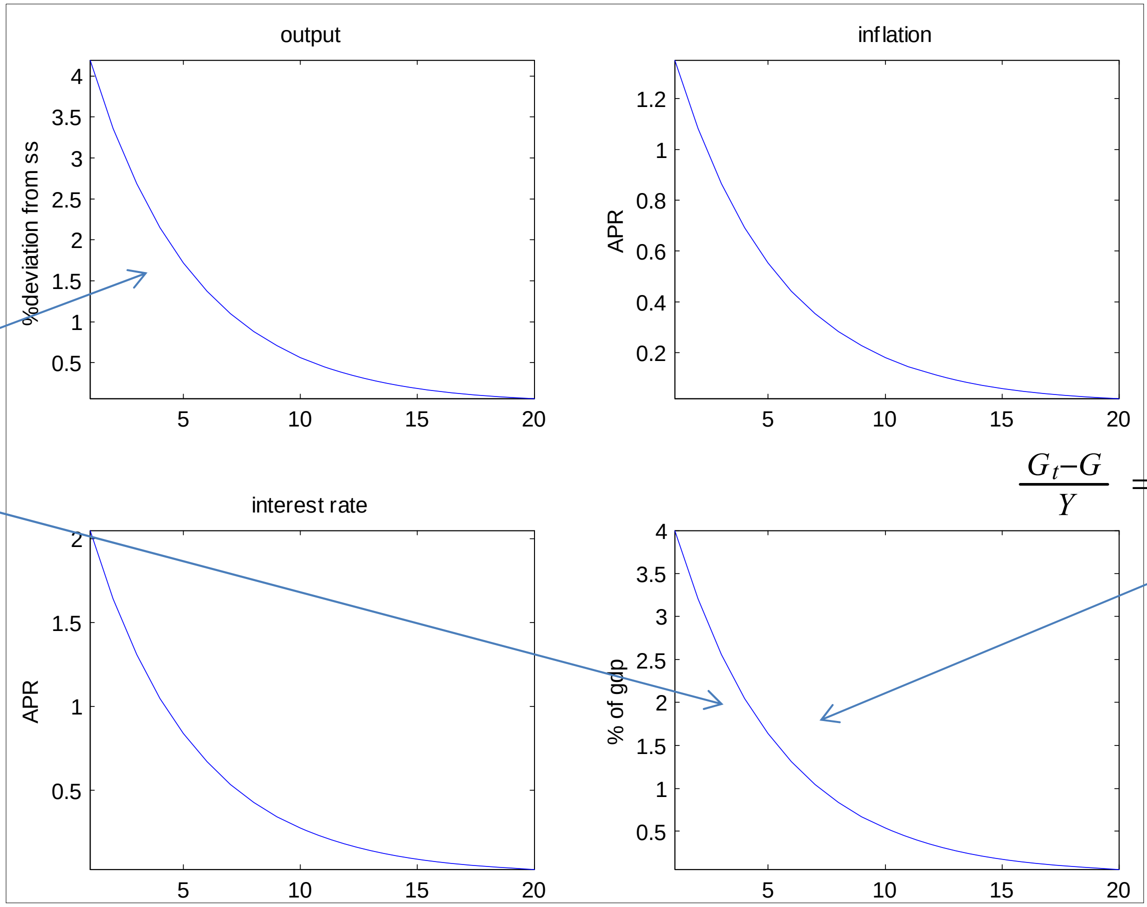
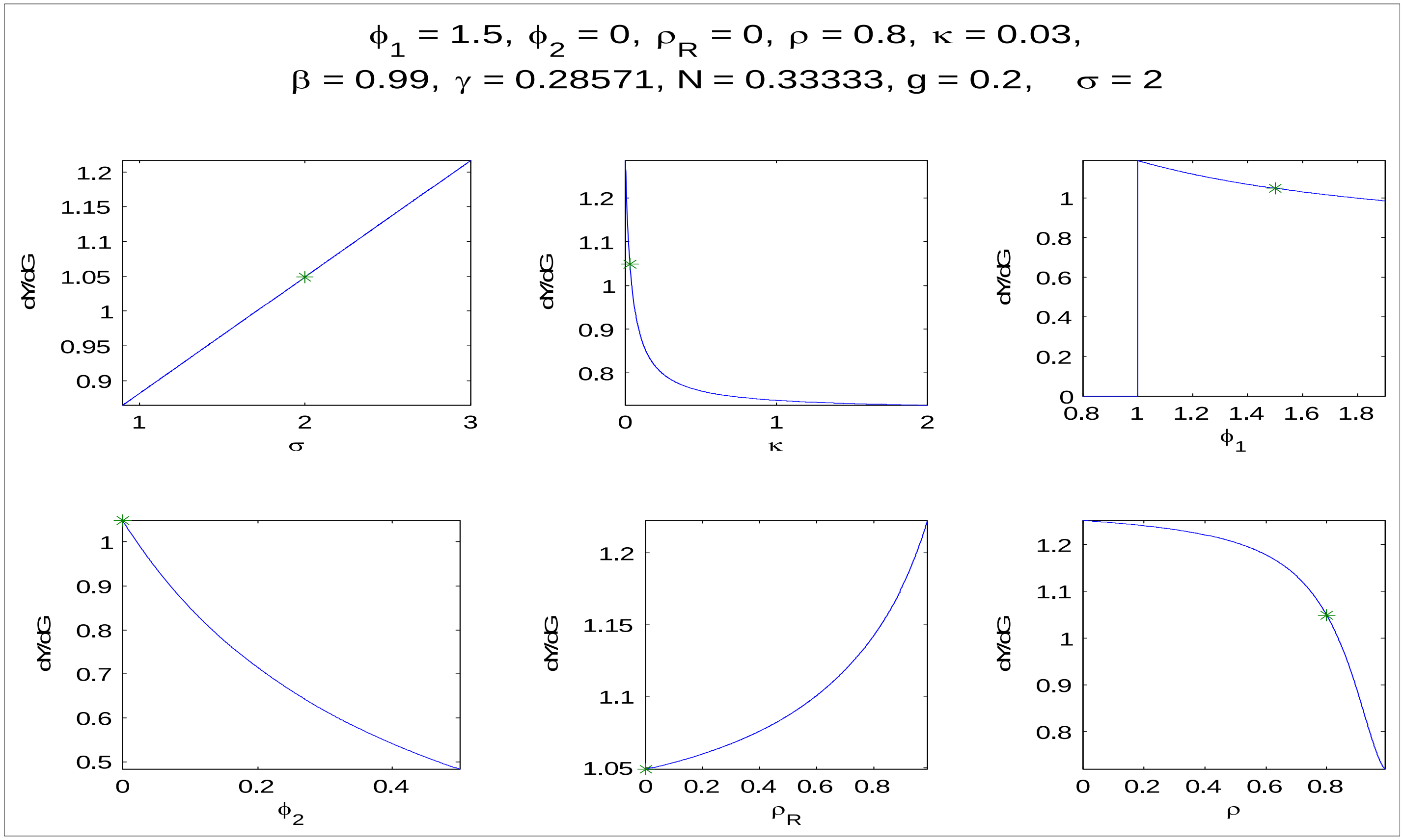


Multiplier = 1.05, constant.

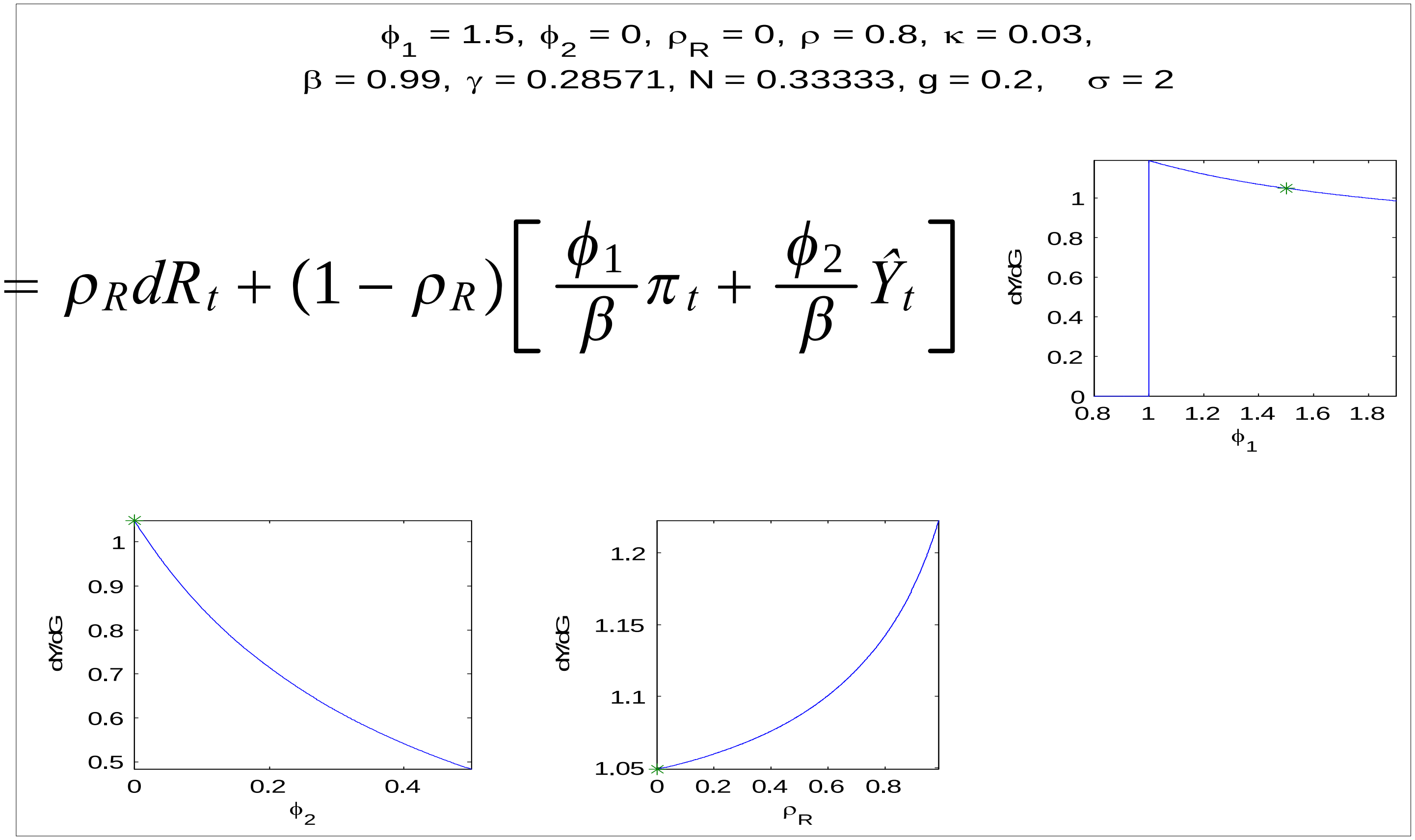
$$\frac{G_t - G}{Y} = \frac{G_t - G}{G}\frac{G}{Y} = \hat{G}_t g$$

Multiplier for Alternative Parameter Values



- Results: multiplier bigger
  - the less monetary policy allows *R* to rise.
  - the more complementary are consumption and labor (i.e., the bigger is σ).
  - the smaller the negative income effect on consumption (i.e., the smaller is ρ).
  - smaller values of κ (i.e., more sticky prices)

Multiplier for Alternative Parameter Values



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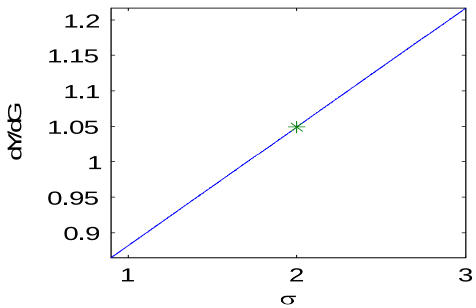
Multiplier for Alternative Parameter Values

$\phi_1 = 1.5,\ \phi_2 = 0,\ \rho_R = 0,\ \rho = 0.8,\ \kappa = 0.03,$
$\beta = 0.99,\ \gamma = 0.28571,\ N = 0.33333,\ g = 0.2,\quad \sigma = 2$



$$u(C_t, N_t) = \frac{\left[C_t^{\gamma}(1-N_t)^{1-\gamma}\right]^{1-\sigma} - 1}{1-\sigma}$$

$$u_{c,t} = C_t^{[\gamma(1-\sigma)-1]}(1-N_t)^{(1-\gamma)(1-\sigma)}$$

- Results: multiplier bigger
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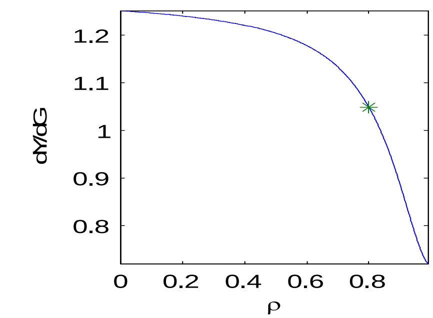
$$\hat{G}_t = \rho \hat{G}_{t-1} + \varepsilon_t$$



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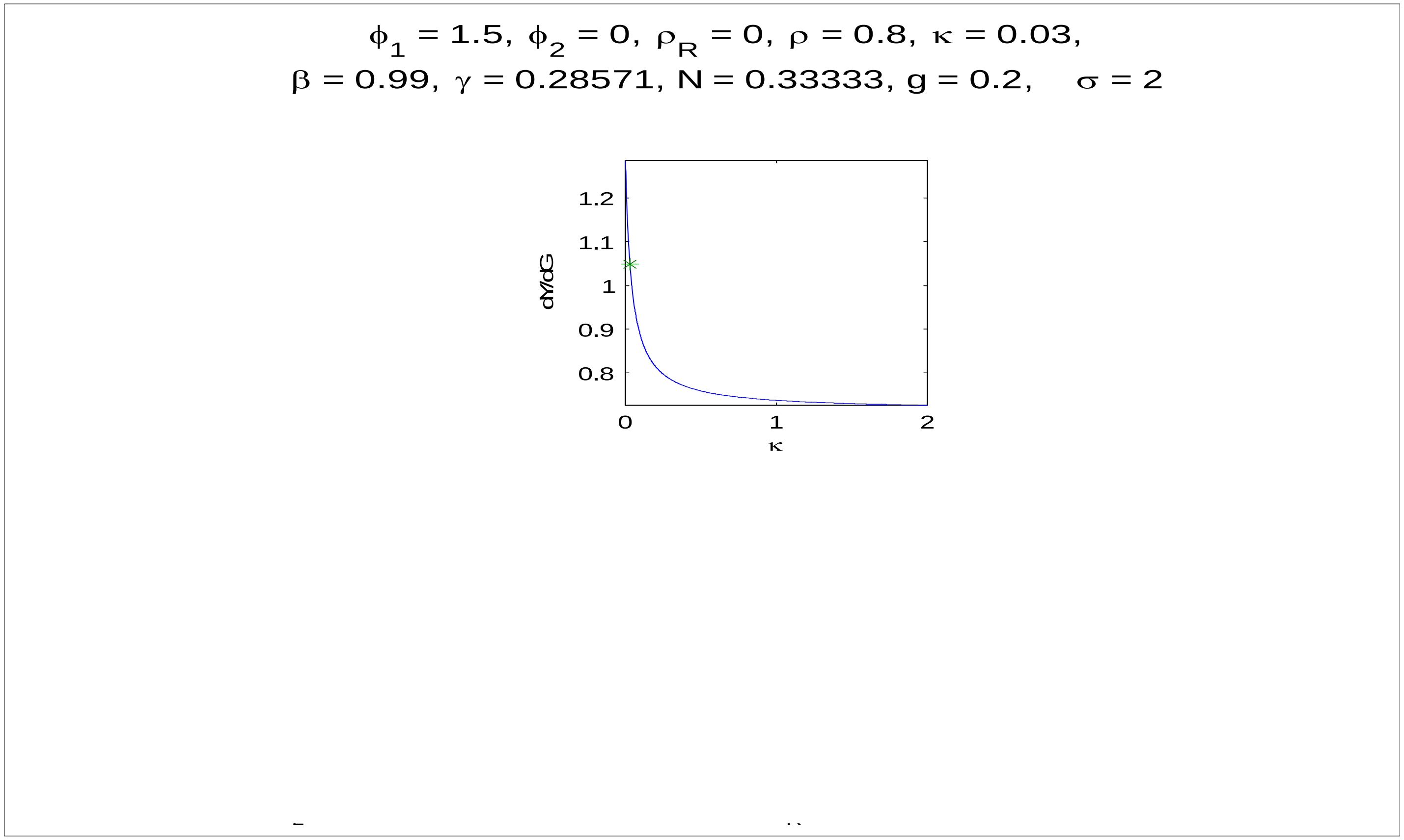


- Results: multiplier bigger
  - smaller values of κ (i.e., more sticky prices)

Analysis of Case when the Non-negativity Constraint on the Nominal Interest Rate is Binding

- Need a shock that puts us into the lower bound.

- One possibility: increased desire to save.
  - Seems particularly relevant at the current time.
  - Other shocks will do it too……

- Discount rate shock.

Monetary Policy

- Monetary policy rule (after linearization)

$$Z_{t+1} = R + \rho_R(R_t - R) + (1-\rho_R)\left[\frac{\phi_1}{\beta}\pi_t + \frac{\phi_2}{\beta}\hat{Y}_t\right]$$

$$\hat{Y}_t = \frac{Y_t - Y}{Y},\ R = \frac{1}{\beta} - 1$$

$$R_{t+1} = \begin{cases} Z_{t+1} & \text{if } Z_{t+1} > 0 \\ 0 & \text{if } Z_{t+1} \leq 0 \end{cases}.$$

← nonlinearity

Eggertsson-Woodford Saving Shock

- Preferences:

$$u(C_0, N_0, G_0) + \frac{1}{1+r_1} E_0 \left\{ u(C_1, N_1, G_1) + \frac{1}{1+r_2} u(C_2, N_2, G_1) + \frac{1}{1+r_2} \frac{1}{1+r_3} u(C_3, N_3, G_3) \ldots \right\}$$

- Before *t=0*
  - System was in non-stochastic, zero inflation steady state,

$$r_{t+1} = R = \frac{1}{\beta} - 1$$

$$R_{t+1} = R$$

$$\hat{G}_t = 0, \text{ for all } t$$

Saving Shock, cnt'd

- At time *t=0,*

$$r_1 = r^l < 0$$

$$\Pr ob[r_{t+1} = r | r_t = r^l] = 1 - p$$

$$\Pr ob[r_{t+1} = r^l | r_t = r^l] = p$$

$$\Pr ob[r_{t+1} = r^l | r_t = r] = 0$$

- "Discount rate drops in *t=0* and is expected to return permanently to its 'normal' level with constant probability, *1-p.*"

Zero Bound Equilibrium

- simple characterization:

$\pi^l, \hat{Y}^l, R = 0,\ Z^l \leq 0$ while discount rate is low

$\pi_t = \hat{Y}_t = 0,\ R = r$ as soon as discount rate snaps back up

Fiscal Policy

- Government spending is set to a constant deviation from steady state, during the zero bound.
- That is,

$$\hat{G}_t \text{ may be nonzero while } r_{t+1} = r^l,\ \hat{G}_t = 0 \text{ when } r_{t+1} = r$$

Equations With Discount Shock

- IS equation:

$$\hat{Y}_t - g[\gamma(\sigma-1)+1]\hat{G}_t = -(1-g)[\beta(R_{t+1}-r_{t+1}) - E_t\pi_{t+1}] + E_t\hat{Y}_{t+1} - g[\gamma(\sigma-1)+1]E_t\hat{G}_{t+1}$$

$$\hat{Y}^l - g[\gamma(\sigma-1)+1]\hat{G}^l = -(1-g)[\beta(0-r^l) - p\pi^l] + p\hat{Y}^l - g[\gamma(\sigma-1)+1]p\hat{G}$$

- Phillips curve:

$$\pi_t = \beta E_t\pi_{t+1} + \kappa\left[\left(\frac{1}{1-g}+\frac{N}{1-N}\right)\hat{Y}_t - \frac{g}{1-g}\hat{G}_t\right]$$

$$\pi^l = \beta p\pi^l + \kappa\left(\frac{1}{1-g}+\frac{N}{1-N}\right)\hat{Y}^l - \frac{g}{1-g}\kappa\hat{G}^l$$

- Monetary Policy:

$$R_{t+1} = 0$$

$$Z_{t+1} = R + \rho_R(R_t - R) + (1-\rho_R)\left[\frac{\phi_1}{\beta}\pi_t + \frac{\phi_2}{\beta}\hat{Y}_t\right] \leq 0$$

Solving for the Zero Bound Allocations

- Is equation:

$$\hat{Y}^l - g[\gamma(\sigma - 1) + 1]\hat{G}^l = -(1-g)[\beta(0 - r^l) - p\pi^l] + p\hat{Y}^l - g[\gamma(\sigma - 1) + 1]p\hat{G}$$

- Phillips curve:

$$\pi^l = \beta p \pi^l + \kappa\left(\frac{1}{1-g} + \frac{N}{1-N}\right)\hat{Y}^l - \frac{g}{1-g}\kappa\hat{G}^l$$

- Two equations in two unknowns!
  - Solve for $\hat{Y}^l, \pi^l$ and verify that $Z^l \leq 0$

Solution

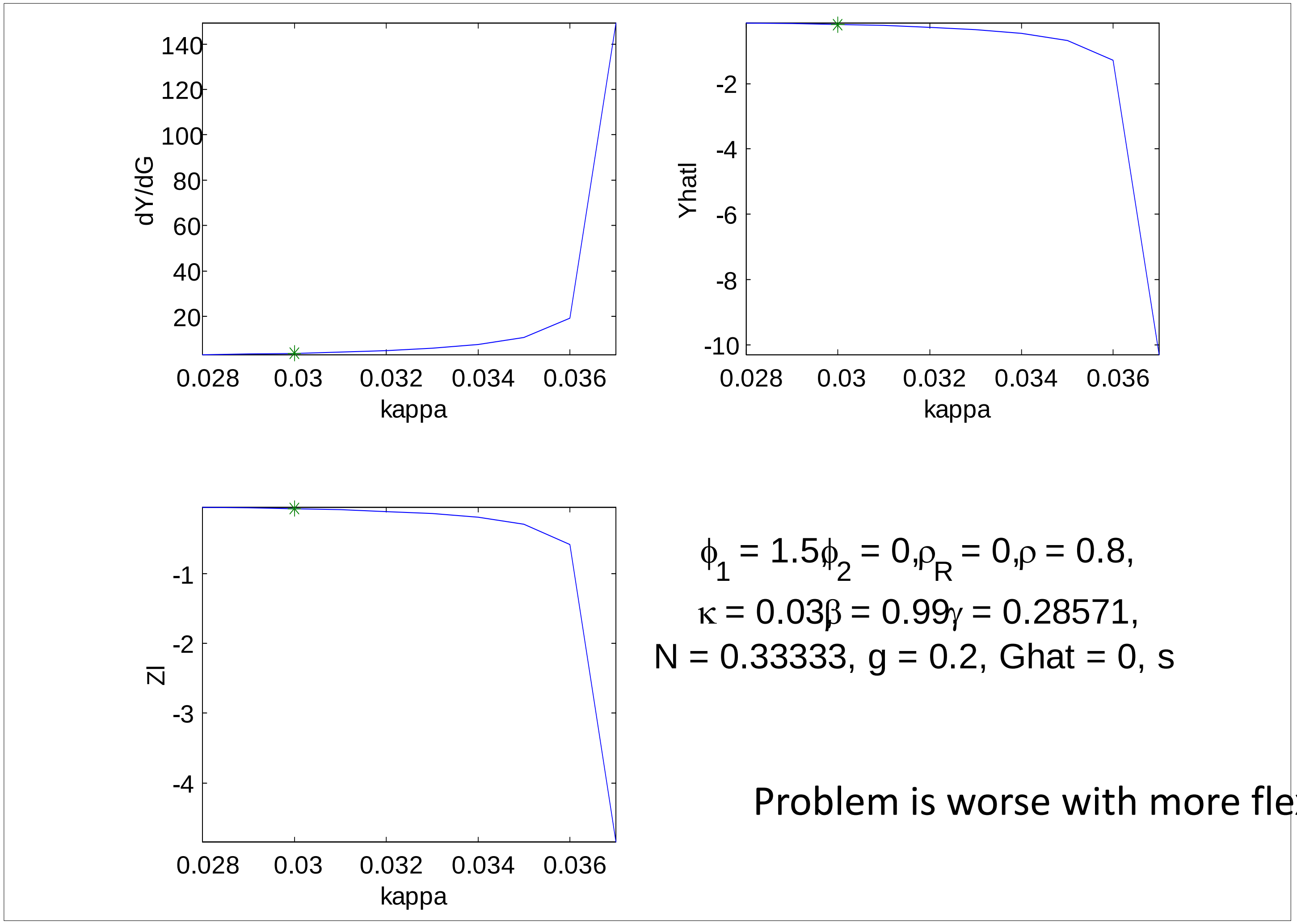
- Inflation:

$$\pi^l = \frac{\kappa\left(\frac{1}{1-g}+\frac{N}{1-N}\right)\left[g[\gamma(\sigma-1)+1]\hat{G}^l+\frac{1-g}{1-p}\beta r^l\right]-\frac{g}{1-g}\kappa\hat{G}^l}{1-\beta p-\kappa\left(\frac{1}{1-g}+\frac{N}{1-N}\right)p\frac{1-g}{1-p}}$$

- Output:

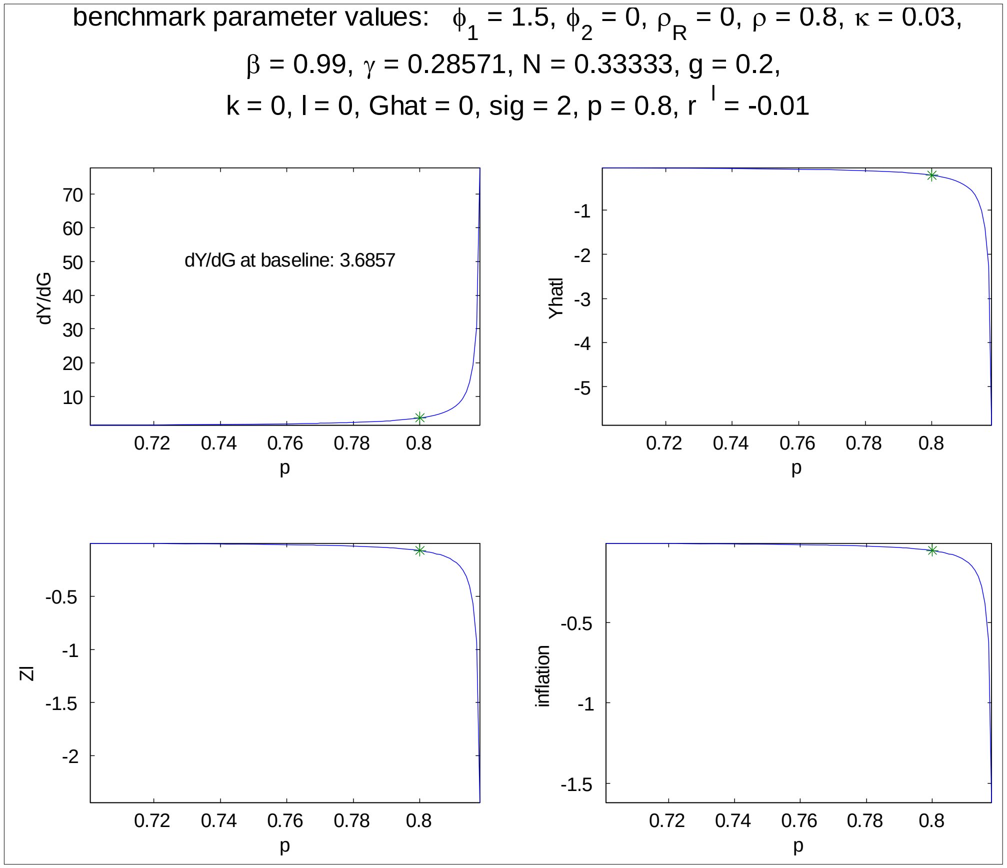
$$\hat{Y}^l = g[\gamma(\sigma-1)+1]\hat{G}^l + \frac{1-g}{1-p}[\beta r^l + p\pi^l]$$

Numerical Simulations



Problem is worse with more flexible prices!

- Results: multiplier 3.7 at benchmark parameter values and may be gigantic.



- As *p* increases, zero-bound becomes more severe…this is because with higher *p*, fall in output is more persistent and resulting negative wealth effect further depresses consumption.

Fiscal Expansion in Zero Bound Highly Effective, But is it *Desirable*?

- Intuition:
  - *Yes*….
    - the vicious cycle produces a huge, inefficient fall in output
    - in the first-best equilibrium, output, consumption and employment are invariant to discount rate shocks
    - If *G* helps to partially undo this inefficiency, then surely it's a good thing

Fiscal Expansion in Zero Bound Highly Effective, But is it *Desirable*?

- Preferences

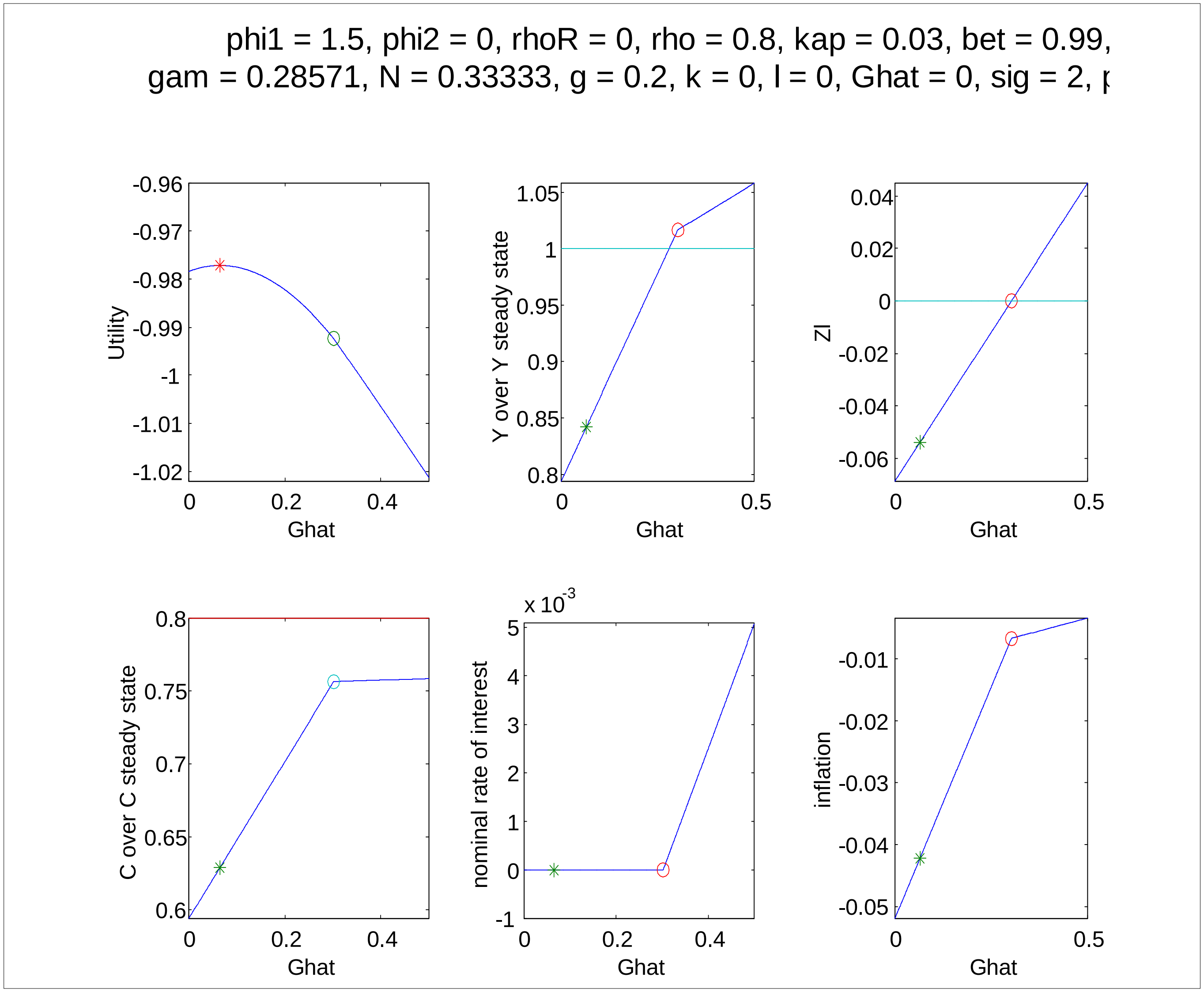
$$\sum_{t=0}^{\infty}\left(\frac{p}{1+r^l}\right)^t\left[\frac{\left[(C^l)^\gamma(1-N^l)^{1-\gamma}\right]^{1-\sigma}-1}{1-\sigma}+v(G^l)\right]$$

$$=\frac{1}{1-\frac{p}{1+r^l}}\left[\frac{\left[(C^l)^\gamma(1-N^l)^{1-\gamma}\right]^{1-\sigma}-1}{1-\sigma}+v(G^l)\right]$$

$$=\frac{1}{1-\frac{p}{1+r^l}}\left[\frac{\left[\left(N\left(\hat{Y}^l+1\right)-Ng\left(\hat{G}^l+1\right)\right)^\gamma\left(1-N\left(\hat{Y}^l+1\right)\right)^{1-\gamma}\right]^{1-\sigma}-1}{1-\sigma}+v\left(Ng\left(\hat{G}^l+1\right)\right)\right]$$

- Compute optimal $\hat{G}^l$
  - (i) $v(G^l)$ = 0,
  - (ii) $v(G)=\psi_g\frac{G^{1-\sigma}}{1-\sigma}$, ψ_g chosen to rationalize $g=0.2$ as optimal in steady state

Case Where G is not Valued



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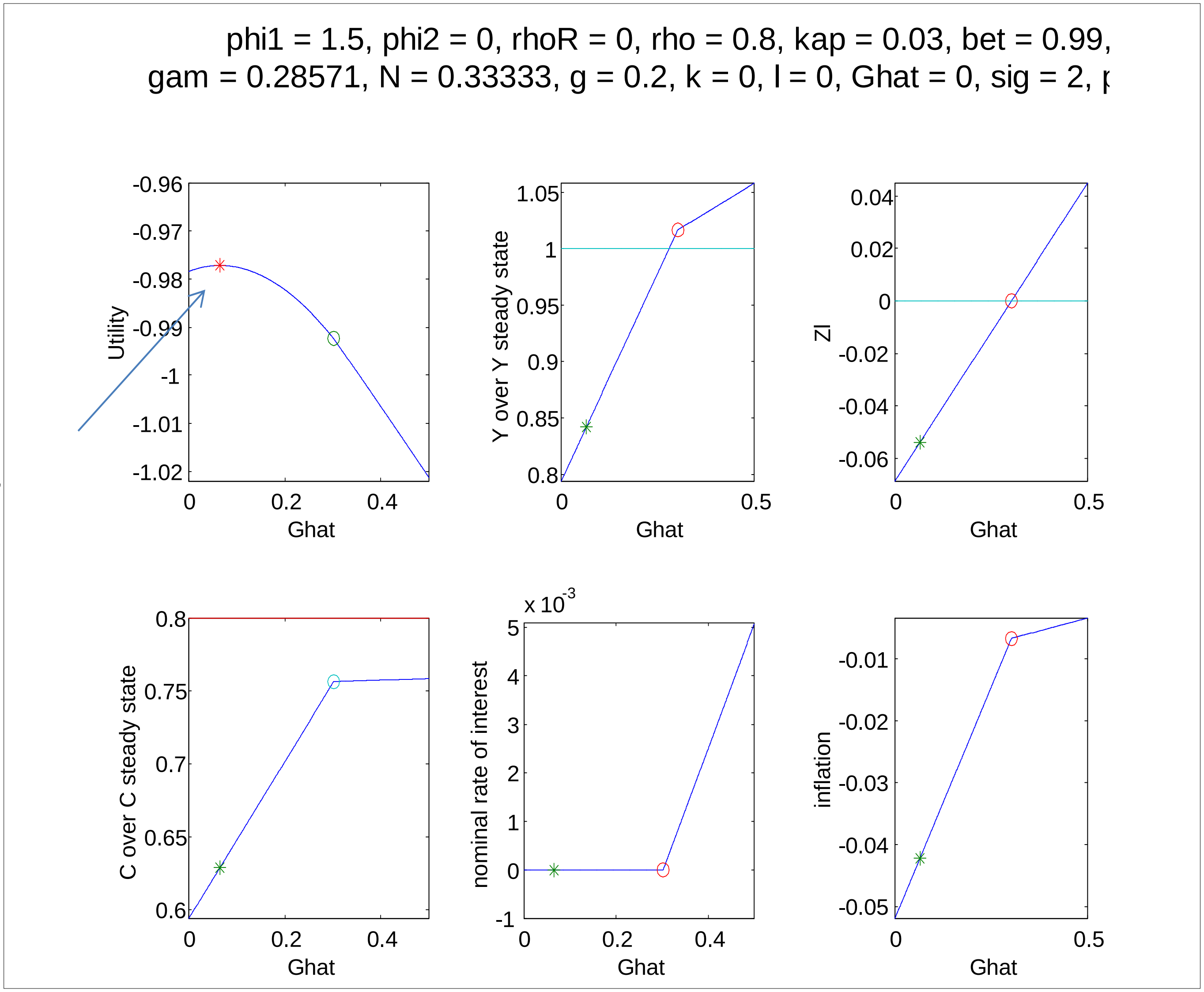
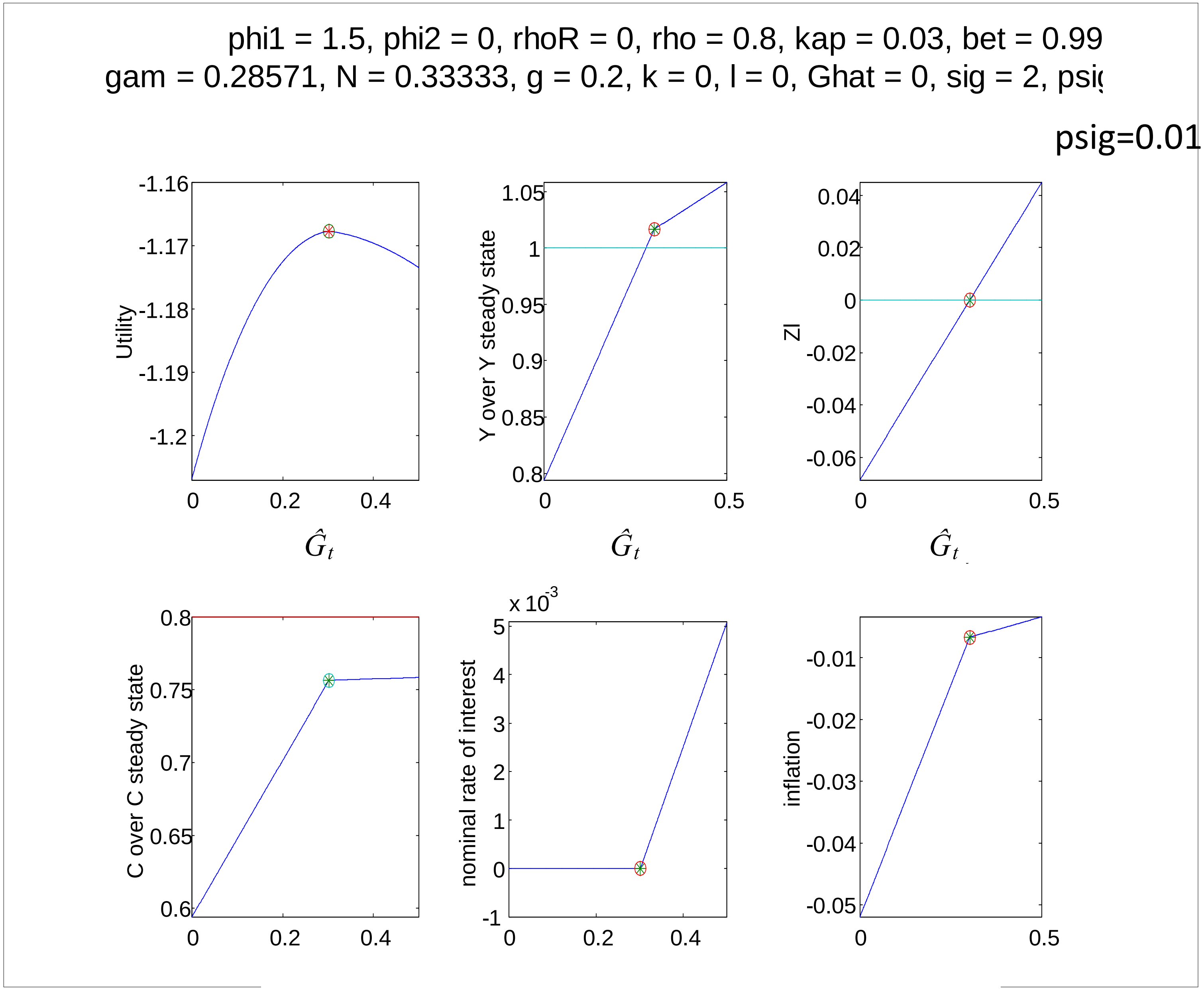


Optimal *G* is substantial, around 5%.

Case Where Gov't Spending is Desirable



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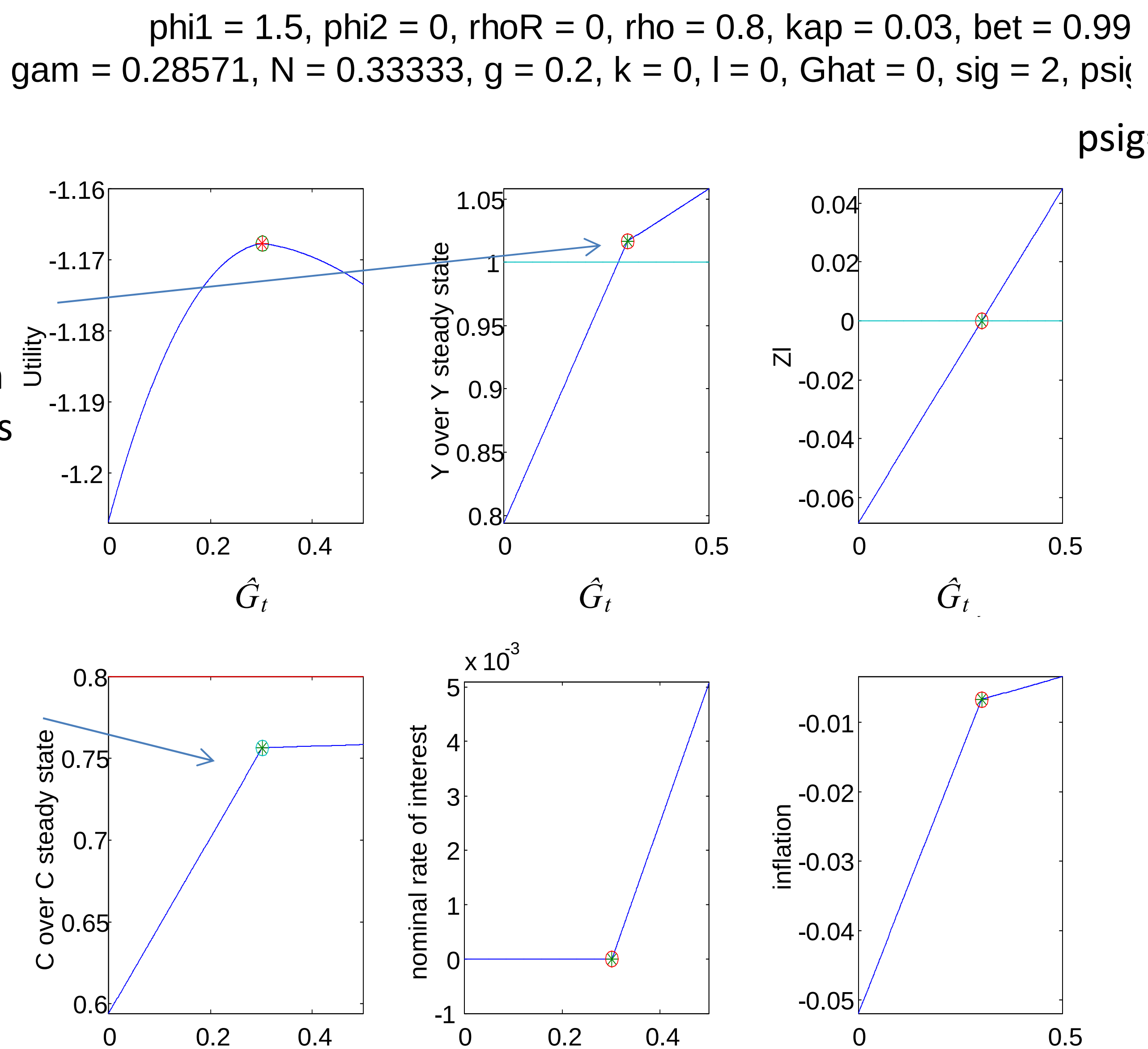

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Optimal Y higher than before crisis

The high level of output is necessary to get partial recovery in consumption

Introducing Investment

- Inclusion of investment does not have a large, qualitative effect.

- Financial frictions could make things much worse.
  – Deflation hurts net worth of investors with nominal debt, and this forces those agents to cut spending by more.

Conclusion of *G* Multiplier Analysis

- Government spending multiplier in a neighborhood of unity in 'normal times'.

- Multiplier can be large when the zero bound is binding (because *R* constant then).

- Increase in *G* is welfare improving during lower bound crisis.

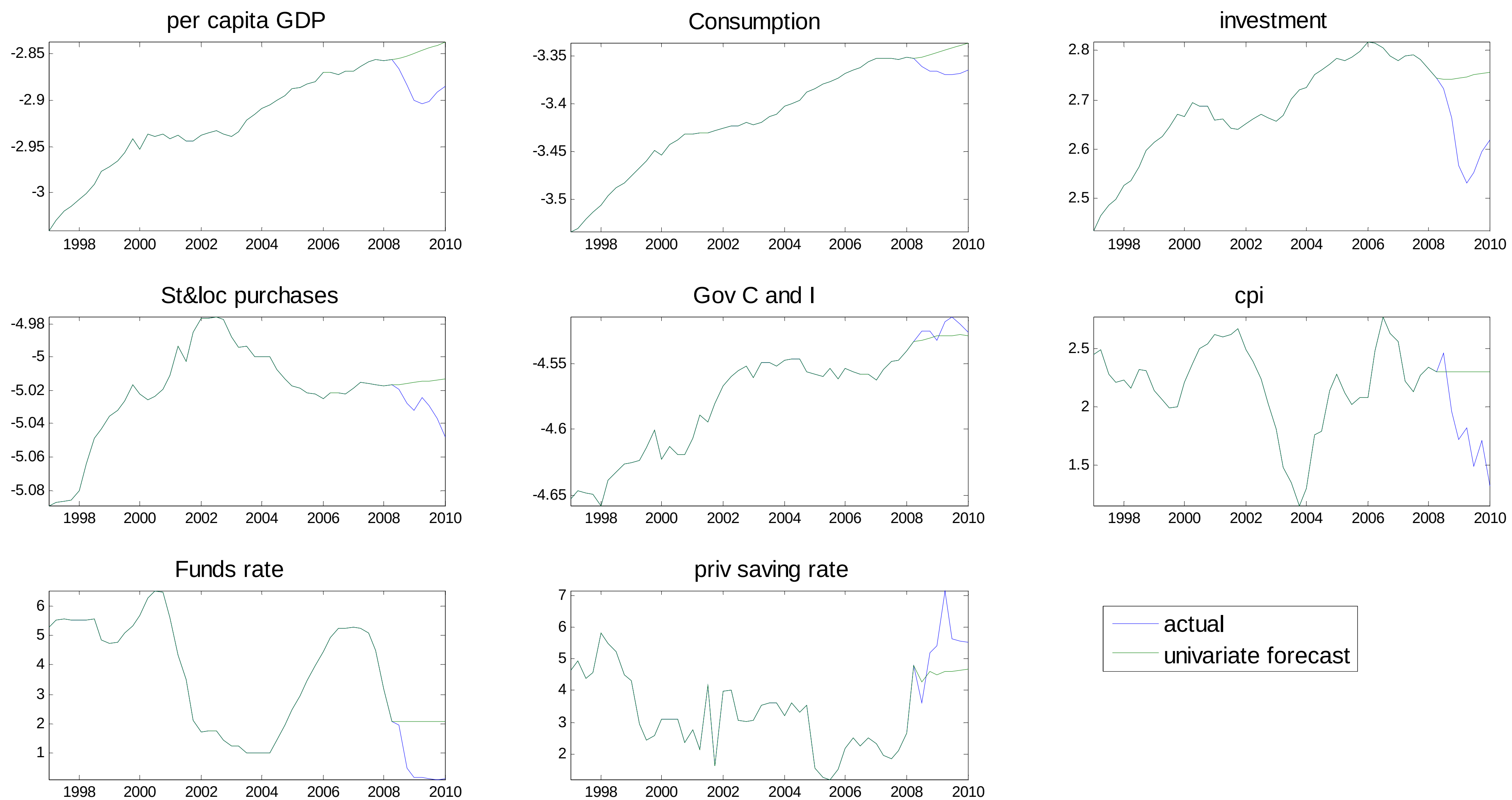
- Caveat: focused exclusively on multiplier
  - Increasing *G* may be bad idea because hard to reverse.
  - May be other ways of accomplishing similar thing (e.g., transition to VAT tax over time).

Can Zero Bound, in Conjunction with Other Shocks Account for Recent Data?

- Suppose that something (thing, *x*) happened in 2008Q3.
- Identify impulse response of economy to *x* by comparing what actually happened with forecast as of 2008Q2.
- Assume *x* is a shock to households' desire to save (the saving rate did go up), and wedge in the rate of return on capital (spreads did go up).

- First, what happened?.....

Actual and 2008Q2 Forecasts



Deterministic Simulation of Medium-Size Model[1]

- Features:
  - Habit persistence in preferences
  - Adjustment costs in change of investment
- Shock to discount rate and to wedge in rate of return on capital

wedge

$$R^k_{t+1} = (1-\tau^k_{t+1})\left[\frac{r^k_{t+1} + P^k_{t+1}(1-\delta)}{P^k_t}\right]$$

- Wedge designed to capture increased financial frictions.

[1] Model taken from Altig-Christiano-Eichenbaum-Linde (2010).

Simulating a Lower Bound Episode

- In periods $t<0$, the system is in deterministic steady state.
- In periods $t=1,2,\ldots,T$ (=10) the shocks that make the lower bound bind occur.
- In periods $t>T$ all shocks go back to their steady state.
- Events in periods $t\geq 0$ completely deterministic.

Monetary Policy

$$Z_t = \frac{1}{\beta} - 1 + \rho_R dR_t + (1 - \rho_R)\left[\frac{\phi_1}{\beta}\pi_t + \frac{\phi_2}{\beta}\hat{Y}_t\right]$$

$$R_{t+1} = \begin{cases} Z_t & Z_t \geq 0 \quad \text{‘zero bound not binding’} \\ 0 & Z_t < 0 \quad \text{‘binding zero bound’} \end{cases}$$

Equilibrium Conditions

Equilibrium Conditions

- Period *t* endogenous variables (simplified):

$$z_t = \begin{pmatrix} dN_t & d\pi_t & dK_{t+1} & dR_{t+1} & dI_t & dZ_t \end{pmatrix} \quad \text{endogenous variables}$$

$$s_t = \begin{pmatrix} dr_{t+1} & d\tau_{t+1}^k & \hat{G}_t \end{pmatrix} \quad \text{exogenous variables}$$

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$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0, \text{ solution: } z_{t+1} = A z_t$$

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  - bottom row of α_0, α_2 composed of zeros.
  - bottom row of α_1:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

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$$\text{for } t = 1,2,\ldots,T: \alpha_0 z_{t+1} + \tilde{\alpha}_1 z_t + \alpha_2 z_{t-1} + d + \beta_0 s_{t+1} + \beta_1 s_t = 0$$

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$s_{T+1}=0$

- Here,

$$\text{bottom row of } \tilde{\alpha}_1 : \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta} - 1 \end{pmatrix}'$$

Shooting Algorithm

- System

$$\text{for } t = 1,2,\ldots,T: \; \alpha_0 z_{t+1} + \tilde{\alpha}_1 z_t + \alpha_2 z_{t-1} + d + \beta_0 s_{t+1} + \beta_1 s_t = 0$$

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- System

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$$\text{for } t > T: \; z_t = A z_{t-1}$$

- Initial conditions: $z_0=0$.
- Fix z_1 arbitrarily.
- Compute $z_2, z_3, \ldots, z_{T+1}$ recursively, to solve:

$$\alpha_0 z_2 + \tilde{\alpha}_1 z_1 + \alpha_2 z_0 + d + \beta_0 s_2 + \beta_1 s_1 = 0$$

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$$\alpha_0 z_{T+1} + \tilde{\alpha}_1 z_T + \alpha_2 z_{T-1} + d + \beta_0 s_{T+1} + \beta_1 s_T = 0$$

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$$\alpha_0 z_{T+1} + \tilde{\alpha}_1 z_T + \alpha_2 z_{T-1} + d + \beta_0 s_{T+1} + \beta_1 s_T = 0$$

This is zero by assumption.

- Adjust z_1 until

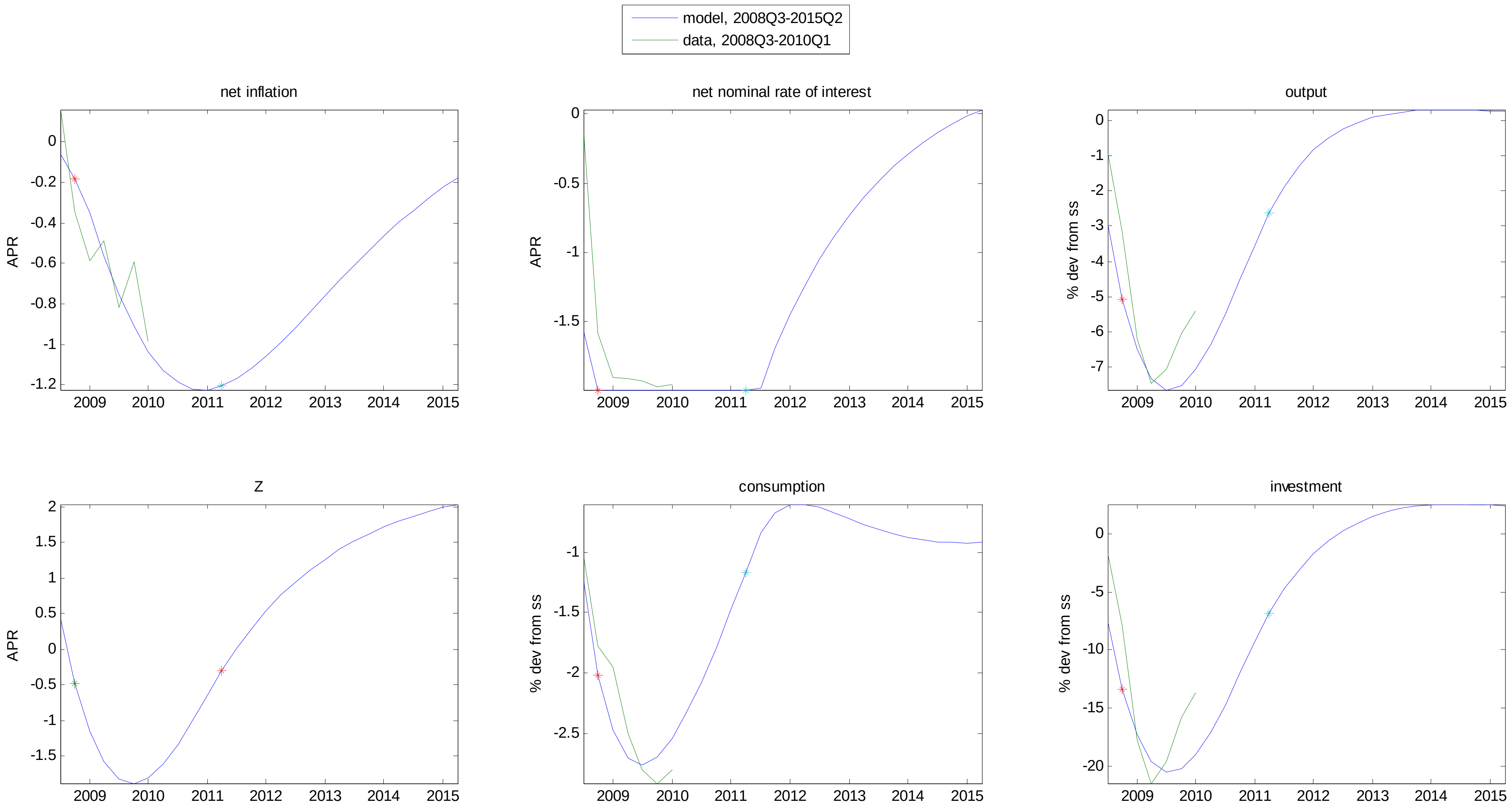
$$z_{T+1} - A z_T = 0$$

Complications

- In practice, α_0 is singular.
  - Apply QZ decomposition.

- In practice, zero bound does not necessarily bind in first and last few periods.
  - Have to adjust equations in an obvious way.

Results: Model Replicates Impulse Responses Reasonably Well



- Government consumption multiplier:

$$\frac{dY_t}{dG} = 0.49, 2.0, 2.2, 2.3, 2.3, 2.3, 2.2, 2.0, 1.8, 1.7, 1.5, 1.3, 0.02, \; t = 1, 2, \ldots, 13.$$

- Denominator: change in G operative while G is up (i.e., periods 2 to 12).

Conclusion of Simulation

- Can account for dynamics of recent data as reflecting the operation of the zero bound and two particular shocks.

- Many people would expect this not to be possible. Mindful of the deflation spiral, they would anticipate that the drop in inflation would be too great.

- In fact, incorporate a very small slope to Phillips curve.
  - However, prices sticky on average only a year.
  - This reflects that capital is firm specific.

- A fully stochastic simulation, with non-linear equilibrium conditions much more interesting.
  - Method for dealing with ‘occasionally binding constraints’ (Christiano-Fisher, JEDC2000).
  - See Adams-Billie, Anton Braun, Villaverde-Juan Rubio.