Perturbation and Projection Methods for Solving DSGE Models

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Discussion of projections taken from Christiano-Fisher, 'Algorithms for Solving Dynamic Models with Occasionally Binding Constraints', 2000, *Journal of Economic Dynamics and Control.*
Discussion of perturbations taken from Judd's textbook.
See also Kim, Kim, Schaumburg and Sims, 2008, cited below.

Outline

- A Toy Example to Illustrate the basic ideas.
  - Functional form characterization of model solution.
  - Use of Projections and Perturbations.
- Neoclassical model.
  - Projection methods
  - Perturbation methods
- Stochastic Simulations and Impulse Responses
  - Focus on perturbation solutions of order greater than unity.
  - The need for pruning.

Simple Example

- Suppose that *x* is some exogenous variable and that the following equation implicitly defines *y*:

$$h(x,y) = 0, \text{ for all } x \in X$$

- Let the solution be defined by the 'policy rule', *g*:

$$y = g(x)$$

- satisfying

'Error function'

$$R(x;g) \equiv h(x,g(x)) = 0$$

- for all $x \in X$

The Need to Approximate

- Finding the policy rule, *g*, is a big problem outside special cases
  - 'Infinite number of unknowns (i.e., one value of *g* for each possible *x*) in an infinite number of equations (i.e., one equation for each possible *x*).'
- Two approaches:
  - projection and perturbation

Projection

- Find a parametric function, $\hat{g}(x;\gamma)$, where γ is a vector of parameters chosen so that it imitates the property of the exact solution, i.e., $R(x;g) = 0$ for all $x \in X$, as well as possible.
- Choose values for γ so that

$$\hat{R}(x;\gamma) = h(x, \hat{g}(x;\gamma))$$

- is close to zero for $x \in X$.

- The method is defined by how 'close to zero' is defined and by the parametric function, $\hat{g}(x;\gamma)$, that is used.

Projection, continued

- Spectral and finite element approximations
  - **Spectral functions**: functions, $\hat{g}(x;\gamma)$, in which each parameter in γ influences $\hat{g}(x;\gamma)$ for all $x \in X$ example:

$$\hat{g}(x;\gamma) = \sum_{i=0}^{n} \gamma_i H_i(x),\ \gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$

$$H_i(x) = x^i \sim \text{ordinary polynominal (not computationaly efficient)}$$

$$H_i(x) = T_i(\varphi(x)),$$

$$T_i(z) : [-1,1] \to [-1,1],\ i^{th} \text{ order Chebyshev polynomial}$$

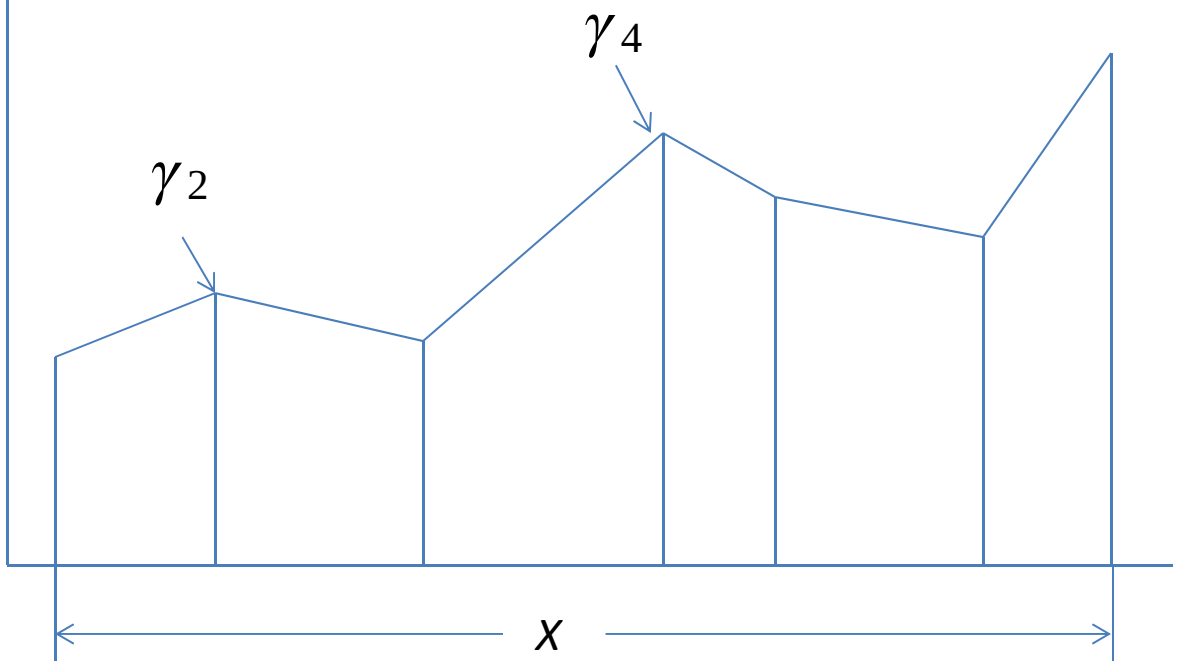
$$\varphi : X \to [-1,1]$$

Projection, continued

– Finite element approximations: functions, $\hat{g}(x;\gamma)$, in which each parameter in γ influences $\hat{g}(x;\gamma)$ over only a subinterval of $x \in X$

$\hat{g}(x;\gamma)$

$$\gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \end{bmatrix}$$



Projection, continued

- 'Close to zero': two methods
- **Collocation**, for *n* values of $x: \; x_1, x_2, \ldots, x_n \in X$ choose *n* elements of $\gamma = \left[\begin{array}{ccc} \gamma_1 & \cdots & \gamma_n \end{array} \right]$ so that

$$\hat{R}(x_i; \gamma) = h(x_i, \hat{g}(x_i; \gamma)) = 0, \; i = 1, \ldots, n$$

  – how you choose the grid of *x*'s matters…

Example of Importance of Grid Points

- Here is an example, taken from a related problem, the problem of *interpolation*.
  - You get to evaluate a function on a set of grid points that you select, and you must guess the shape of the function between the grid points.

- Consider the function,

$$f(k) = \frac{1}{1+k^2},\ k \in [-5, 5]$$

- Next slide shows what happens when you select 11 equally-spaced grid points and interpolate by fitting a 10^{th} order polynomial.
  - As you increase the number of grid points on a fixed interval grid, oscillations in tails grow more and more violent.

- Chebyshev approximation theorem: distribute more points in the tails (by selecting zeros of Chebyshev polynomial) and get convergence in sup norm.

How You Select the Grid Points Matters

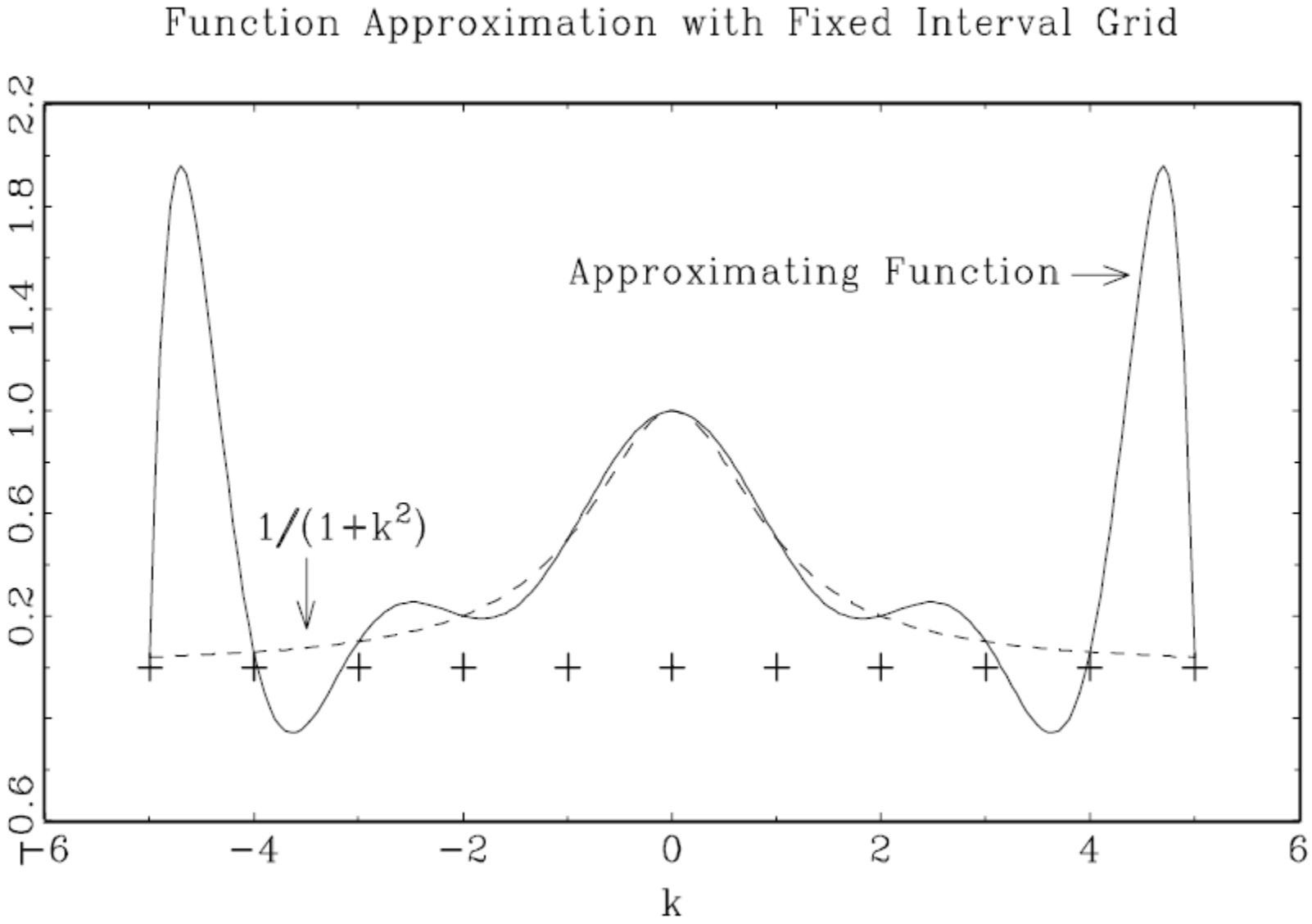
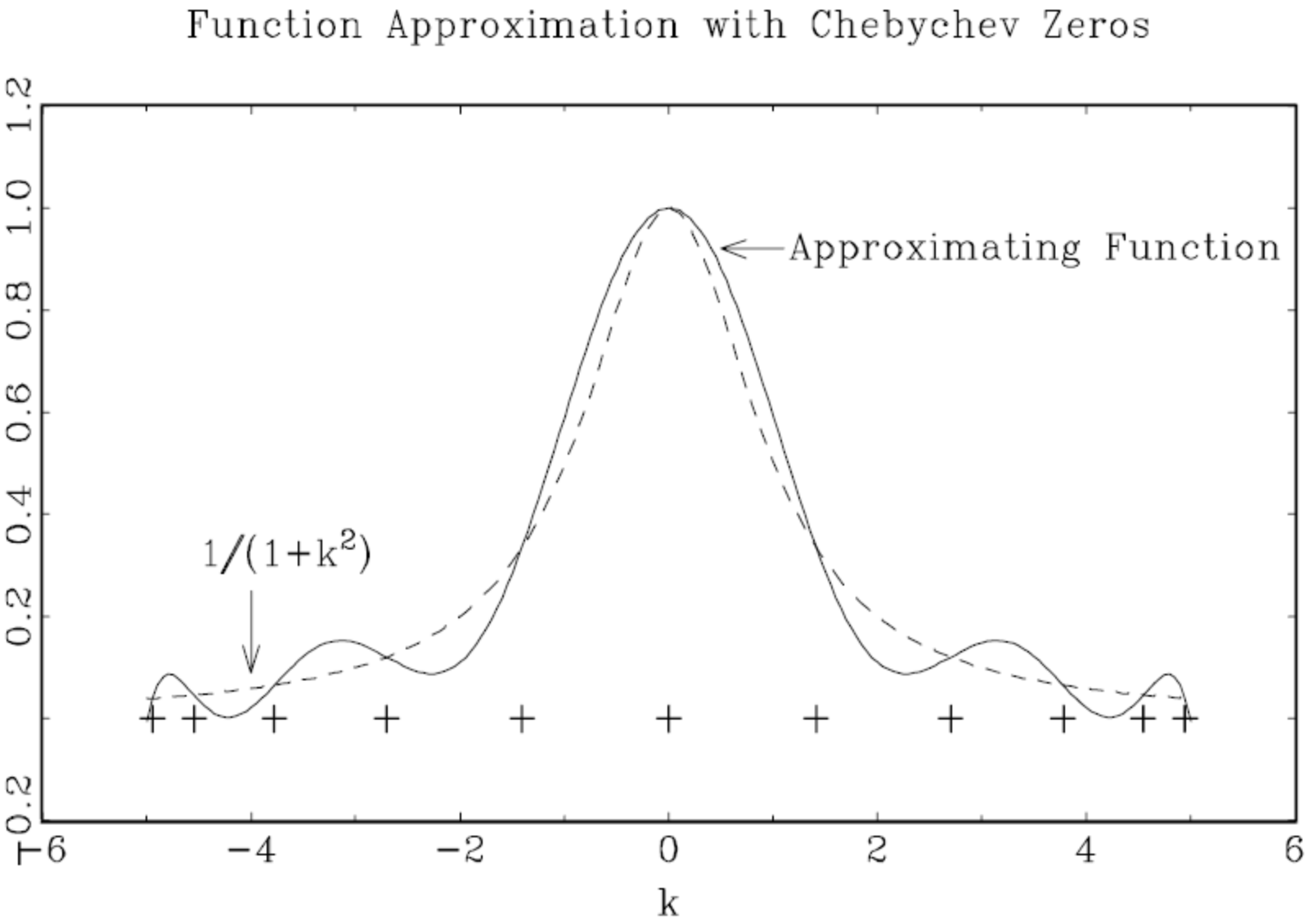


Figure from Christiano-Fisher, JEDC, 1990


Function Approximation with Chebychev Zeros
1.2
1.0
0.8
0.6
0.4
0.2
0.2
Approximating Function
1/(1+k²)
−6
−4
−2
0
2
4
6
k

Projection, continued

- 'Close to zero': two methods
- **Collocation**, for *n* values of $x: x_1, x_2, \ldots, x_n \in X$ choose *n* elements of $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$ so that

$$\hat{R}(x_i; \gamma) = h(x_i, \hat{g}(x_i; \gamma)) = 0,\ i = 1, \ldots, n$$

  – how you choose the grid of *x*'s matters…

- **Weighted Residual**, for *m>n* values of $x: x_1, x_2, \ldots, x_m \in X$ choose the *n* γ_i's

$$\sum_{j=1}^{m} w_j^i h(x_j, \hat{g}(x_j; \gamma)) = 0,\ i = 1, \ldots, n$$

Perturbation

- Projection uses the 'global' behavior of the functional equation to approximate solution.
  - Problem: requires finding zeros of non-linear equations. Iterative methods for doing this are a pain.
  - Advantage: can easily adapt to situations the policy rule is not continuous or simply non-differentiable (e.g., occasionally binding zero lower bound on interest rate).

- Perturbation method uses local properties of functional equation and Implicit Function/Taylor's theorem to approximate solution.
  - Advantage: can implement it using non-iterative methods.
  - Possible disadvantages:
    - may require enormously high derivatives to achieve a decent global approximation.
    - Does not work when there are important non-differentiabilities (e.g., occasionally binding zero lower bound on interest rate).

Perturbation, cnt'd

- Suppose there is a point, $x^* \in X$, where we know the value taken on by the function, *g*, that we wish to approximate:

$$g(x^*) = g^*, \text{ some } x^*$$

- Use the implicit function theorem to approximate *g* in a neighborhood of x^*
- Note:

$$R(x;g) = 0 \text{ for all } x \in X$$

$$\rightarrow$$

$$R^{(j)}(x;g) \equiv \frac{d^j}{dx^j} R(x;g) = 0 \text{ for all } j, \text{ all } x \in X.$$

Perturbation, cnt'd

- Differentiate *R* with respect to x and evaluate the result at x^* :

$$R^{(1)}(x^*) = \frac{d}{dx} h(x, g(x))|_{x=x^*} = h_1(x^*, g^*) + h_2(x^*, g^*) g'(x^*) = 0$$

$$\rightarrow g'(x^*) = -\frac{h_1(x^*, g^*)}{h_2(x^*, g^*)}$$

- Do it again!

$$R^{(2)}(x^*) = \frac{d^2}{dx^2} h(x, g(x))|_{x=x^*} = h_{11}(x^*, g^*) + 2h_{12}(x^*, g^*) g'(x^*)$$
$$h_{22}(x^*, g^*)[g'(x^*)]^2 + h_2(x^*, g^*) g''(x^*).$$

$\rightarrow$ Solve this linearly for $g''(x^*)$.

Perturbation, cnt'd

- Preceding calculations deliver (assuming enough differentiability, appropriate invertibility, a high tolerance for painful notation!), recursively:

$$g'(x^*), g''(x^*), \ldots, g^{(n)}(x^*)$$

- Then, have the following Taylor's series approximation:

$$g(x) \approx \hat{g}(x)$$

$$\hat{g}(x) = g^* + g'(x^*) \times (x - x^*) + \frac{1}{2} g''(x^*) \times (x - x^*)^2 + \ldots + \frac{1}{n!} g^{(n)}(x^*) \times (x - x^*)^n$$

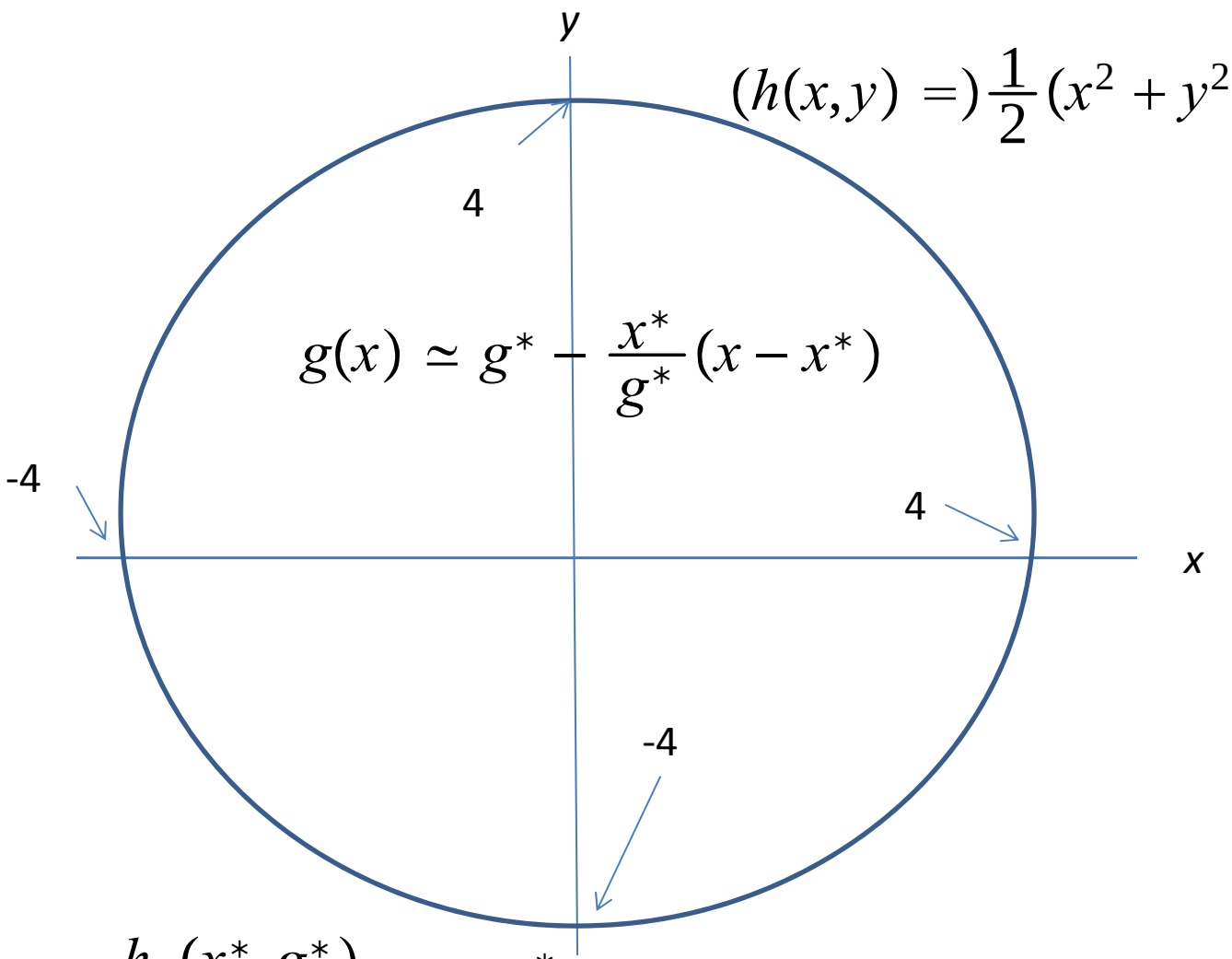
Perturbation, cnt'd

- Check….
- Study the graph of

$$R(x;\hat{g})$$

  - over $x \in X$ to verify that it is everywhere close to zero (or, at least in the region of interest).

Example of Implicit Function Theorem



$$(h(x,y) =)\frac{1}{2}(x^2+y^2)-8=0$$

$$g(x) \simeq g^* - \frac{x^*}{g^*}(x-x^*)$$

$$g'(x^*) = -\frac{h_1(x^*,g^*)}{h_2(x^*,g^*)} = -\frac{x^*}{g^*} \quad (h_2 \text{ had better not be zero!})$$

Neoclassical Growth Model

- Objective:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),\ u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

- Constraints:

$$c_t + \exp(k_{t+1}) \leq f(k_t, a_t),\ t = 0, 1, 2, \ldots.$$

$$a_t = \rho a_{t-1} + \varepsilon_t,\ \varepsilon_t \sim E\varepsilon_t = 0,\ E\varepsilon_t^2 = V_\varepsilon$$

$$f(k_t, a_t) = \exp(\alpha k_t)\exp(a_t) + (1-\delta)\exp(k_t)$$

Why *Log* Capital?

- Might hope to get an accurate solution.
- Consider the special case, $\alpha = \gamma = 1$
  - In this case, we know the solution is given by:

$$K_{t+1} = \beta\alpha \exp(a_t) K_t^{\alpha},\ K_t \equiv \exp(k_t)$$

  - So, in terms of log capital, the solution is *exactly* linear:

$$k_{t+1} = \log(\beta\alpha) + a_t + \alpha k_t$$

  - Solution methods often work with polynomials (the perturbation method always does!) , so in case, $\alpha = \gamma = 1$, you would get exactly the right answer.
- We will return to this issue later.

Efficiency Condition

$$E_t[u'\left(\overbrace{f(k_t, a_t) - \exp(k_{t+1})}^{c_{t+1}}\right)$$

$$-\beta u'\left(\overbrace{f(k_{t+1}, \rho a_t + \sigma\varepsilon_{t+1}) - \exp(k_{t+2})}^{c_{t+1}}\right) \overbrace{f_K(k_{t+1}, \rho a_t + \sigma\varepsilon_{t+1})}^{\text{period } t+1 \text{ marginal product of capital}} \quad] \quad = 0.$$

- Here,
  - k_t, a_t ~given numbers
  - ε_{t+1} ~random variable
  - time t choice variable, k_{t+1}
- Parameter, σ , indexes a set of models, with the model of interest corresponding to

$$\sigma = 1$$

Solution

- A policy rule,

$$k_{t+1} = g(k_t, a_t, \sigma).$$

- With the property:

$$R(k_t, a_t, \sigma; g) \equiv E_t\{u'\left(\overbrace{f(k_t, a_t) - \exp[g(k_t, a_t, \sigma)]}^{c_t}\right)$$

$$- \beta u'\left(\overbrace{f\left(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma\varepsilon_{t+1}}^{a_{t+1}}\right) - \exp\left[g\left(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma\varepsilon_{t+1}}^{a_{t+1}}, \sigma\right)\right]}^{c_{t+1}}\right)$$

$$\times f_K\left(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma\varepsilon_{t+1}}^{a_{t+1}}\right)\} = 0,$$

- for all $a_t, k_t, \sigma.$

Projection Methods

- Let

$$\hat{g}(k_t, a_t, \sigma; \gamma)$$

  - be a function with finite parameters (could be either spectral or finite element, as before).

- Choose parameters, γ, to make

$$R(k_t, a_t, \sigma; \hat{g})$$

  - as close to zero as possible, over a range of values of the state.
  - use weighted residuals or Collocation.

Occasionally Binding Constraints

- Suppose we add the non-negativity constraint on investment:

$$\exp(g(k_t, a_t, \sigma)) - (1-\delta)\exp(k_t) \geq 0$$

- Express problem in Lagrangian form and optimum is characterized in terms of equality conditions with a multiplier and with a complementary slackness condition associated with the constraint.

- Conceptually straightforward to apply preceding method. For details, see Christiano-Fisher, 'Algorithms for Solving Dynamic Models with Occasionally Binding Constraints', 2000, *Journal of Economic Dynamics and Control.*
  - This paper describes a wide range of strategies, including those based on parameterizing the expectation function, that may be easier, when constraints are occasionally bind.

Perturbation Approach

- Straightforward application of the perturbation approach, as in the simple example, requires knowing the value taken on by the policy rule at a point.

- The overwhelming majority of models used in macro do have this property.

  - In these models, can compute non-stochastic steady state without any knowledge of the policy rule, *g*.
  - Non-stochastic steady state is k^* such that

$$k^* = g\left(k^*, \overbrace{0}^{a=0 \text{ (nonstochastic steady state in no uncertainty case)}}, \overbrace{0}^{\sigma=0 \text{ (no uncertainty)}} \right)$$

  - and

$$k^* = \log\left\{ \left[\frac{\alpha\beta}{1-(1-\delta)\beta} \right]^{\frac{1}{1-\alpha}} \right\}.$$

Perturbation

- Error function:

$$R(k_t, a_t, \sigma; g) \equiv E_t\{u'\left(\overbrace{f(k_t, a_t) - \exp[g(k_t, a_t, \sigma)]}^{c_t}\right)$$

$$- \beta u'\left[\overbrace{f(g(k_t, a_t, \sigma), \rho a_t + \sigma\varepsilon_{t+1}) - \exp[g(g(k_t, a_t, \sigma), \rho a_t + \sigma\varepsilon_{t+1}, \sigma)]}^{c_{t+1}}\right]$$

$$\times f_K(g(k_t, a_t, \sigma), \rho a_t + \sigma\varepsilon_{t+1})\} = 0,$$

  - for all values of k_t, a_t, σ.

- So, all order derivatives of *R* with respect to its arguments are zero (assuming they exist!).

Four (Easy to Show) Results About Perturbations

- Taylor series expansion of policy rule:

$$g(k_t, a_t, \sigma) \simeq \overbrace{k + g_k(k_t - k) + g_a a_t + g_\sigma \sigma}^{\text{linear component of policy rule}}$$

$$+\overbrace{\frac{1}{2}[g_{kk}(k_t - k)^2 + g_{aa}a_t^2 + g_{\sigma\sigma}\sigma^2] + g_{ka}(k_t - k)a_t + g_{k\sigma}(k_t - k)\sigma + g_{a\sigma}a_t\sigma + \ldots}^{\text{second and higher order terms}}$$

  - $g_\sigma = 0$: to a first order approximation, 'certainty equivalence'
  - All terms found by solving linear equations, except coefficient on past endogenous variable, g_k ,which requires solving for eigenvalues
  - To second order approximation: slope terms certainty equivalent –

$$g_{k\sigma} = g_{a\sigma} = 0$$

  - Quadratic, higher order terms computed recursively.

First Order Perturbation

- Working out the following derivatives and evaluating at $k_t = k^*, a_t = \sigma = 0$

$$R_k(k_t, a_t, \sigma; g) = R_a(k_t, a_t, \sigma; g) = R_\sigma(k_t, a_t, \sigma; g) = 0$$

- Implies:

'problematic term'

Source of certainty equivalence
In linear approximation

$$R_k = u''(f_k - e^g g_k) - \beta u' f_{Kk} g_k - \beta u''(f_k g_k - e^g g_k^2) f_K = 0$$

$$R_a = u''(f_a - e^g g_a) - \beta u'[f_{Kk} g_a + f_{Ka}\rho] - \beta u''(f_k g_a + f_a \rho - e^g[g_k g_a + g_a \rho]) f_K = 0$$

$$R_\sigma = -[u' e^g + \beta u''(f_k - e^g g_k) f_K] g_\sigma = 0$$

Absence of arguments in these functions reflects they are evaluated in $k_t = k^*, a_t = \sigma = 0$

Technical notes for following slide

$$u''(f_k - e^g g_k) - \beta u' f_{Kk} g_k - \beta u'' (f_k g_k - e^g g_k^2) f_K = 0$$

$$\frac{1}{\beta}(f_k - e^g g_k) - u' \frac{f_{Kk}}{u''} g_k - (f_k g_k - e^g g_k^2) f_K = 0$$

$$\frac{1}{\beta} f_k - \left[\frac{1}{\beta} e^g + u' \frac{f_{Kk}}{u''} + f_k f_K \right] g_k + e^g g_k^2 f_K = 0$$

$$\frac{1}{\beta} \frac{f_k}{e^g f_K} - \left[\frac{1}{\beta f_K} + \frac{u'}{u''} \frac{f_{Kk}}{e^g f_K} + \frac{f_k}{e^g} \right] g_k + g_k^2 = 0$$

$$\frac{1}{\beta} - \left[1 + \frac{1}{\beta} + \frac{u'}{u''} \frac{f_{Kk}}{e^g f_K} \right] g_k + g_k^2 = 0$$

- Simplify this further using:

$$\begin{aligned}
&\beta f_K \sim \text{steady state equation} \\
f_K &= \alpha K^{\alpha-1} \exp(a) + (1-\delta),\ K \equiv \exp(k) \\
&= \alpha \exp[(\alpha-1)k + a] + (1-\delta) \\
f_k &= \alpha \exp[\alpha k + a] + (1-\delta)\exp(k) = f_K e^g \\
f_{Kk} &= \alpha(\alpha-1)\exp[(\alpha-1)k + a] \\
f_{KK} &= \alpha(\alpha-1)K^{\alpha-2}\exp(a) = \alpha(\alpha-1)\exp[(\alpha-2)k + a] = f_{Kk} e^{-g}
\end{aligned}$$

- to obtain polynomial on next slide.

First Order, cont'd

- Rewriting $R_k = 0$ term:

$$\frac{1}{\beta} - \left[1 + \frac{1}{\beta} + \frac{u'}{u''} \frac{f_{KK}}{f_K} \right] g_k + g_k^2 = 0$$

- There are two solutions, $0 < g_k < 1,\ g_k > \frac{1}{\beta}$
  - Theory (see Stokey-Lucas) tells us to pick the smaller one.
  - In general, could be more than one eigenvalue less than unity: multiple solutions.
- Conditional on solution to g_k, g_a solved for linearly using $R_a = 0$ equation.
- These results all generalize to multidimensional case

Numerical Example

- Parameters taken from Prescott (1986):

$$\beta = 0.99,\ \gamma = 2(20),\ \alpha = 0.36,\ \delta = 0.02,\ \rho = 0.95,\ V_\varepsilon = 0.01^2$$

- Second order approximation:

$$\hat{g}(k_t, a_{t-1}, \varepsilon_t, \sigma) = \overbrace{k^*}^{3.88} + \overbrace{g_k}^{0.98\ (0.996)}(k_t - k^*) + \overbrace{g_a}^{0.06\ (0.07)} a_t + \overbrace{g_\sigma}^{0}\sigma$$

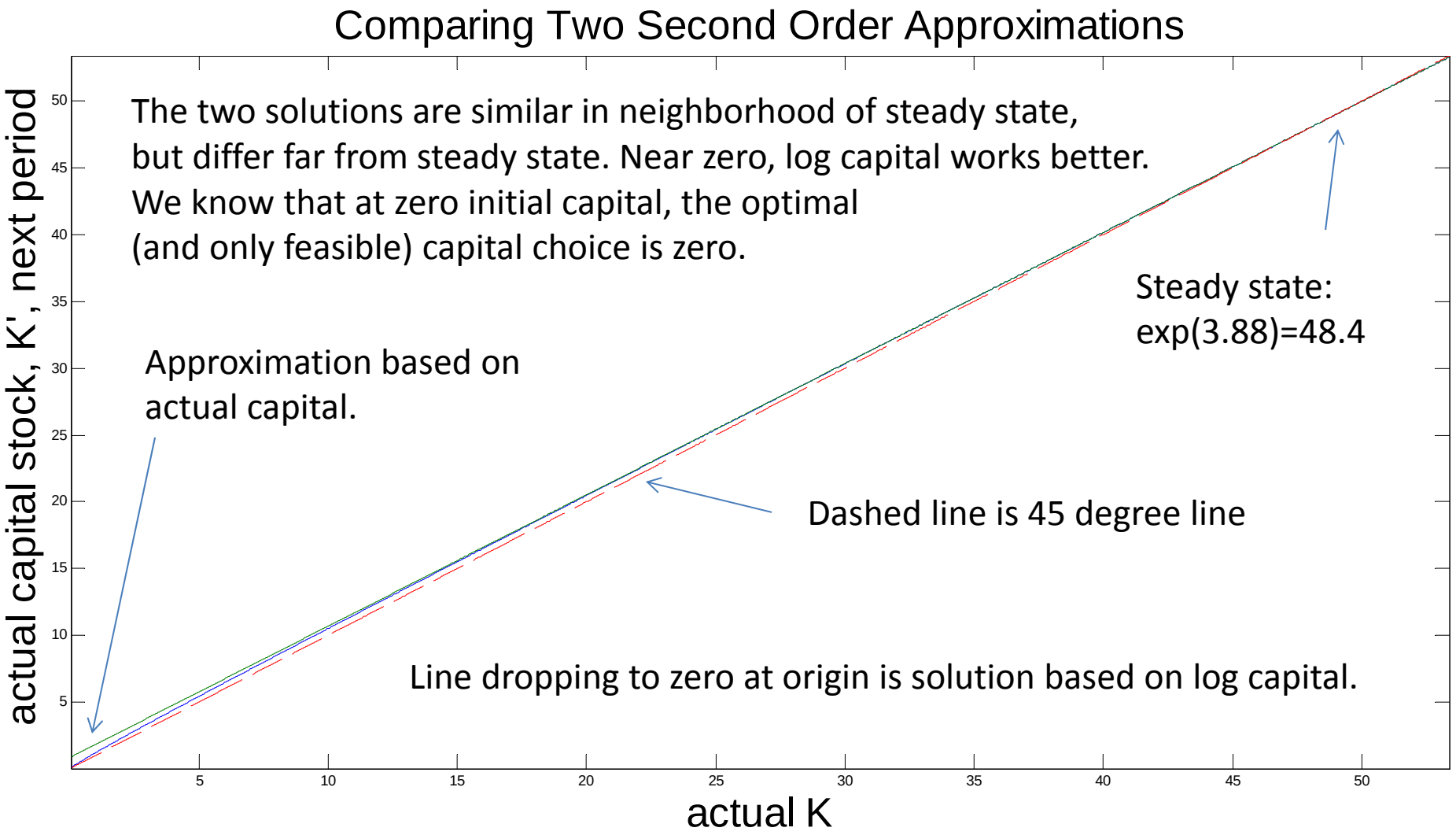
$$+ \frac{1}{2}\Big[\overbrace{g_{kk}}^{0.014\ (0.00017)}(k_t - k)^2 + \overbrace{g_{aa}}^{0.067\ (0.079)} a_t^2 + \overbrace{g_{\sigma\sigma}}^{0.000024\ (0.00068)}\overbrace{\sigma^2}^{1}\Big]$$

$$+ \overbrace{g_{ka}}^{-0.035\ (-0.028)}(k_t - k)a_t + \overbrace{g_{k\sigma}}^{0}(k_t - k)\sigma + \overbrace{g_{a\sigma}}^{0} a_t\sigma$$

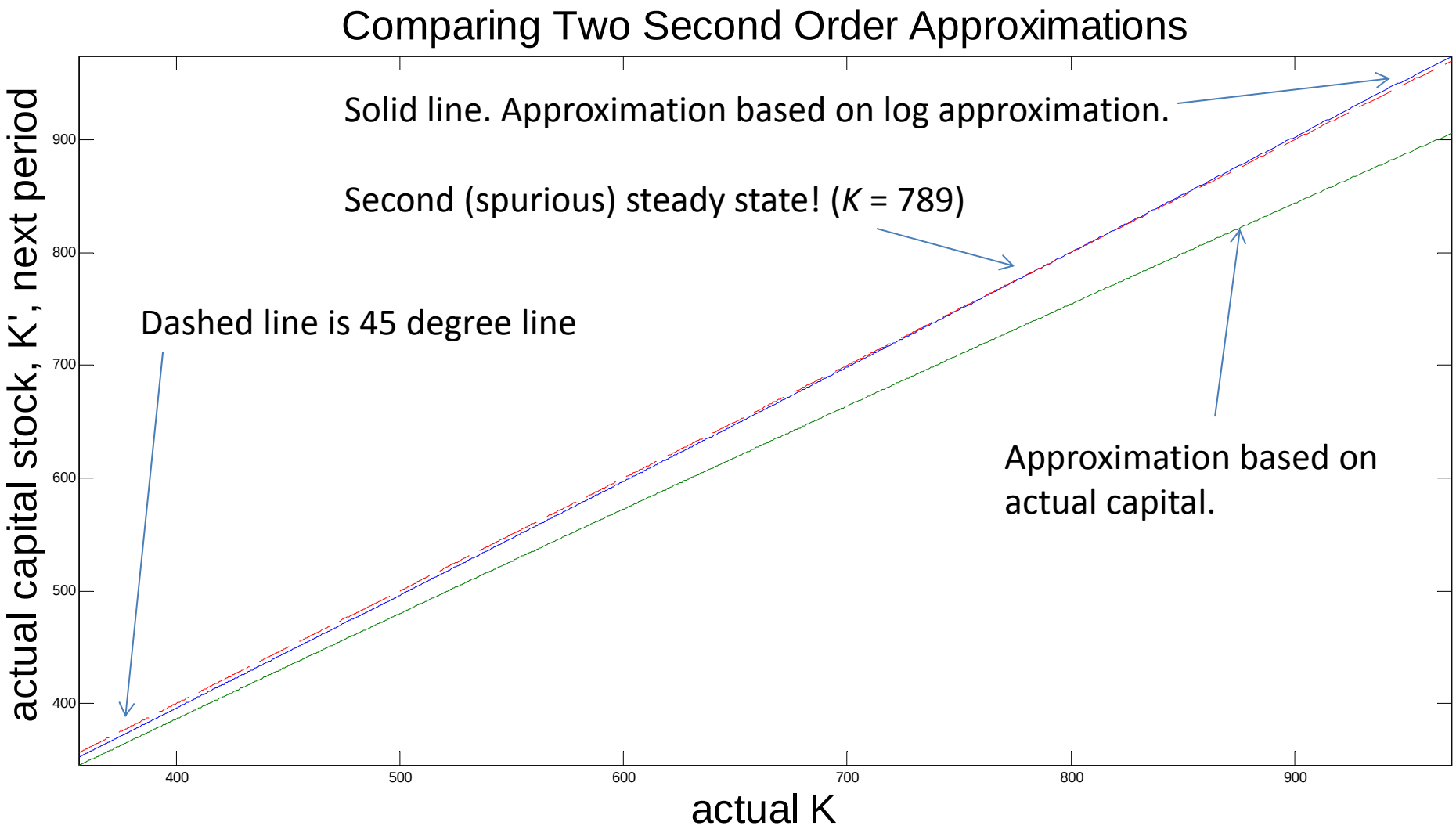
A Closer Look at the 2nd Order Approximation: Road Map

- Log, versus linear approximation
- Spurious steady state problem.
- Comparison of 1st and 2nd order approximation.
- Impulse response functions (IRF) and spurious steady state problem: toy example.
  - Standard definition of IRF
  - Linear approximation: easy!
  - Nonlinear approximation:
    - Computing IRF hard - requires stochastic simulation.
  - Alternative definition of IRF.
- Addressing the spurious steady state problem: pruning.

Was *Log* Capital a Good Idea? Maybe



Was *Log* Capital a Good Idea? Maybe Not



A Closer Look at Spurious Steady State in Approximation based on Log *K*

- Because of scale problem, it is hard to see the policy rule when graphed in the 'natural way', k_{t+1} against k_t .

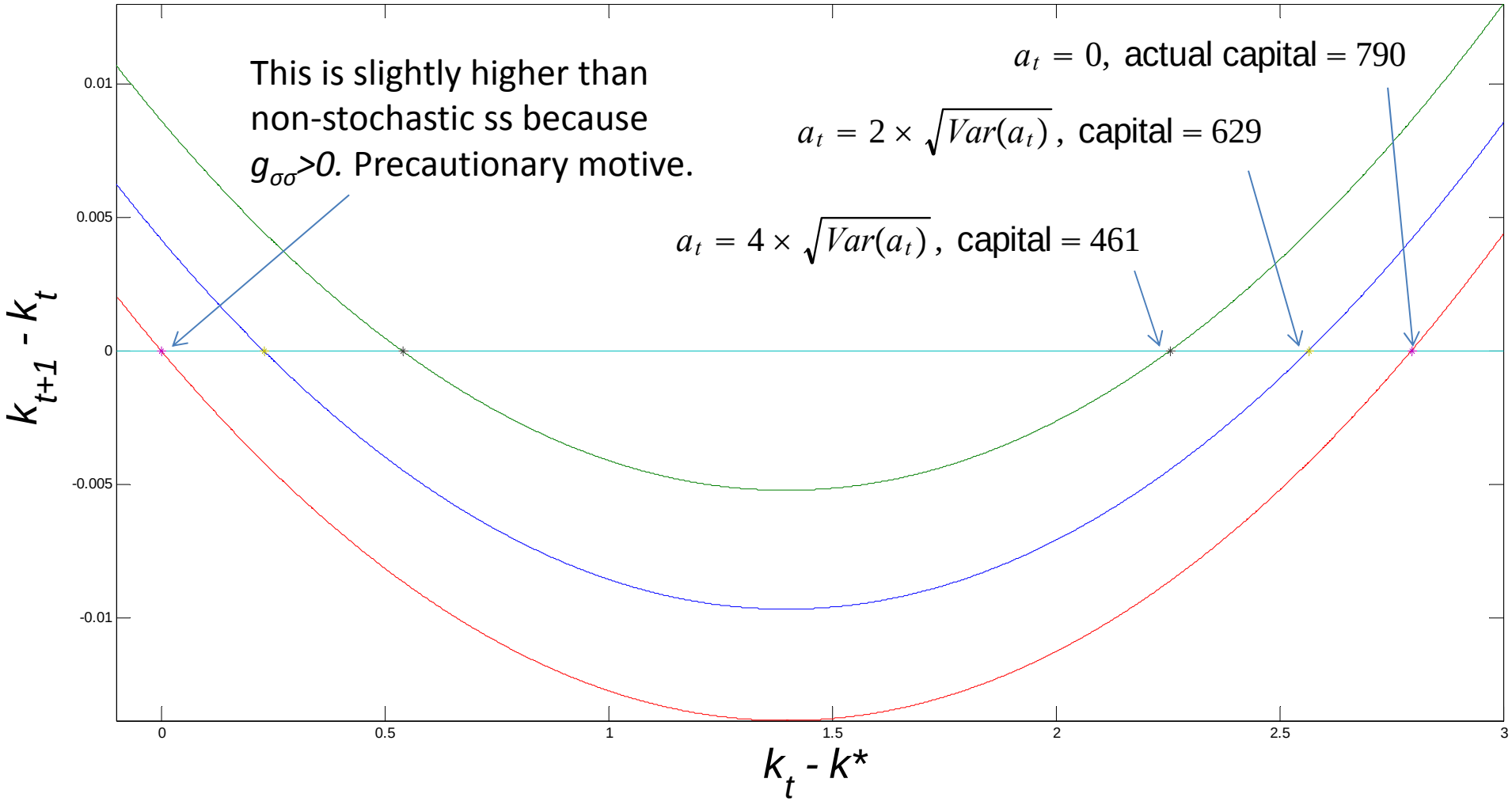
- Instead, will graph:

  - k_{t+1} - k_t against k_t –k^* (recall, k_t = log(K_t)).

$$y_{t+1} = \overbrace{g_k}^{0.98} y_t + \overbrace{g_a}^{0.063} a_t + \frac{1}{2}\left[\overbrace{g_{kk}}^{0.014} \times y_t^2 + \overbrace{g_{aa}}^{0.067} a_t^2 + \overbrace{g_{\sigma\sigma}}^{0.000024} \right] - \overbrace{g_{ka}}^{-0.035} \times y_t a_t$$

$y_t \equiv k_t - k^*$, k_t ~log, capital stock

Second order approximation of policy rule for log-linear capital



Spurious Steady State

- Setting $a_t=0$ and ignoring $g_{\sigma\sigma}$ (it's small anyway), the 2nd order approximation to the policy rule is:

$$y_{t+1} = g_k y_t + \frac{1}{2} g_{kk} y_t^2$$

- This has two steady states: $y_t=0$ and

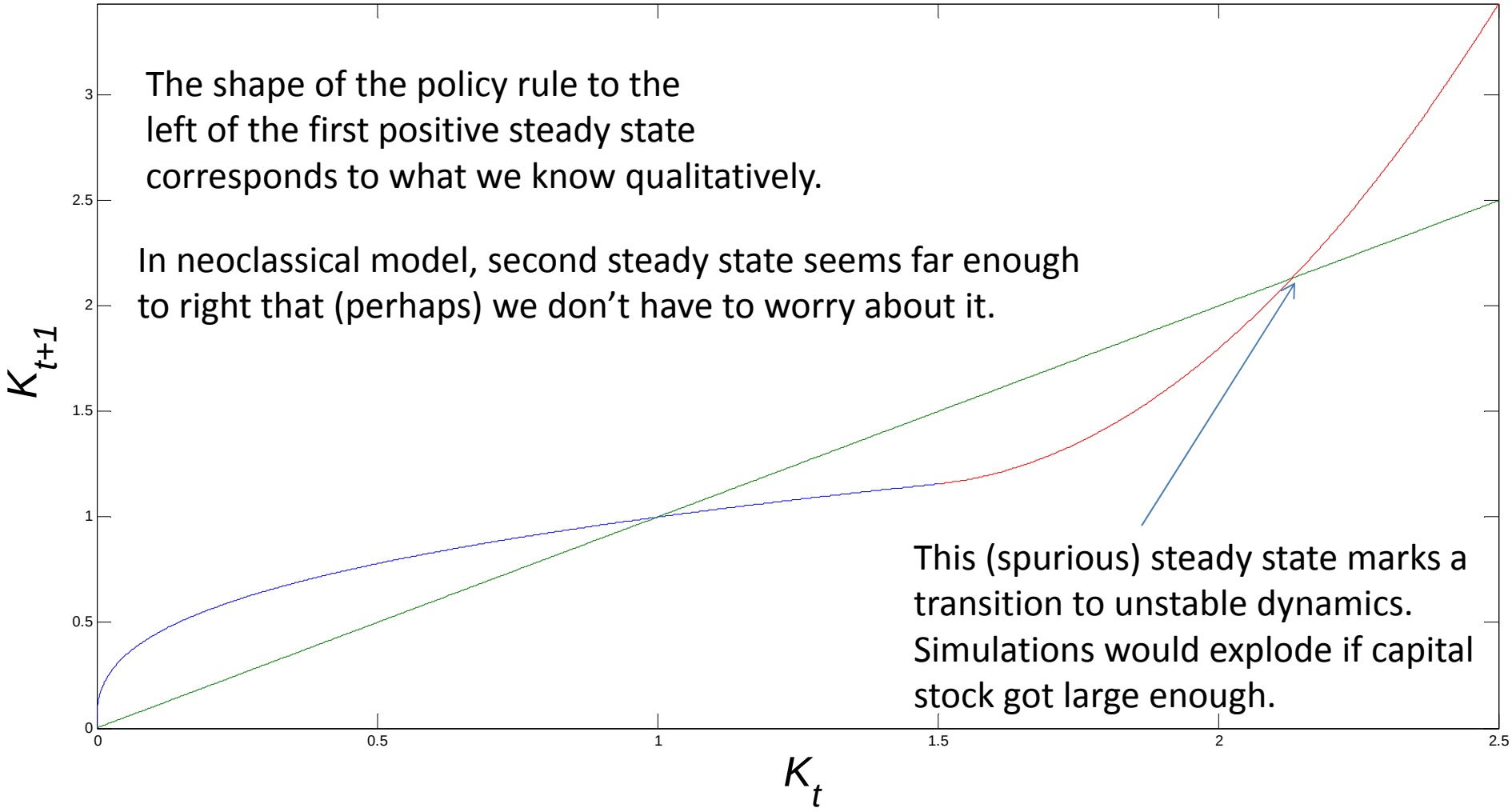
$$y = \frac{2(1-g_k)}{g_{kk}} = \frac{2(1-0.98)}{0.014} = 2.86$$

- This corresponds to the following value of the capital stock:

$$y = k - k^* = \log(K/K^*)$$

$$K = \exp(y + k^*) \overbrace{\approx}^{\text{because } g_{\sigma\sigma} \text{ ignored}} \exp(2.86 + 3.9) = 790.3 \text{ (after rounding)}$$

Stylized Representation of 2nd Order Approximation of Policy Rule



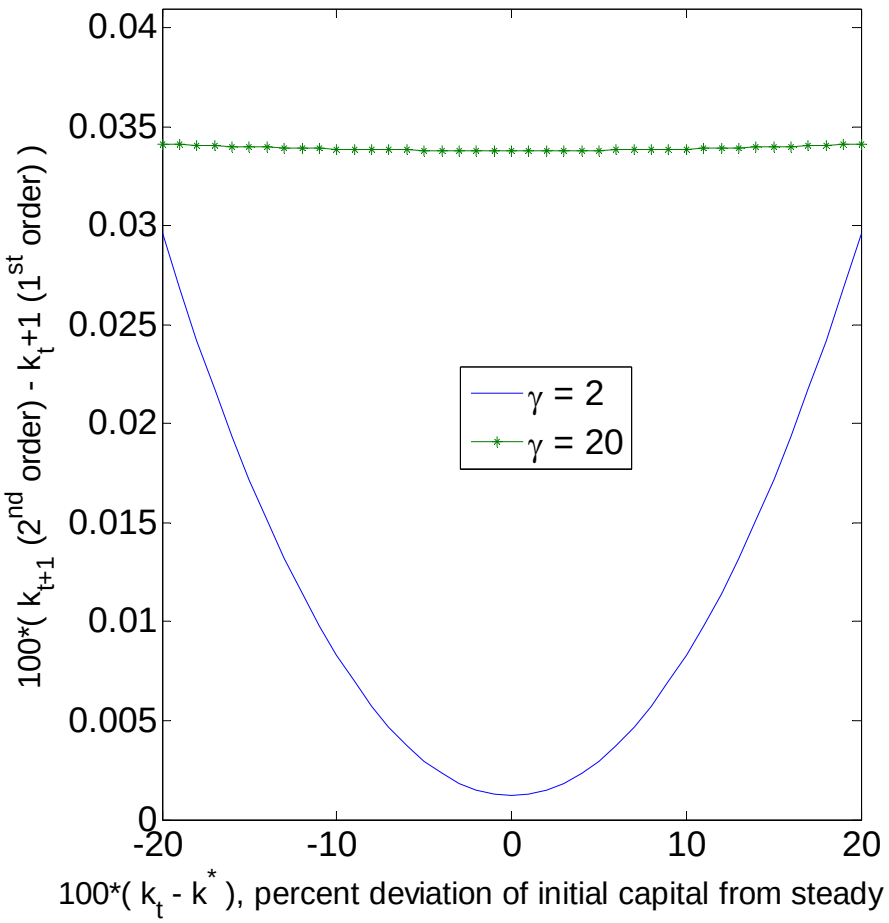
Comparing 1st and 2nd Order Approximation

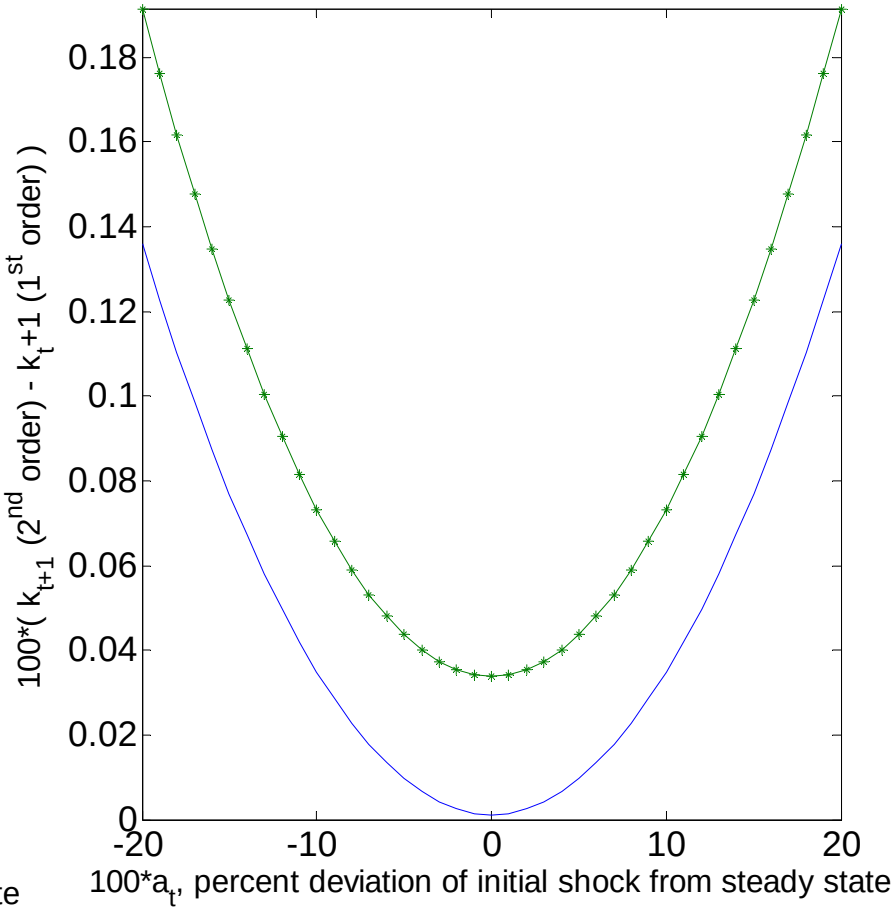
- Following is a graph that compares the policy rules implied by the first and second order perturbation.

- The graph itself corresponds to the baseline parameterization, and results are reported in parentheses for risk aversion equal to 20.

'If initial capital is 20 percent away from steady state, then capital choice differs by 0.03 (0.035) percent between the two approximations.'

'If shock is 6 standard deviations away from its mean, then capital choice differs by 0.14 (0.18) percent between the two approximations'





Number in parentheses at top correspond to $\gamma = 20$.

Impulse Response Functions, Simulations, Pruning

- Impulse response function:
  - Impact of a shock on expectation of future variables.

$$E[y_{t+j}|\Omega_{t-1}, \text{shock}_t \neq 0] - E[y_{t+j}|\Omega_{t-1}, \text{shock}_t = 0],\ j = 0, 1, 2, \ldots$$

  - Impulse responses are useful for building intuition about the economic properties of a model.

  - Can also used for model estimation, if you have the empirical analogs from VAR analysis.

Impulse Response Function, cnt'd

- Example:

$$y_t = \overbrace{\rho y_{t-1}}^{\Omega_{t-1}} + \varepsilon_t$$

- Obviously:

$$\overbrace{E[y_t|\Omega_{t-1}, \varepsilon_t \neq 0]}^{\rho y_{t-1}+\varepsilon_t} - \overbrace{E[y_t|\Omega_{t-1}, \varepsilon_t = 0]}^{\rho y_{t-1}} = \varepsilon_t$$

- Also

$$\begin{aligned} y_{t+1} &= \rho y_t + \varepsilon_{t+1} \\ &= \rho^2 y_{t-1} + \varepsilon_{t+1} + \rho\varepsilon_t \end{aligned}$$

- So that:

$$E[y_{t+1}|\Omega_{t-1}, \varepsilon_t] = \rho^2 y_{t-1} + \rho\varepsilon_t,\ E[y_{t+1}|\Omega_{t-1}, \varepsilon_t = 0] = \rho^2 y_{t-1}$$

$$\rightarrow E[y_{t+1}|\Omega_{t-1}, \varepsilon_t] - E[y_{t+1}|\Omega_{t-1}, \varepsilon_t = 0] = \rho\varepsilon_t$$

- In general:

$$E[y_{t+j}|\Omega_{t-1}, \varepsilon_t \neq 0] - E[y_{t+j}|\Omega_{t-1}, \varepsilon_t = 0] = \rho^j \varepsilon_t$$

Impulse Responses, cnt'd

- Easy in the linear system!
  - Impulse responses not even a function of Ω_{t-1}
- Different story in our 2nd order approximation, especially because of the spurious steady state..
- Example:

Same form as our 2nd order approximation

$$y_t = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t$$

- Obviously:

Easy…

$$\overbrace{E[y_t|\Omega_{t-1}, \varepsilon_t]}^{\overbrace{\rho y_{t-1}+\alpha y_{t-1}^2+\varepsilon_t}^{\Omega_{t-1}}} - \overbrace{E[y_t|\Omega_{t-1}, \varepsilon_t = 0]}^{\rho y_{t-1}+\alpha y_{t-1}^2} = \varepsilon_t$$

IRF's, cnt'd

- Too hard to compute IRF's by analytic formulas, when equations are not linear.

$$y_t = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t$$

- What we need:
  - Fix a value for $\Omega_{t-1} = \underbrace{\rho y_{t-1} + \alpha y_{t-1}^2}_{\text{our example}}$
  - Compute:

$$E[y_{t+j}|\Omega_{t-1}, \varepsilon_t],\ j = 1, 2, 3, \ldots, T,\ \text{for a given value of } \varepsilon_t > 0$$

$$E[y_{t+j}|\Omega_{t-1}, \varepsilon_t = 0],\ j = 1, 2, 3, \ldots, T.$$

  - Subtract:

$$E[y_{t+j}|\Omega_{t-1}, \varepsilon_t] - E[y_{t+j}|\Omega_{t-1}, \varepsilon_t = 0],\ j = 1, 2, 3, \ldots, T$$

IRF's, cnt'd

- Computational strategy
  - From a random number generator, draw:

$$\varepsilon_{t+1}^{(1)}, \varepsilon_{t+2}^{(1)}, \ldots\ldots, \varepsilon_{t+T}^{(1)}$$

  - Using the stochastic equation, $\rho y_{t-1} + \alpha y_{t-1}^2$ and the given ε_t compute

$$y_{t+1}^{(1)}, y_{t+2}^{(1)}, \ldots, y_{t+T}^{(1)}$$

  - Repeat this, over and over again, *R* (big) times, to obtain

$$y_{t+1}^{(1)}, y_{t+2}^{(1)}, \ldots, y_{t+T}^{(1)}$$

$$\ldots$$

$$y_{t+1}^{(R)}, y_{t+2}^{(R)}, \ldots, y_{t+T}^{(R)}$$

  - Finally,

$$E[y_{t+j} | \Omega_{t-1}, \varepsilon_t] = \frac{1}{R} \sum_{l=1}^{R} y_{t+j}^{(l)}, \; j = 1, 2, \ldots, T$$

IRF's, cnt'd

- To get $E[y_{t+j}|\Omega_{t-1}, \varepsilon_t = 0]$, $j = 1, 2, 3, \ldots, T$, just repeat the preceding calculations, except set $\varepsilon_t = 0$

- To do the previous calculations, need *R* and *T*.
  - Dynare will do these calculations.
  - In the stoch_simul command,
    - *R* is set by including the argument, *replic=R*.
    - *T* is set by including *irf=T*.

A Different Type of Impulse Response Function

- The previous concept of an impulse response function required specifying the information set, Ω_{t-1} .
  - How to specify this is not often discussed…in part because with linear solutions it is irrelevant.
  - With nonlinear solutions, Ω_{t-1} makes a difference.
  - How to choose Ω_{t-1} ?
  - One possibility: nonstochastic steady state.
  - Another possibility: stochastic mean.

A Different IRF

- Note that

$$E[y_{t+j}|\Omega_{t-1}, \varepsilon_t] - E[y_{t+j}|\Omega_{t-1}, \varepsilon_t = 0],\ j = 1, 2, 3, \ldots, T$$

  - is a function of Ω_{t-1}.
  - Evaluate the IRF at the mean of Ω_{t-1} as follows
    - Suppose there is date *0*, date *t* and date *T*, where *T>t* and *t* is itself large.
    - Draw R sets of shocks (no need to draw ε_t)

$$\varepsilon_0^{(1)}, \varepsilon_1^{(1)}, \ldots, \varepsilon_{t-1}^{(1)}, \varepsilon_{t+1}^{(1)}, \varepsilon_{t+2}^{(1)}, \ldots\ldots, \varepsilon_{t+T}^{(1)}$$

$$\ldots$$

$$\varepsilon_0^{(R)}, \varepsilon_1^{(R)}, \ldots, \varepsilon_{t-1}^{(R)}, \varepsilon_{t+1}^{(R)}, \varepsilon_{t+2}^{(R)}, \ldots\ldots, \varepsilon_{t+T}^{(R)}$$

A Different IRF

- Using $\varepsilon_t \neq 0,\ \varepsilon_t = 0$ together with

$$\varepsilon_0^{(1)}, \varepsilon_1^{(1)}, \ldots, \varepsilon_{t-1}^{(1)}, \varepsilon_{t+1}^{(1)}, \varepsilon_{t+2}^{(1)}, \ldots\ldots, \varepsilon_{t+T}^{(1)}$$

$$\ldots$$

$$\varepsilon_0^{(R)}, \varepsilon_1^{(R)}, \ldots, \varepsilon_{t-1}^{(R)}, \varepsilon_{t+1}^{(R)}, \varepsilon_{t+2}^{(R)}, \ldots\ldots, \varepsilon_{t+T}^{(R)}$$

- Compute *two* sets:

$$y_0^{(1)}, y_1^{(1)}, \ldots, y_{t-1}^{(1)}, y_t^{(1)}, y_{t+1}^{(1)}, y_{t+2}^{(1)}, \ldots, y_{t+T}^{(1)}$$

$$\ldots$$

$$y_0^{(R)}, y_1^{(R)}, \ldots, y_{t-1}^{(R)}, y_t^{(R)}, y_{t+1}^{(R)}, y_{t+2}^{(R)}, \ldots, y_{t+T}^{(R)}$$

- The period *t+j* IRF is computed by averaging across *l=1,…,R*, for given *t+j, j=0,1,…,T*. Then, subtract, as before.
- In Dynare, *t* is set with *drop=t* parameter in stoch_simul command.

Simulations

- The computation of impulse responses, when the solution is nonlinear, involves simulations.

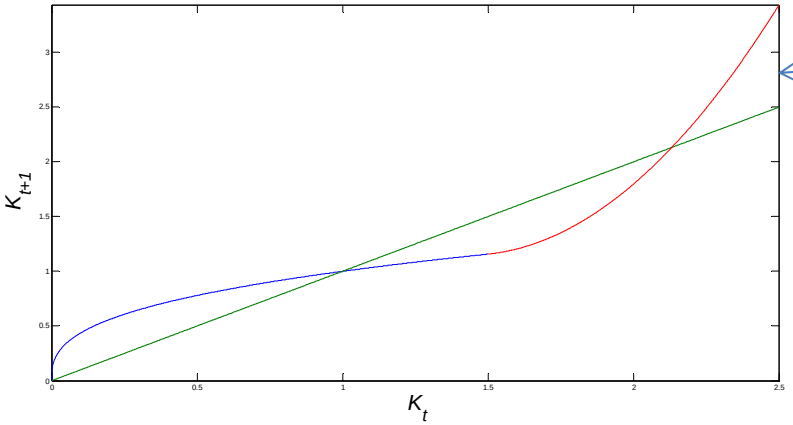
- The simulations can be a basis for computing second moments and other statistics.

- The simulated data can be compared with actual data, to evaluate the empirical fit of the model.

Pruning

- All these simulations must confront a potentially major problem.
  - The additional spurious steady states introduced by 2nd order approximations introduce the possibility of explosive behavior.

Actual capital



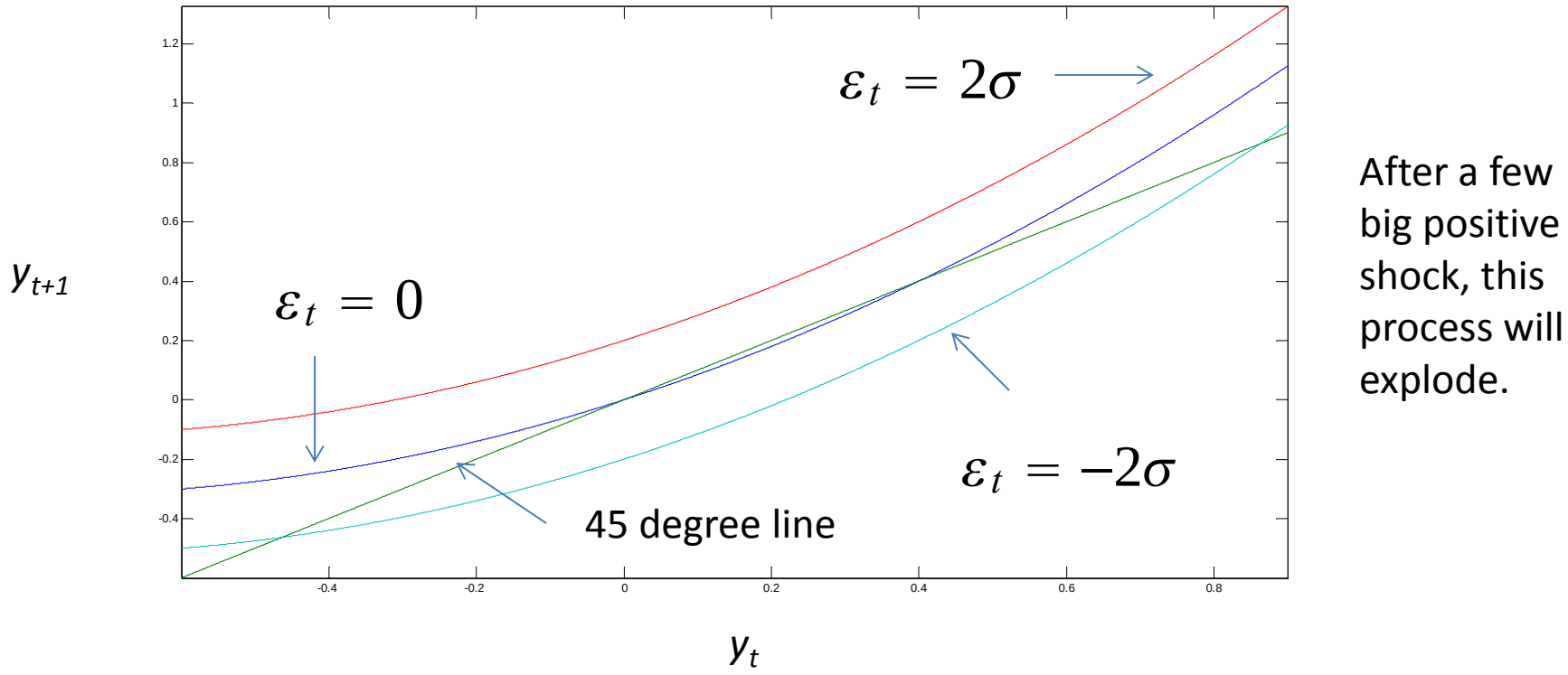
Convex, so once you get into this region, you really blow up, generating NaN and Inf in MATLAB.

  - Although we have seen that this explosiveness is not so likely in neoclassical model, that model is nevertheless useful for thinking about explosiveness and spurious steady states.
    - Quite likely in larger sized model, even for USA-sized shocks.

Pruning, cnt'd

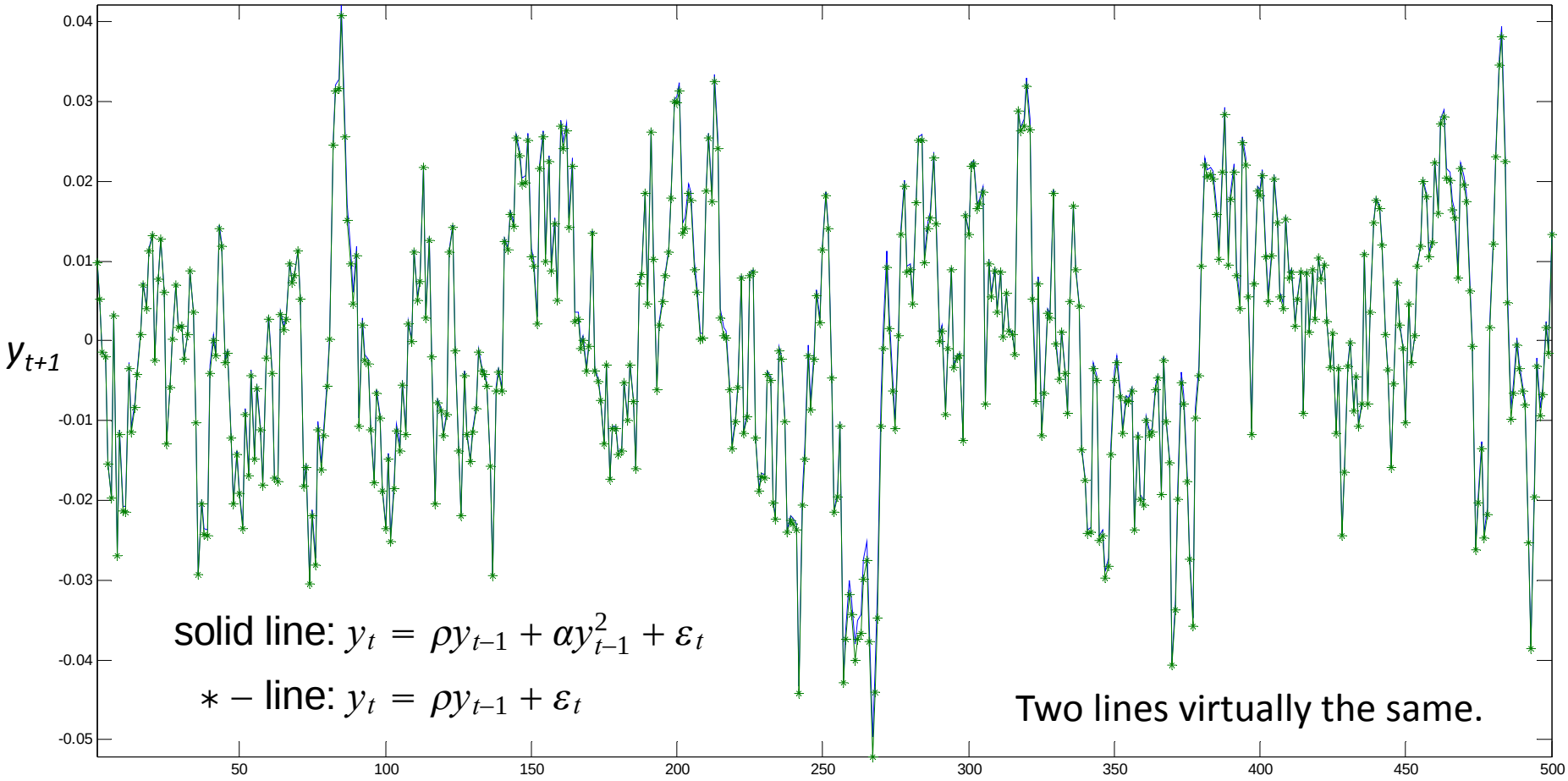
- Example (closely related to neoclassical model solution):

$$y_t = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t,\ \rho = 0.8,\ \alpha = 0.5,\ E\varepsilon_t^2 = \sigma,\ \sigma = 0.10$$

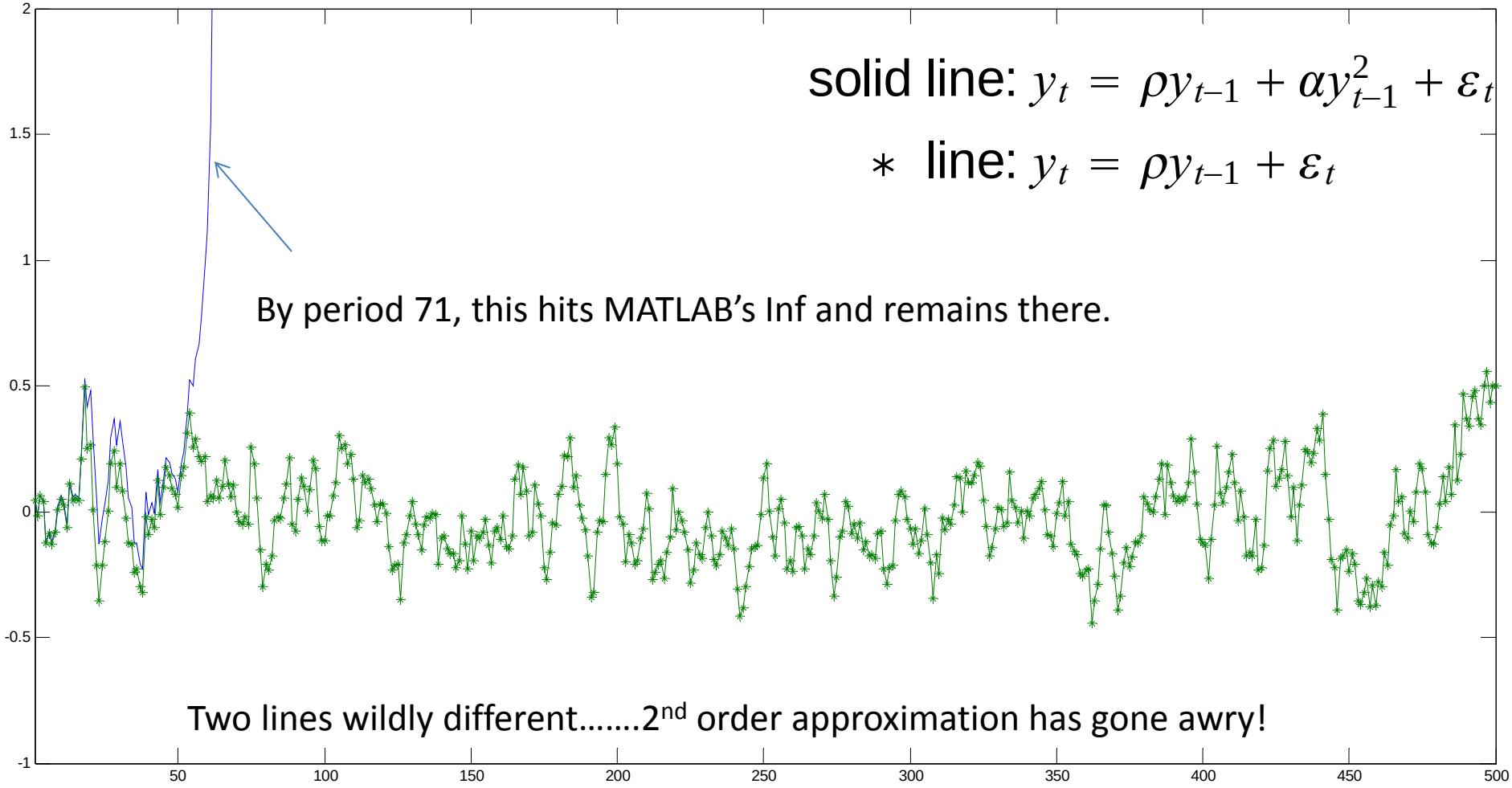


After a few big positive shock, this process will explode.

Simulations, std dev = σ/10



Simulations , std dev = σ



Pruning, cnt'd

- Must do something about the explosive behavior.
- Dynare and others take the approach described in :
  - Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher A. Sims, 2008, "Calculating and using second-order accurate solutions of discrete time dynamic equilibrium models," *Journal of Economic Dynamics and Control*, 32(11), 3397–3414.

Pruning

- This is a procedure for simulating the nonlinear difference equation, that avoids the explosions.

- First, draw a sequence, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T$

- Next, solve for $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_T$ in the linear component of the process:

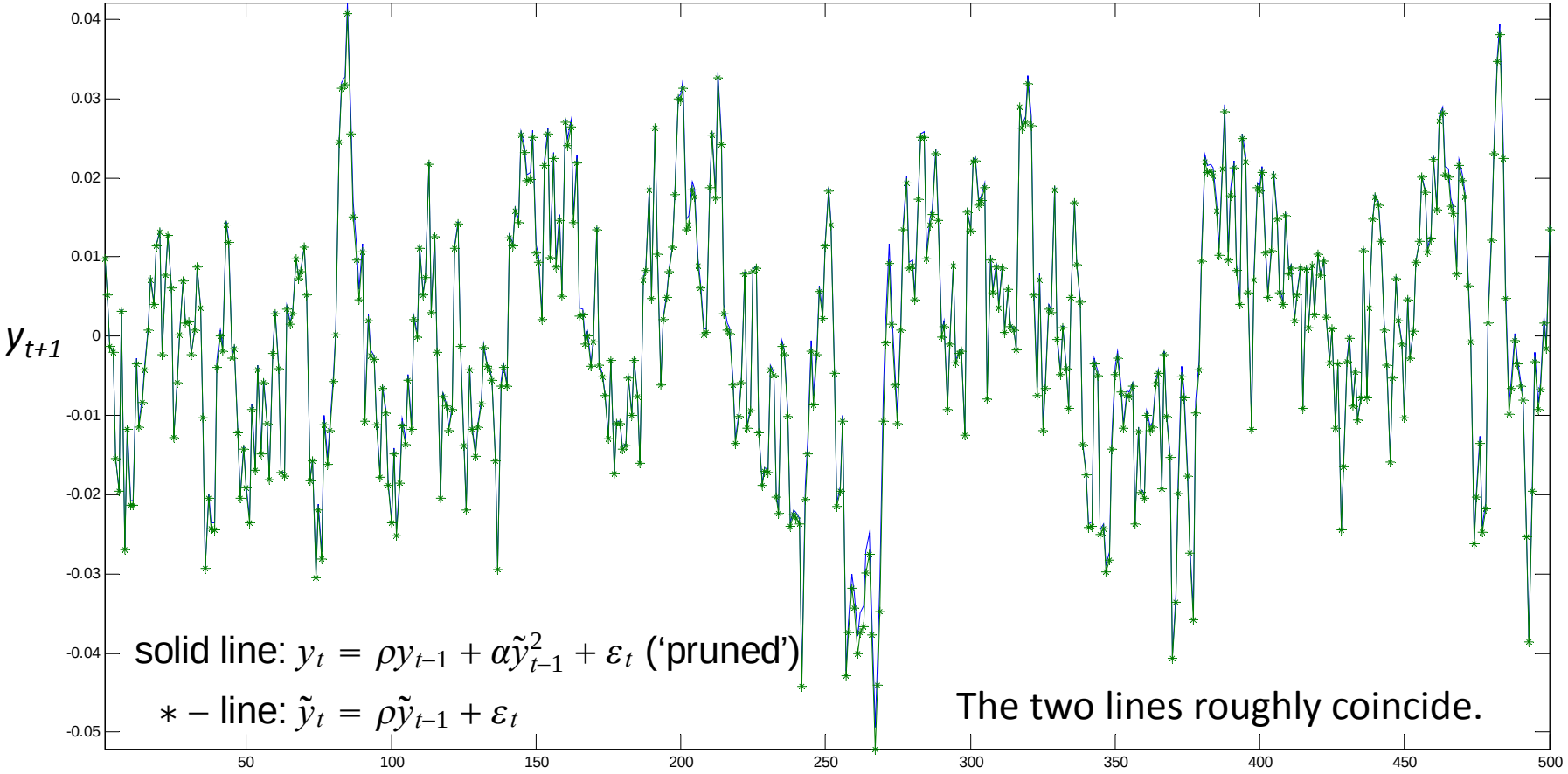
$$\tilde{y}_t = \rho \tilde{y}_{t-1} + \varepsilon_t$$

- The ('pruned') solution to the 2nd order difference equation is $y_1, y_2, \ldots, y_T$ in

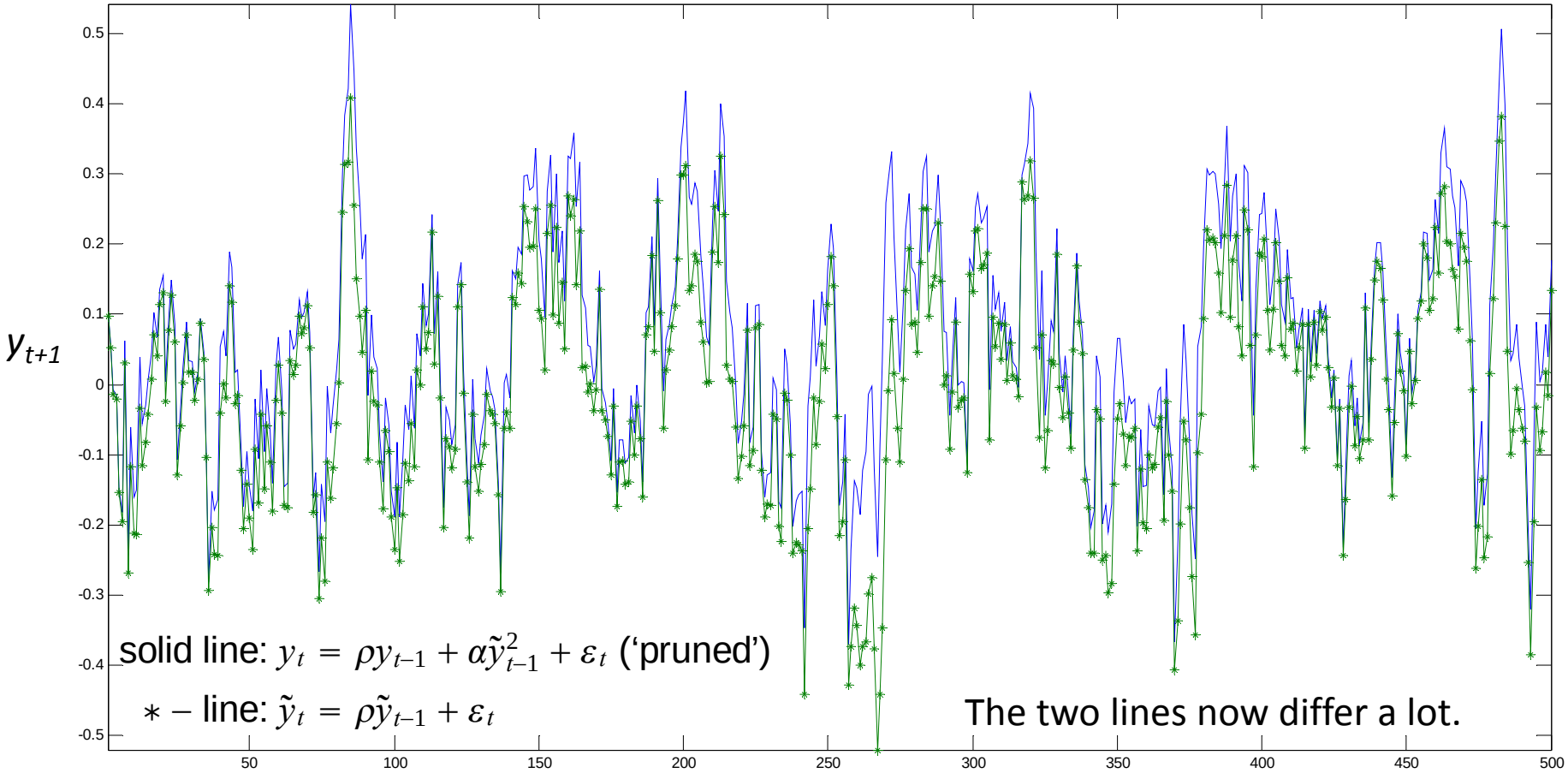
$$y_t = \rho y_{t-1} + \alpha \tilde{y}_{t-1}^2 + \varepsilon_t$$

- Note that this cannot explode.

Simulations, std dev = σ/10



Simulations, std dev = σ



Conclusion

- For modest US-sized fluctuations and for aggregate quantities, it may be reasonable to work with first order perturbations.
  - This assumption deserves much further testing.
  - Can do this by studying the error function.
  - Also, try fancier approximations and see if it changes your results.

- One alternative to first order perturbations is higher order perturbations.
  - These must be handled with care, as they are characterized by spurious steady states, which may be the transition point to unstable dynamics.
  - Must do some sort of pruning to compute IRF's, or just to simulate data.

- An alternative is to apply projection methods.
  - Perhaps these have less problems with spurious steady states.
  - Computation of solutions is more cumbersome in this case.

- First order perturbation: linearize (or, log-linearize) equilibrium conditions around non-stochastic steady state and solve the resulting system.
  - This approach assumes 'certainty equivalence'. Ok, as a first order approximation.

Solution by Linearization

List of endogenous variables determined at *t*

- (log) Linearized Equilibrium Conditions:

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

- Posit Linear Solution:

$$z_t = A z_{t-1} + B s_t$$

$$s_t - P s_{t-1} - \epsilon_t = 0.$$

Exogenous shocks

- To satisfy equil conditions, *A* and *B* must:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \quad F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

- If there is exactly one *A* with eigenvalues less than unity in absolute value, that's the solution. Otherwise, multiple solutions.

- Conditional on *A*, solve linear system for *B*.