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Solving Dynamic General Equilibrium Models Using Log Linear Approximation

Log-linearization strategy

- Example #1: A Simple RBC Model.
  - Define a Model 'Solution'
  - Motivate the Need to Somehow Approximate Model Solutions
  - Describe Basic Idea Behind Log Linear Approximations
  - Some Strange Examples to be Prepared For

    'Blanchard-Kahn conditions not satisfied'
- Example #2: Bringing in uncertainty.
- Example #3: Stochastic RBC Model with Hours Worked (Matrix Generalization of Previous Results)

Example #1: Nonstochastic RBC Model

$$\text{Maximize}_{\{c_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to:

$$C_t + K_{t+1} - (1-\delta)K_t \; = \; K_t^{\alpha}, \; K_0 \text{ given}$$

First order condition:

$$C_t^{-\sigma} - \beta C_{t+1}^{-\sigma}\left[\alpha K_{t+1}^{\alpha-1} + (1-\delta)\right],$$

or, after substituting out resource constraint:

$$v(K_t, K_{t+1}, K_{t+2}) \; = \; 0, \; t = 0, 1, ..., \text{ with } K_0 \text{ given.}$$

Example #1: Nonstochastic RBC Model ...

- 'Solution': a function, $K_{t+1} = g(K_t)$, such that

$$v(K_t, g(K_t), g[g(K_t)]) \;=\; 0, \text{ for all } K_t.$$

- Problem:

This is an Infinite Number of Equations
(one for each possible K_t)
in an Infinite Number of Unknowns
(a value for g for each possible K_t)

- With Only a Few Rare Exceptions this is Very Hard to Solve Exactly
  - Easy cases:
    - If $\sigma = 1, \delta = 1 \Rightarrow g(K_t) = \alpha\beta K_t^{\alpha}$.
    - If v is linear in K_t, K_{t+1}, K_{t+1}.
  - Standard Approach: Approximate v by a Log Linear Function.

Approximation Method Based on Linearization

- Three Steps
  - Compute the Steady State
  - Do a Log Linear Expansion About Steady State
  - Solve the Resulting Log Linearized System
- Step 1: Compute Steady State -
  - Steady State Value of K, K^* -

$$
\begin{aligned}
C^{-\sigma} - \beta C^{-\sigma}\left[\alpha K^{\alpha-1} + (1-\delta)\right] &= 0, \\
\Rightarrow\ \alpha K^{\alpha-1} + (1-\delta) &= \frac{1}{\beta} \\
\Rightarrow\ K^* &= \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

  - K^* satisfies:

$$v(K^*, K^*, K^*) = 0.$$

- Step 2:
  - Replace v by First Order Taylor Series Expansion About Steady State:

$$v_1(K_t - K^*) + v_2(K_{t+1} - K^*) + v_3(K_{t+2} - K^*) \; = \; 0,$$

  - Here,

$$v_1 \; = \; \frac{dv_u(K_t, K_{t+1}, K_{t+2})}{dK_t}, \text{ at } K_t = K_{t+1} = K_{t+2} = K^*.$$

  - Conventionally, do *Log-Linear Approximation*:

$$\begin{aligned}(v_1 K)\,\hat{K}_t + (v_2 K)\,\hat{K}_{t+1} + (v_3 K)\,\hat{K}_{t+2} \; &= \; 0, \\ \hat{K}_t \equiv \frac{K_t - K^*}{K^*}&.\end{aligned}$$

  - Write this as:

$$\alpha_2 \hat{K}_t + \alpha_1 \hat{K}_{t+1} + \alpha_0 \hat{K}_{t+2} \; = \; 0,$$
$$\alpha_2 = v_1 K, \; \alpha_1 = v_2 K, \; \alpha_0 = v_3 K$$

- Step 3: Solve
  - Posit the Following Policy Rule:

    $$\hat{K}_{t+1} = A\hat{K}_t,$$

    Where A is to be Determined.
  - Compute A :

    $$\alpha_2\hat{K}_t + \alpha_1 A\hat{K}_t + \alpha_0 A^2\hat{K}_t = 0,$$

    or

    $$\alpha_2 + \alpha_1 A + \alpha_0 A^2 = 0.$$
  - A is the Eigenvalue of Polynomial
- In General: Two Eigenvalues.
  - Can Show: In RBC Example, One Eigenvalue is Explosive. The Other Not.
  - There Exist Theorems (see Stokey-Lucas, chap. 6) That Say You Should Ignore the Explosive A.

Some Strange Examples to be Prepared For

- Other Examples Are Possible:
  - Both Eigenvalues Explosive
  - Both Eigenvalues Non-Explosive

Example #2: RBC Model With Uncertainty

- Model

$$\text{Maximize } E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma},$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t \; = \; K_t^{\alpha}\varepsilon_t,$$

where ε_t is a stochastic process with $E\varepsilon_t = \varepsilon$, say. Let

$$\hat{\varepsilon}_t \; = \; \frac{\varepsilon_t - \varepsilon}{\varepsilon},$$

and suppose

$$\hat{\varepsilon}_t \; = \; \rho\hat{\varepsilon}_{t-1} + e_t, \; e_t \tilde{\ } N(0, \sigma_e^2).$$

- First Order Condition:

$$E_t \left\{ C_t^{-\sigma} - \beta C_{t+1}^{-\sigma} \left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta \right] \right\} = 0.$$

Example #2: RBC Model With Uncertainty ...

- First Order Condition:

$$E_t v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0,$$

where

$$\begin{aligned} & v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\ = \ & (K_t^{\alpha}\varepsilon_t + (1-\delta)K_t - K_{t+1})^{-\sigma} \\ & -\beta\left(K_{t+1}^{\alpha}\varepsilon_{t+1} + (1-\delta)K_{t+1} - K_{t+2}\right)^{-\sigma} \\ & \times \left[\alpha K_{t+1}^{\alpha-1}\varepsilon_{t+1} + 1 - \delta\right]. \end{aligned}$$

- Solution: a $g(K_t, \varepsilon_t)$, Such That

$$E_t v\left(g(g(K_t,\varepsilon_t),\varepsilon_{t+1}), g(K_t,\varepsilon_t), K_t, \varepsilon_{t+1}, \varepsilon_t\right) \ = \ 0,$$

For All K_t, ε_t.

- Hard to Find g, Except in Special Cases
  - One Special Case: v is Log Linear.

- Log Linearization Strategy:
  - Step 1: Compute Steady State of K_t when ε_t is Replaced by $E\varepsilon_t$
  - Step2: Replace v By its Taylor Series Expansion About Steady State.
  - Step 3: Solve Resulting Log Linearized System.
- Logic: If Actual Stochastic System Remains in a Neighborhood of Steady State, Log Linear Approximation Good

- Caveat: Strategy not accurate in all conceivable situations.
  - Example: suppose that where I live -

$$\varepsilon \equiv \text{temperature} = \begin{cases} \text{140 Fahrenheit, 50 percent of time} \\ \text{0 degrees Fahrenheit the other half} \end{cases}.$$

  - On average, temperature quire nice: $E\varepsilon = 70$ (like parts of California)

  - Let $K =$ capital invested in heating and airconditioning

    * EK *very*, *very* large!
    * Economist who predicts investment based on replacing ε by $E\varepsilon$ would predict $K = 0$ (as in many parts of California)

  - In standard model this is not a big problem, because shocks are not so big....steady state value of K (i.e., the value that results eventually when ε is replaced by $E\varepsilon$) is approximately $E\varepsilon$ (i.e., the average value of K when ε is stochastic).

- Step 1: Steady State:

$$K^* = \left[\frac{\alpha \varepsilon}{\frac{1}{\beta} - (1-\delta)} \right]^{\frac{1}{1-\alpha}}.$$

- Step 2: Log Linearize -

$$\begin{aligned}
& v(K_{t+2}, K_{t+1}, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
\simeq\ & v_1 (K_{t+2} - K^*) + v_2 (K_{t+1} - K^*) + v_3 (K_t - K^*) \\
& + v_3 (\varepsilon_{t+1} - \varepsilon) + v_4 (\varepsilon_t - \varepsilon) \\
=\ & v_1 K^* \left(\frac{K_{t+2} - K^*}{K^*} \right) + v_2 K^* \left(\frac{K_{t+1} - K^*}{K^*} \right) + v_3 K^* \left(\frac{K_t - K^*}{K^*} \right) \\
& + v_3 \varepsilon \left(\frac{\varepsilon_{t+1} - \varepsilon}{\varepsilon} \right) + v_4 \varepsilon \left(\frac{\varepsilon_t - \varepsilon}{\varepsilon} \right) \\
=\ & \alpha_0 \hat{K}_{t+2} + \alpha_1 \hat{K}_{t+1} + \alpha_2 \hat{K}_t + \beta_0 \hat{\varepsilon}_{t+1} + \beta_1 \hat{\varepsilon}_t.
\end{aligned}$$

- Step 3: Solve Log Linearized System
  - Posit:
$$\hat{K}_{t+1} = A\hat{K}_t + B\hat{\varepsilon}_t.$$
  - Pin Down A and B By Condition that log-linearized Euler Equation Must Be Satisfied.
    * Note:
$$\begin{aligned} \hat{K}_{t+2} &= A\hat{K}_{t+1} + B\hat{\varepsilon}_{t+1} \\ &= A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}. \end{aligned}$$
    * Substitute Posited Policy Rule into Log Linearized Euler Equation:
$$E_t\left[\alpha_0\hat{K}_{t+2} + \alpha_1\hat{K}_{t+1} + \alpha_2\hat{K}_t + \beta_0\hat{\varepsilon}_{t+1} + \beta_1\hat{\varepsilon}_t\right] = 0,$$
so must have:
$$\begin{aligned} &E_t\{\alpha_0\left[A^2\hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}\right] \\ &+\alpha_1\left[A\hat{K}_t + B\hat{\varepsilon}_t\right] + \alpha_2\hat{K}_t + \beta_0\rho\hat{\varepsilon}_t + \beta_0 e_{t+1} + \beta_1\hat{\varepsilon}_t\} = 0 \end{aligned}$$

∗ Then,

$$
\begin{aligned}
& E_t\left[\alpha_0 \hat{K}_{t+2} + \alpha_1 \hat{K}_{t+1} + \alpha_2 \hat{K}_t + \beta_0 \hat{\varepsilon}_{t+1} + \beta_1 \hat{\varepsilon}_t\right] \\
& = E_t\{\alpha_0\left[A^2 \hat{K}_t + AB\hat{\varepsilon}_t + B\rho\hat{\varepsilon}_t + Be_{t+1}\right] \\
& +\alpha_1\left[A\hat{K}_t + B\hat{\varepsilon}_t\right] + \alpha_2 \hat{K}_t + \beta_0 \rho\hat{\varepsilon}_t + \beta_0 e_{t+1} + \beta_1 \hat{\varepsilon}_t\} \\
& = \alpha(A)\hat{K}_t + F\hat{\varepsilon}_t \\
& = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(A) &= \alpha_0 A^2 + \alpha_1 A + \alpha_2, \\
F &= \alpha_0 AB + \alpha_0 B\rho + \alpha_1 B + \beta_0\rho + \beta_1
\end{aligned}
$$

∗ Find A and B that Satisfy:

$$\alpha(A) = 0,\ F = 0.$$

Example #3 RBC Model With Hours Worked and Uncertainty

- Maximize

$$E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to

$$C_t + K_{t+1} - (1-\delta)K_t = f(K_t, N_t, \varepsilon_t)$$

and

$$E\varepsilon_t = \varepsilon,$$

$$\hat{\varepsilon}_t = \rho\hat{\varepsilon}_{t-1} + e_t, \; e_t \tilde{} N(0, \sigma_e^2)$$

$$\hat{\varepsilon}_t = \frac{\varepsilon_t - \varepsilon}{\varepsilon}.$$

- First Order Conditions:

$$E_t v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) = 0$$

and

$$v_N(K_{t+1}, N_t, K_t, \varepsilon_t) = 0.$$

where

$$\begin{aligned}
& v_K(K_{t+2}, N_{t+1,} K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
= \; & U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\
& -\beta U_c\left(f(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, N_{t+1}\right) \\
& \times \left[f_K(K_{t+1}, N_{t+1}, \varepsilon_{t+1}) + 1 - \delta\right]
\end{aligned}$$

and,

$$\begin{aligned}
& v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\
= \; & U_N\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\
& +U_c\left(f(K_t, N_t, \varepsilon_t) + (1-\delta)K_t - K_{t+1}, N_t\right) \\
& \times f_N(K_t, N_t, \varepsilon_t).
\end{aligned}$$

- Steady state K^* and N^* such that Equilibrium Conditions Hold with $\varepsilon_t \equiv \varepsilon$.

- Log-Linearize the Equilibrium Conditions:

$$
\begin{aligned}
& v_K(K_{t+2}, N_{t+1,}K_{t+1}, N_t, K_t, \varepsilon_{t+1}, \varepsilon_t) \\
= \; & v_{K,1}K^*\hat{K}_{t+2} + v_{K,2}N^*\hat{N}_{t+1} + v_{K,3}K^*\hat{K}_{t+1} + v_{K,4}N^*\hat{N}_t + v_{K,5}K^*\hat{K}_t \\
& + v_{K,6}\varepsilon\hat{\varepsilon}_{t+1} + v_{K,7}\varepsilon\hat{\varepsilon}_t
\end{aligned}
$$

$v_{K,j}$ ~ Derivative of v_K with respect to j^{th} argument, evaluated in steady state.

$$
\begin{aligned}
& v_N(K_{t+1}, N_t, K_t, \varepsilon_t) \\
= \; & v_{N,1}K^*\hat{K}_{t+1} + v_{N,2}N^*\hat{N}_t + v_{N,3}K^*\hat{K}_t + v_{N,4}\varepsilon\hat{\varepsilon}_{t+1}
\end{aligned}
$$

$v_{N,j}$ ~ Derivative of v_N with respect to j^{th} argument, evaluated in steady state.

- Representation Log-linearized Equilibrium Conditions
  - Let
$$z_t = \begin{pmatrix} \hat{K}_{t+1} \\ \hat{N}_t \end{pmatrix},\ s_t = \hat{\varepsilon}_t,\ \epsilon_t = e_t.$$
  - Then, the linearized Euler equation is:
$$\begin{aligned} E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] &= 0, \\ s_t &= P s_{t-1} + \epsilon_t,\ \epsilon_t \sim\ N(0, \sigma_e^2),\ P = \rho. \end{aligned}$$
  - Here,
$$\begin{aligned} \alpha_0 &= \begin{bmatrix} \upsilon_{K,1} K^* & \upsilon_{K,2} N^* \\ 0 & 0 \end{bmatrix},\ \alpha_1 = \begin{bmatrix} \upsilon_{K,3} K^* & \upsilon_{K,4} N^* \\ \upsilon_{N,1} K^* & \upsilon_{N,2} N^* \end{bmatrix}, \\ \alpha_2 &= \begin{bmatrix} \upsilon_{K,5} K^* & 0 \\ \upsilon_{N,3} K^* & 0 \end{bmatrix}, \end{aligned}$$
$$\beta_0 = \begin{pmatrix} \upsilon_{K,6}\varepsilon \\ 0 \end{pmatrix},\ \beta_1 = \begin{pmatrix} \upsilon_{K,7}\varepsilon \\ \upsilon_{N,4}\varepsilon \end{pmatrix}.$$
- Previous is a Canonical Representation That Essentially All Log Linearized Models Can be Fit Into (See Christiano (2002).)

- Again, Look for Solution

$$z_t = Az_{t-1} + Bs_t,$$

where A and B are pinned down by log-linearized Equilibrium Conditions.

- Now, A is *Matrix* Eigenvalue of *Matrix* Polynomial:

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0.$$

- Also, B Satisfies Same System of Log Linear Equations as Before:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0.$$

- Go for the 2 Free Elements of B Using 2 Equations Given by

$$F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Finding the Matrix Eigenvalue of the Polynomial Equation,

$$\alpha(A) = 0,$$

  and Determining if A is Unique is a Solved Problem.

- See Anderson, Gary S. and George Moore, 1985, ‘A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models,’ *Economic Letters*, 17, 247-52 or Articles in Computational Economics, October, 2002. See also, the program, DYNARE.

- Solving for B
  - Given A, Solve for B Using Following (Log Linear) System of Equations:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

  - To See this, Use

$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1)\, vec(A_2),$$

to Convert $F = 0$ Into

$$vec(F') = d + q\delta = 0, \; \delta = vec(B').$$

  - Find B By First Solving:

$$\delta = -q^{-1} d.$$