

# Foundations for the New Keynesian Model

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# Objective

- Describe a very simple model economy with no monetary frictions.
  - Describe its properties.
  - ‘markets work well’
- Modify the model to include price setting frictions.
  - Now markets won’t necessarily work so well, unless monetary policy is good.

# Model

- Household preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\},$$

$$\tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iid N(0, \sigma_\varepsilon^2)$$

# Production

- Final output requires lots of intermediate inputs:

$$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \varepsilon > 1$$

- Production of intermediate inputs:

$$Y_{i,t} = e^{a_t} N_{i,t}, \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a, \varepsilon_t^a \sim iid N(0, \sigma_a^2)$$

- Constraint on allocation of labor:

$$\int_0^1 N_{it} di = N_t$$

# Efficient Allocation of Total Labor

- Suppose total labor,  $N_t$ , is fixed.
- What is the best way to allocate  $N_t$  among the various activities,  $0 \leq i \leq 1$ ?
- Answer:
  - allocate labor equally across all the activities

$$N_{it} = N_t, \text{ all } i$$

# Suppose Labor *Not* Allocated Equally

- Example:

$$N_{it} = \begin{cases} 2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\ 2(1 - \alpha)N_t & i \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad 0 \leq \alpha \leq 1.$$

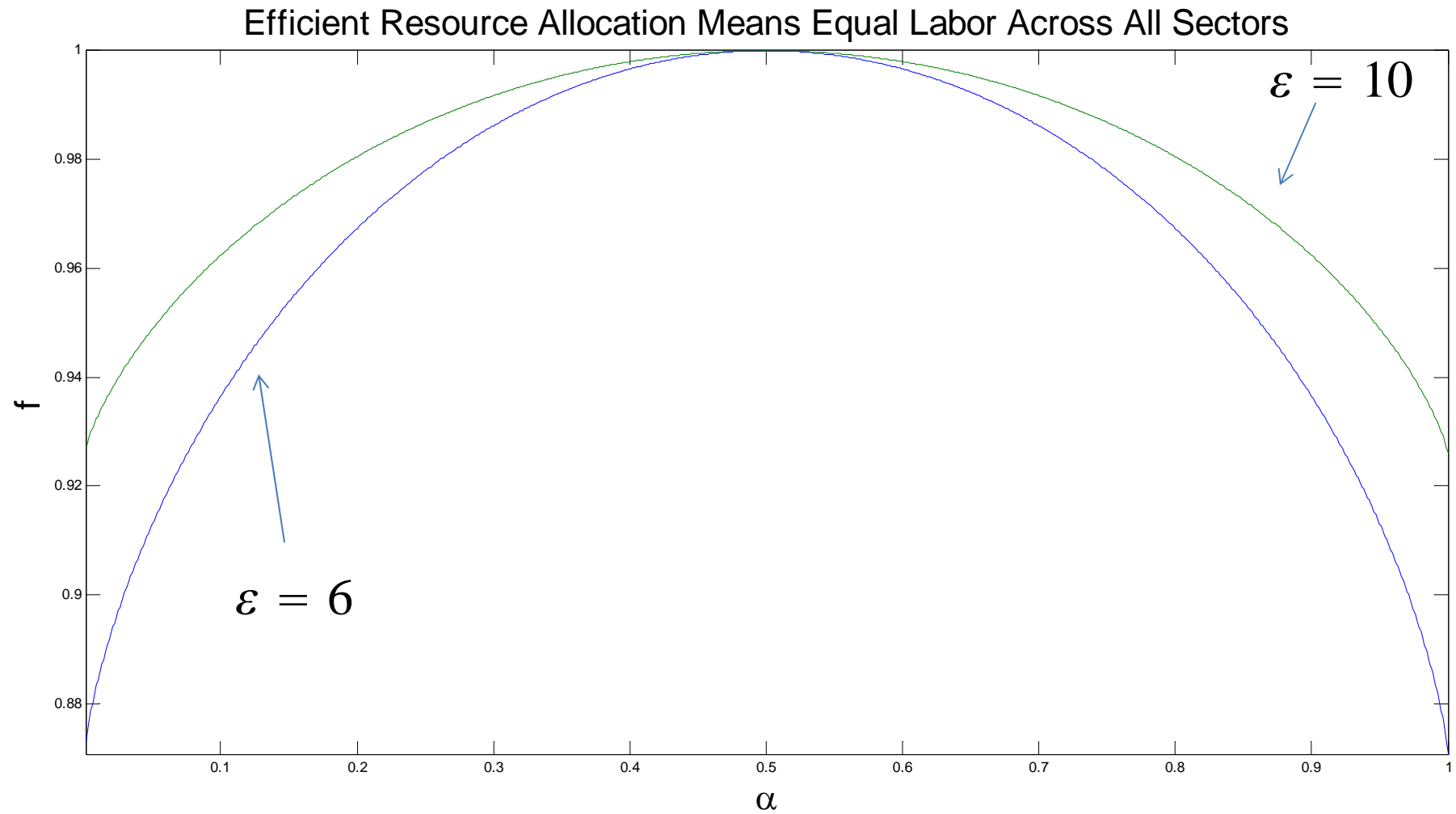
- Note that this is a particular distribution of labor across activities:

$$\int_0^1 N_{it} di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2(1 - \alpha)N_t = N_t$$

# Labor *Not* Allocated Equally, cnt'd

$$\begin{aligned} Y_t &= \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \left[ \int_0^{\frac{1}{2}} Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= e^{a_t} \left[ \int_0^{\frac{1}{2}} N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= e^{a_t} \left[ \int_0^{\frac{1}{2}} (2\alpha N_t)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 (2(1-\alpha)N_t)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= e^{a_t} N_t \left[ \int_0^{\frac{1}{2}} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= e^{a_t} N_t \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= e^{a_t} N_t f(\alpha) \end{aligned}$$

$$f(\alpha) = \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$





# Economy with Efficient $N$ Allocation

- Efficiency dictates

$$N_{it} = N_t \text{ all } i$$

- So, with efficient production:

$$Y_t = e^{a_t} N_t$$

- Resource constraint:

$$C_t \leq Y_t$$

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iid,$$

# Efficient Determination of Labor

- Lagrangian:

$$\max_{C_t, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l} = \log C_t - \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\phi} \\ \underbrace{u(C_t, N_t, \tau_t)} + \lambda_t [e^{a_t} N_t - C_t] \end{array} \right\}$$

- First order conditions:

$$u_c(C_t, N_t, \tau_t) = \lambda_t, \quad u_n(C_t, N_t, \tau_t) + \lambda_t e^{a_t} = 0$$

- or:

$$u_{n,t} + u_{c,t} e^{a_t} = 0$$

marginal cost of labor in consumption units =  $-\frac{du}{dN_t} = \frac{dC_t}{dN_t}$

$$\frac{\overbrace{-u_{n,t}}}{u_{c,t}} = \text{marginal product of labor} \quad \overbrace{e^{a_t}}$$

# Efficient Determination of Labor, cont'd

- Solving the fnc's:

$$\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}$$

$$C_t \exp(\tau_t) N_t^\varphi = e^{a_t}$$

$$e^{a_t} N_t \exp(\tau_t) N_t^\varphi = e^{a_t}$$

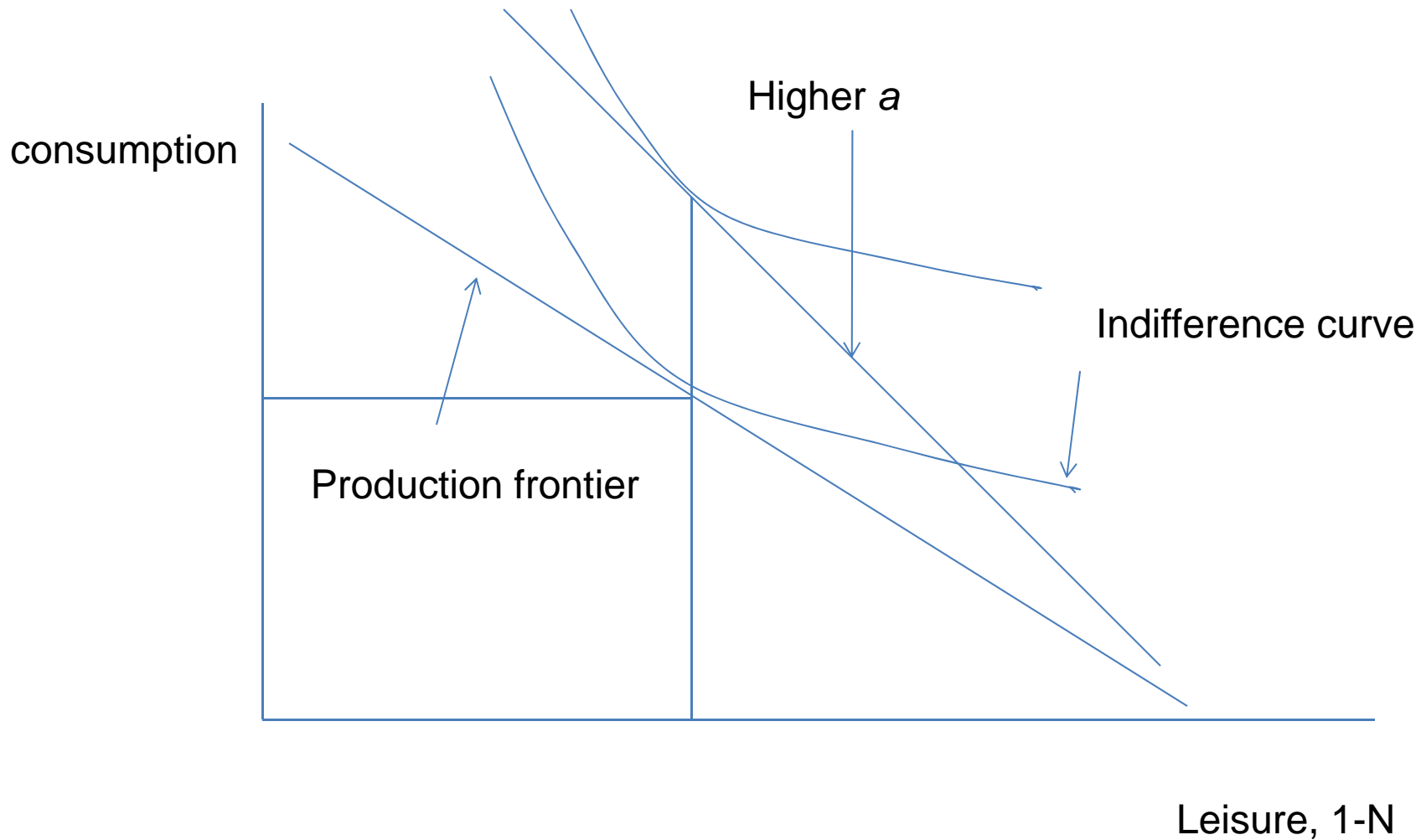
$$\rightarrow N_t = \exp\left(\frac{-\tau_t}{1 + \varphi}\right)$$

$$\rightarrow C_t = \exp\left(a_t - \frac{\tau_t}{1 + \varphi}\right)$$

- Note:

– Labor responds to preference shock, *not* to tech shock

# Response to a Jump in $a$



# Decentralizing the Model

- Give households budget constraints and place them in markets.
- Give the production functions to firms and suppose that they seek to maximize profits.

# Households

- Solve:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\},$$

- Subject to:

$$C_t + \overbrace{B_{t+1}}^{\text{bonds purchases in } t} \leq \overbrace{w_t}^{\text{wage rate}} N_t + \overbrace{\pi_t}^{\text{profits}} + \overbrace{r_{t-1}}^{\text{(real) interest on bonds}} B_t$$

- First order conditions:

$$\frac{-u_{n,t}}{u_{c,t}} = C_t \exp(\tau_t) N_t^\varphi = w_t \quad \text{'marginal cost of working equals marginal benefit'}$$

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \quad \text{'marginal cost of saving equals marginal benefit'}$$

# Final Good Firms

- Final good firms buy  $Y_{i,t}$ ,  $i \in (0, 1)$ , at given prices,  $P_{i,t}$ , to maximize profits:

$$Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Subject to

$$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

- Foc's:

$$P_{i,t} = \left( \frac{Y_t}{Y_{i,t}} \right)^{\frac{1}{\varepsilon}}$$

$$\rightarrow Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t, \quad 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$

# Intermediate Good Firms

- For each  $Y_{i,t}$  there is a single producer who is a monopolist in the product market and hires labor,  $N_{i,t}$  in competitive labor markets.

- Marginal cost of production:

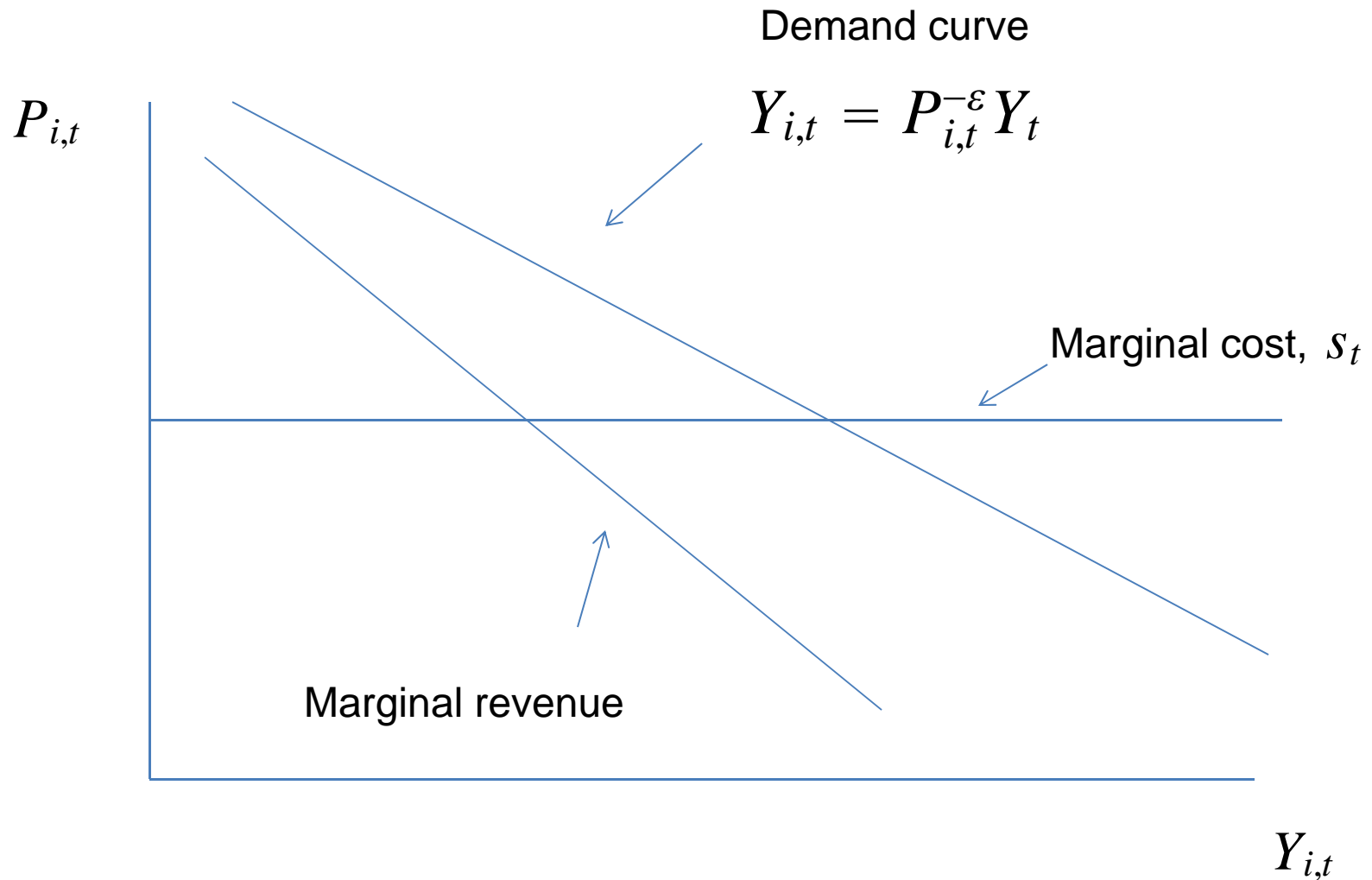
$$\text{(real) marginal cost} = s_t = \frac{\frac{d\text{Cost}}{d\text{worker}}}{\frac{d\text{output}}{d\text{worker}}} = \frac{\left(1 - \underbrace{\quad}_{\nu}\right) w_t}{\exp(a_t)}$$

subsidy payment to firm

- Subsidy will be required to ensure markets work efficiently.



# Intermediate Good Firms



# *ith* Intermediate Good Firm

- Problem:  $\max_{N_{it}} P_{it} Y_{it} - s_t Y_{it}$
- Subject to demand for  $Y_{i,t}$ :  $Y_{i,t} = P_{i,t}^{-\varepsilon} Y_t$

- Problem:

$$\max_{N_{it}} P_{it} P_{i,t}^{-\varepsilon} Y_t - s_t P_{i,t}^{-\varepsilon} Y_t$$

$$\text{fonc : } (1 - \varepsilon) P_{it}^{-\varepsilon} Y_t + \varepsilon s_t P_{i,t}^{-\varepsilon-1} Y_t = 0$$

$$P_{it} = \frac{\varepsilon}{\varepsilon - 1} s_t \text{ 'price is markup over marginal cost'}$$

- Note: all prices are the same, so resources allocated efficiently across intermediate good firms.

$$P_{i,t} = P_{j,t} = 1, \text{ because } 1 = \int_0^1 P_{i,t}^{1-\varepsilon} di$$

# Equilibrium

- Pulling things together:

$$1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1 - \nu)w_t}{\exp(a_t)}$$

household fnc  
 $\underbrace{\hspace{10em}}_{=}$

$$\frac{\varepsilon(1 - \nu)}{\varepsilon - 1} \frac{\frac{-u_{n,t}}{u_{c,t}}}{\exp(a_t)}$$

if  $\frac{\varepsilon(1-\nu)}{\varepsilon-1} = 1$   
 $\underbrace{\hspace{10em}}_{=}$

$$\frac{\frac{-u_{n,t}}{u_{c,t}}}{\exp(a_t)}.$$

If proper subsidy is provided to monopolists, employment is efficient:

$$\text{if } 1 - \nu = \frac{\varepsilon - 1}{\varepsilon}, \text{ then } \frac{-u_{n,t}}{u_{c,t}} = \exp(a_t)$$

# Equilibrium Allocations

- With efficient subsidy,

$$\frac{-u_{n,t}}{u_{c,t}} \quad \underbrace{\quad}_{\text{functional form}} \quad C_t \exp(\tau_t) N_t^\varphi \quad \underbrace{\quad}_{\text{resource constraint}} \quad \exp(a_t) \exp(\tau_t) N_t^{1+\varphi} = \exp(a_t)$$
$$\rightarrow N_t = \exp\left(\frac{-\tau_t}{1+\varphi}\right)$$
$$C_t = e^{a_t} N_t = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)$$

- Bond market clearing implies:

$$B_t = 0 \text{ always}$$

# Interest Rate in Equilibrium

- Interest rate backed out of household intertemporal Euler equation:

$$u_{c,t} = \beta E_t u_{c,t+1} r_t \rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} r_t$$

$$\rightarrow r_t = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp[c_t - c_{t+1}]} = \frac{1}{\beta E_t \exp\left[a_t - a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right]}$$

$$= \frac{1}{\beta \exp\left[E_t\left(-\Delta a_{t+1} - \frac{\tau_t - \tau_{t+1}}{1+\varphi}\right) + \frac{1}{2}V\right]}, \quad V = \sigma_a^2 + \left(\frac{1}{1+\varphi}\right)^2 \sigma_\lambda^2$$

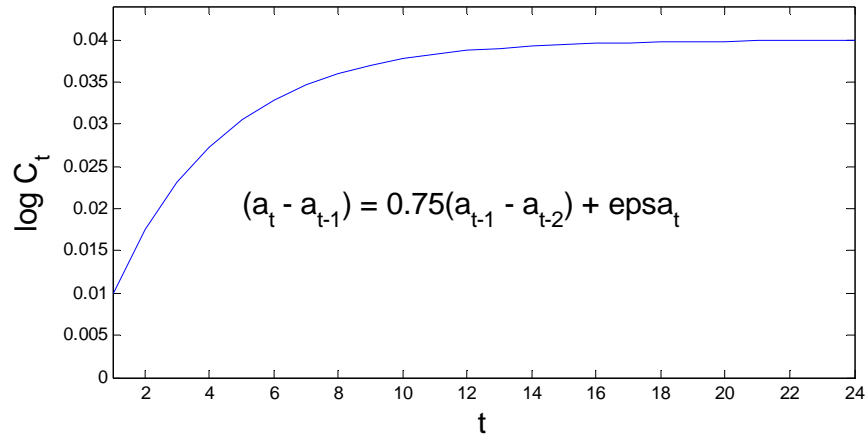
$$\log r_t = -\log \beta + E_t \left( \overbrace{\Delta a_{t+1} - \frac{\tau_{t+1} - \tau_t}{1+\varphi}}^{c_{t+1} - c_t} \right) + \frac{1}{2}V$$

using assumptions about  $\Delta a_t$  and  $\tau_t$

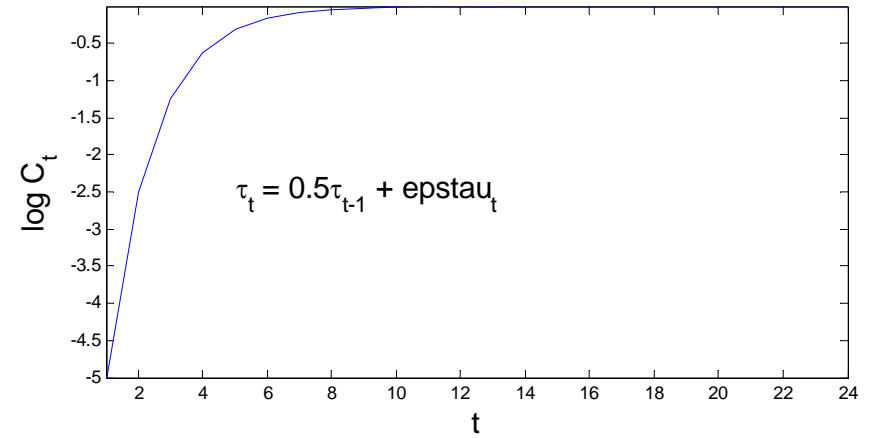
$$\underbrace{\hspace{1.5cm}} = -\log \beta + \rho \Delta a_t - \frac{(\lambda-1)\tau_t}{1+\varphi} + \frac{1}{2}V$$

# Dynamic Properties of the Model

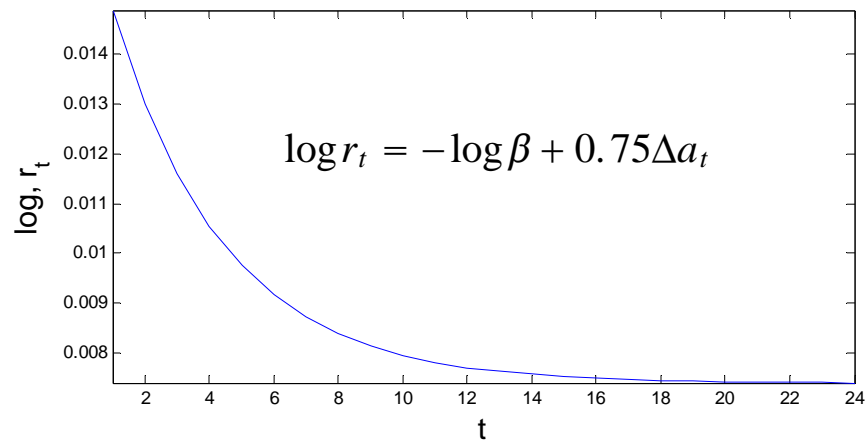
Response to .01 Technology Shock in Period 1



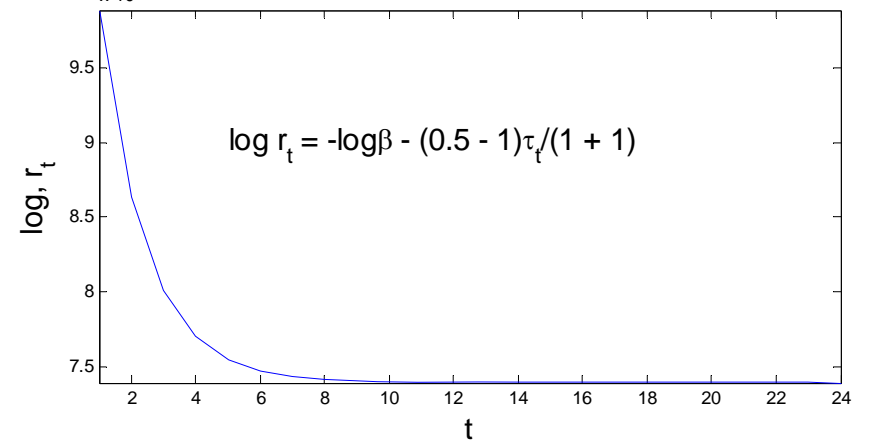
$\times 10^{-3}$  Response to .01 Preference Shock in Period 1



interest rate



interest rate



# Key Features of Equilibrium Allocations

- Allocations *efficient* (as long as monopoly power neutralized)
- Employment does not respond to technology
  - Improvement in technology raises marginal product of labor and marginal cost of labor by same amount.
- First best consumption not a function of intertemporal considerations
  - Discount rate irrelevant.
  - Anticipated future values of shocks irrelevant.
- Natural rate of interest steers consumption and employment towards their natural levels.

# Introducing Price Setting Frictions (Clarida-Gali-Gertler Model)

- Households maximize:

$$E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim iid,$$

- Subject to:

Profits, net of taxes raised by  
Government to finance subsidies.

$$P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + T_t$$

- Intratemporal first order condition:

$$C_t \exp(\tau_t) N_t^\varphi = \frac{W_t}{P_t}$$



# Household Intertemporal FONC

- Condition:

$$1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \frac{R_t}{1 + \pi_{t+1}}$$

– or

$$\begin{aligned} 1 &= \beta E_t \frac{C_t}{C_{t+1}} \frac{R_t}{1 + \pi_{t+1}} \\ &= \beta E_t \exp[\log(R_t) - \log(1 + \pi_{t+1}) - \Delta c_{t+1}] \\ &\simeq \beta \exp[\log(R_t) - E_t \pi_{t+1} - E_t \Delta c_{t+1}], \quad c_t \equiv \log(C_t) \end{aligned}$$

– take log of both sides:

$$0 = \log(\beta) + r_t - E_t \pi_{t+1} - E_t \Delta c_{t+1}, \quad r_t = \log(R_t)$$

– or

$$c_t = -\log(\beta) - [r_t - E_t \pi_{t+1}] + c_{t+1}$$

# Final Good Firms

- Buy  $Y_{i,t}$ ,  $i \in [0, 1]$  at prices  $P_{i,t}$  and sell  $Y_t$  for  $P_t$
- Take all prices as given (competitive)
- Profits:

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Production function:

$$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \varepsilon > 1,$$

- First order condition:

$$Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \quad \rightarrow \quad P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}$$

# Intermediate Good Firms

- Each  $i$ th good produced by a single monopoly producer.
- Demand curve:

$$Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon}$$

- Technology:

$$Y_{i,t} = \exp(a_t) N_{i,t}, \quad \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a,$$

- Calvo Price-setting Friction

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1 - \theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases}$$

# Marginal Cost

$$\begin{aligned} \text{real marginal cost} &= s_t = \frac{\frac{d\text{Cost}}{d\text{worker}}}{\frac{d\text{Output}}{d\text{worker}}} = \frac{(1 - \nu)W_t/P_t}{\exp(a_t)} \\ &= \frac{\overbrace{(1 - \nu)}^{=\frac{\varepsilon-1}{\varepsilon} \text{ in efficient setting}}}{\exp(a_t)} C_t \exp(\tau_t) N_t^\varphi \end{aligned}$$

# The Intermediate Firm's Decisions

- *ith* firm is required to satisfy whatever demand shows up at its posted price.
- Its only real decision is to adjust price whenever the opportunity arises.

# Intermediate Good Firm

- Present discounted value of  $i^{th}$  firm profits:

$$E_t^i \sum_{j=0}^{\infty} \beta^j \underbrace{v_{t+j}}_{\text{marginal value of dividends to household} = u_{c,t+j}/P_{t+j}} \underbrace{\left[ \overbrace{P_{i,t+j} Y_{i,t+j}}^{\text{revenues}} - \overbrace{P_{t+j} S_{t+j} Y_{i,t+j}}^{\text{total cost}} \right]}_{\text{period } t+j \text{ profits sent to household}}$$

Conditional expectation over price and aggregate uncertainty (e.g.,  $\Delta a_t$  and  $\tau_t$ ).

- Each of the  $1 - \theta$  firms that can optimize price choose  $\tilde{P}_t$  to optimize

Conditional expectation over aggregate uncertainty alone.

in selecting price, firm only cares about future states in which it can't reoptimize

$$E_t \sum_{j=0}^{\infty} \beta^j \underbrace{\theta^j}_{\text{probability of not reoptimizing}} v_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} S_{t+j} Y_{i,t+j}]$$

# Intermediate Good Firm Problem

- Substitute out the demand curve:

$$\begin{aligned}
 & E_t \sum_{j=0}^{\infty} (\beta\theta)^j v_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}] \\
 & = E_t \sum_{j=0}^{\infty} (\beta\theta)^j v_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon}].
 \end{aligned}$$

- Differentiate with respect to  $\tilde{P}_t$  :

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j v_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [(1 - \varepsilon)(\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1}] = 0,$$

- or

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j v_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

# Intermediate Good Firm Problem

- Objective:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0.$$

$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j P_{t+j}^{\varepsilon} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_t}, X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j} \bar{\pi}_{t+j-1} \dots \bar{\pi}_{t+1}}, & j \geq 1 \\ 1, & j = 0. \end{cases}, X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}}, j > 0$$



# Intermediate Good Firm Problem

- Want  $\tilde{p}_t$  in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0$$

- Solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

- But, still need expressions for  $K_t$ ,  $F_t$ .

$$\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left( \frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta \overbrace{E_t E_{t+1}}^{=E_t \text{ by LIME}} \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \overbrace{\sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j}}^{\text{exactly } K_{t+1}!} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon-1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}
\end{aligned}$$

- From previous slide:

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

- Substituting out for marginal cost:

$$\begin{aligned} \frac{\varepsilon}{\varepsilon - 1} s_t &= \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\overbrace{W_t/P_t}^{d\text{Cost}/d\text{labor}}}{\underbrace{\exp(a_t)}_{d\text{Output}/d\text{labor}}} \\ &= \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\overbrace{\exp(\tau_t) N_t^\varphi C_t}^{= \frac{W_t}{P_t} \text{ by household optimization}}}{\exp(a_t)}. \end{aligned}$$

# In Sum

- solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} S_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},$$

- Where:

$$K_t = (1 - v_t) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t) N_t^\varphi C_t}{\exp(a_t)} + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

$$F_t \equiv E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{1-\varepsilon} F_{t+1}$$

# To Characterize Equilibrium

- Have equations characterizing optimization by firms and households.
- Still need:
  - Expression for all the prices. Prices,  $P_{i,t}$ ,  $0 \leq i \leq 1$ , will all be different because of the price setting frictions.
  - Relationship between aggregate employment and aggregate output not simple because of price distortions:

$$Y_t \neq e^{a_t} N_t, \text{ in general}$$

- This part of the analysis is the reason why it made Calvo famous – it's not easy.

# Going for Prices

- Aggregate price relationship

$$P_t = \left[ \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

$$= \left[ \int_{\text{firms that reoptimize price}} P_{i,t}^{(1-\varepsilon)} di + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

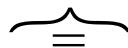
$$\underbrace{\text{all reoptimizers choose same price}}_{\equiv} \left[ (1 - \theta) \tilde{P}_t^{(1-\varepsilon)} + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

- In principle, to solve the model need all the prices,  $P_t, P_{i,t}, 0 \leq i \leq 1$ 
  - Fortunately, that won't be necessary.

# Key insight

$$\int_{\text{firms that don't reoptimize price in } t} P_{i,t}^{(1-\varepsilon)} di$$

add over prices, weighted by # of firms posting that price



$$\int \left[ \begin{array}{c} \text{'number' of firms that had price, } P(\omega), \text{ in } t-1 \text{ and were not able to reoptimize in } t \\ \overbrace{f_{t-1,t}(\omega)} \\ P(\omega)^{(1-\varepsilon)} \end{array} \right] d\omega$$

# Applying the Insight

- By Calvo randomization assumption

$$f_{t-1,t}(\omega) = \theta \times \overbrace{f_{t-1}(\omega)}^{\text{total 'number' of firms with price } P(\omega) \text{ in } t-1}, \text{ for all } \omega$$

- Substituting:

$$\begin{aligned} \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di &= \int f_{t-1,t}(\omega) P(\omega)^{(1-\varepsilon)} d\omega \\ &= \theta \int f_{t-1}(\omega) P(\omega)^{(1-\varepsilon)} d\omega \\ &= \theta P_{t-1}^{(1-\varepsilon)} \end{aligned}$$



# Expression for $\tilde{p}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[ (1 - \theta)\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

– Only two variables here!

- Divide by  $P_t$ :

$$1 = \left[ (1 - \theta)\tilde{p}_t^{(1-\varepsilon)} + \theta \left( \frac{1}{\bar{\pi}_t} \right)^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{p}_t = \left[ \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}}$$

# Relation Between Aggregate Output and Aggregate Inputs

- Technically, there is no ‘aggregate production function’ in this model
  - If you know how many people are working,  $N$ , and the state of technology,  $a$ , you don’t have enough information to know what  $Y$  is.
  - Price frictions imply that resources will not be efficiently allocated among different inputs.
    - Implies  $Y$  low for given  $a$  and  $N$ . How low?
    - Tak Yun (JME) gave a simple answer.

# Tak Yun Algebra

$$Y_t^* = \int_0^1 Y_{i,t} di \left( \overset{\text{labor market clearing}}{=} \int_0^1 A_t N_{i,t} di = A_t N_t \right)$$

$$\overset{\text{demand curve}}{=} Y_t \int_0^1 \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^\varepsilon \int_0^1 (P_{i,t})^{-\varepsilon} di$$

$$= Y_t P_t^\varepsilon (P_t^*)^{-\varepsilon}$$

Calvo insight

- Where: 
$$P_t^* \equiv \left[ \int_0^1 P_{i,t}^{-\varepsilon} di \right]^{\frac{-1}{\varepsilon}} = [(1 - \theta)\tilde{P}_t^{-\varepsilon} + \theta(P_{t-1}^*)^{-\varepsilon}]^{\frac{-1}{\varepsilon}}$$

# Relationship Between Agg Inputs and Agg Output

- Rewriting previous equation:

$$Y_t = \left( \frac{P_t^*}{P_t} \right)^\varepsilon Y_t^*$$

$$= p_t^* e^{a_t} N_t,$$

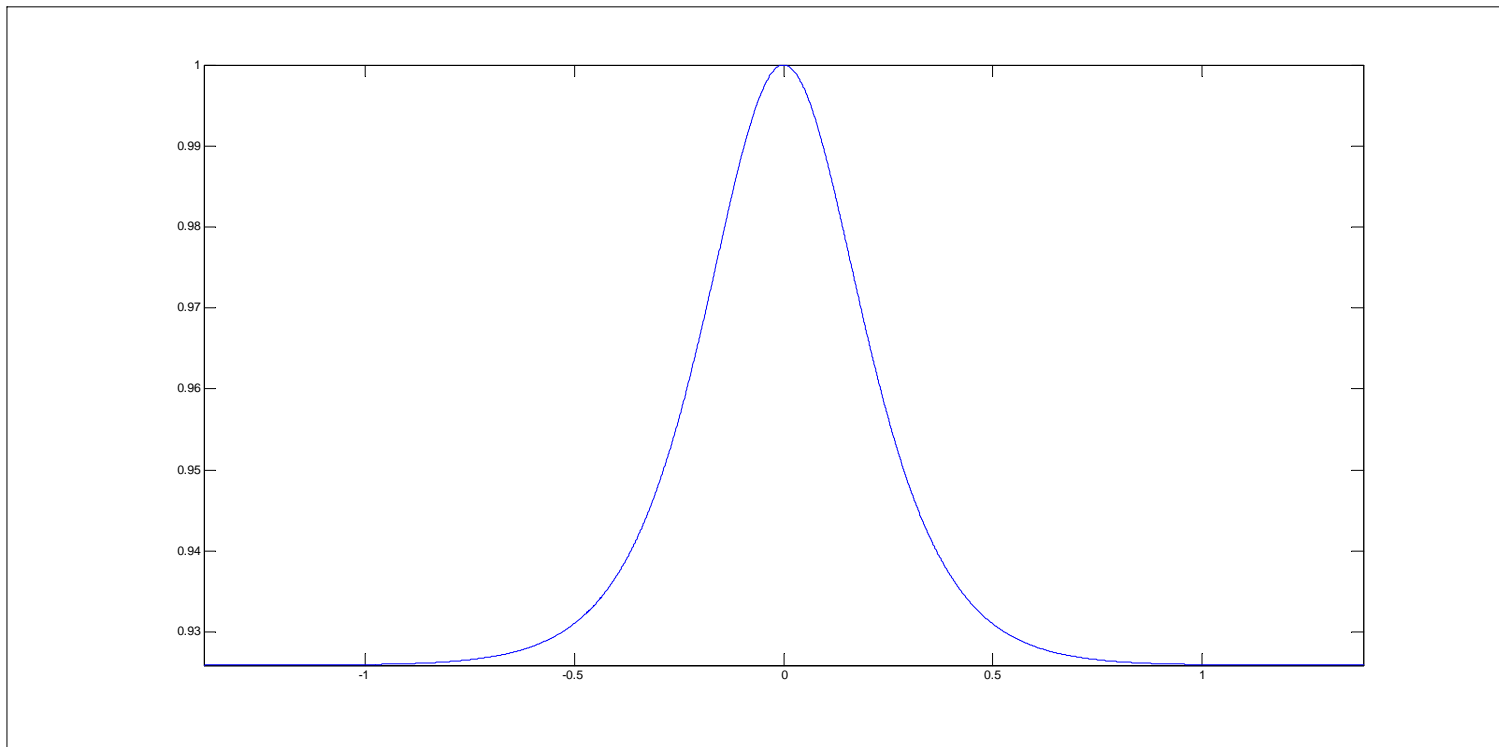
- ‘efficiency distortion’:

$$p_t^* : \begin{cases} \leq 1 \\ = 1 & P_{i,t} = P_{j,t}, \text{ all } i, j \end{cases}$$

# Example of Efficiency Distortion

$$P_{j,t} = \begin{cases} P^1 & 0 \leq j \leq \alpha \\ P^2 & \alpha \leq j \leq 1 \end{cases} \cdot p_t^* = \left( \frac{P_t^*}{P_t} \right)^\varepsilon = \left( \frac{\left[ \alpha + (1 - \alpha) \left( \frac{P^2}{P^1} \right)^{-\varepsilon} \right]^{\frac{-1}{\varepsilon}}}{\left[ \alpha + (1 - \alpha) \left( \frac{P^2}{P^1} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}} \right)^\varepsilon$$

$\alpha = 0.5, \varepsilon = 10$



$\log P^1/P^2$

# Collecting Equilibrium Conditions

- Price setting:

$$K_t = (1 - \nu) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t) N_t^\varphi C_t}{A_t} + \beta \theta E_t \bar{\pi}_{t+1}^\varepsilon K_{t+1} \quad (1)$$

$$F_t = 1 + \beta \theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \quad (2)$$

- Intermediate good firm optimality and restriction across prices:

$$\overbrace{\frac{K_t}{F_t}}^{=\tilde{p}_t \text{ by firm optimality}} = \overbrace{\left[ \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}}}^{=\tilde{p}_t \text{ by restriction across prices}} \quad (3)$$

# Equilibrium Conditions

- Law of motion of (Tak Yun) distortion:

$$p_t^* = \left[ (1 - \theta) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right]^{-1} \quad (4)$$

- Household Intertemporal Condition:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

- Aggregate inputs and output:

$$C_t = p_t^* e^{a_t} N_t \quad (6)$$

- 6 equations, 8 unknowns:

$$v, C_t, p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t$$

- System under determined!

# Underdetermined System

- Not surprising: we added a variable, the nominal rate of interest.
- Also, we're counting subsidy as among the unknowns.
- Have two extra policy variables.
- One way to pin them down: compute optimal policy.



# Ramsey-Optimal Policy

- 6 equations in 8 unknowns.....
  - Many configurations of the 8 unknowns that satisfy the 6 equations.
  - Look for the best configurations (Ramsey optimal)
    - Value of tax subsidy and of  $R$  represent optimal policy
- Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.

# Ramsey Problem

$$\begin{aligned}
 & \max_{v, p_t^*, C_t, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right. \\
 & + \lambda_{1t} \left[ \frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
 & + \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1-\theta) \left( \frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \\
 & + \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t] \\
 & + \lambda_{4t} \left[ (1-\nu) \frac{\varepsilon}{\varepsilon-1} \frac{C_t \exp(\tau_t) N_t^{\varphi}}{e^{a_t}} + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right] \\
 & + \lambda_{5t} \left[ F_t \left( \frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \\
 & \left. + \lambda_{6t} [C_t - p_t^* e^{a_t} N_t] \right\}
 \end{aligned}$$

# Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\begin{aligned}
 & \max_{v, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right. \\
 \text{defines } R & \rightarrow + \lambda_{1t} \left[ \frac{1}{p_t^* N_t} - E_t \frac{e^{a_t} \beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
 & + \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1-\theta) \left( \frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right) \right] \\
 \text{defines } F & \rightarrow + \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t] \\
 \text{defines tax} & \rightarrow + \lambda_{4t} \left[ (1-v) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta \theta \bar{\pi}_{t+1}^\varepsilon K_{t+1} - K_t \right] \\
 \text{defines } K & \rightarrow + \lambda_{5t} \left[ F_t \left( \frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \left. \right\}
 \end{aligned}$$

# Solving the Ramsey Problem, cnt'd

- Simplified problem:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) + \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1-\theta) \left( \frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \right\}$$

- First order conditions with respect to  $p_t^*$ ,  $\bar{\pi}_t$ ,  $N_t$

$$p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^{\varepsilon} = \lambda_{2t}, \quad \bar{\pi}_t = \left[ \frac{(p_{t-1}^*)^{\varepsilon-1}}{1-\theta + \theta(p_{t-1}^*)^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}}, \quad N_t = \exp\left(-\frac{\tau_t}{\varphi+1}\right)$$

- Substituting the solution for inflation into law of motion for price distortion:

$$p_t^* = \left[ (1-\theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}.$$

# Solution to Ramsey Problem

Eventually, price distortions eliminated, regardless of shocks

$$p_t^* = \left[ (1 - \theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}$$

When price distortions gone, so is inflation.

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

Efficient ('first best') allocations in real economy

$$N_t = \exp\left(-\frac{\tau_t}{1 + \varphi}\right)$$

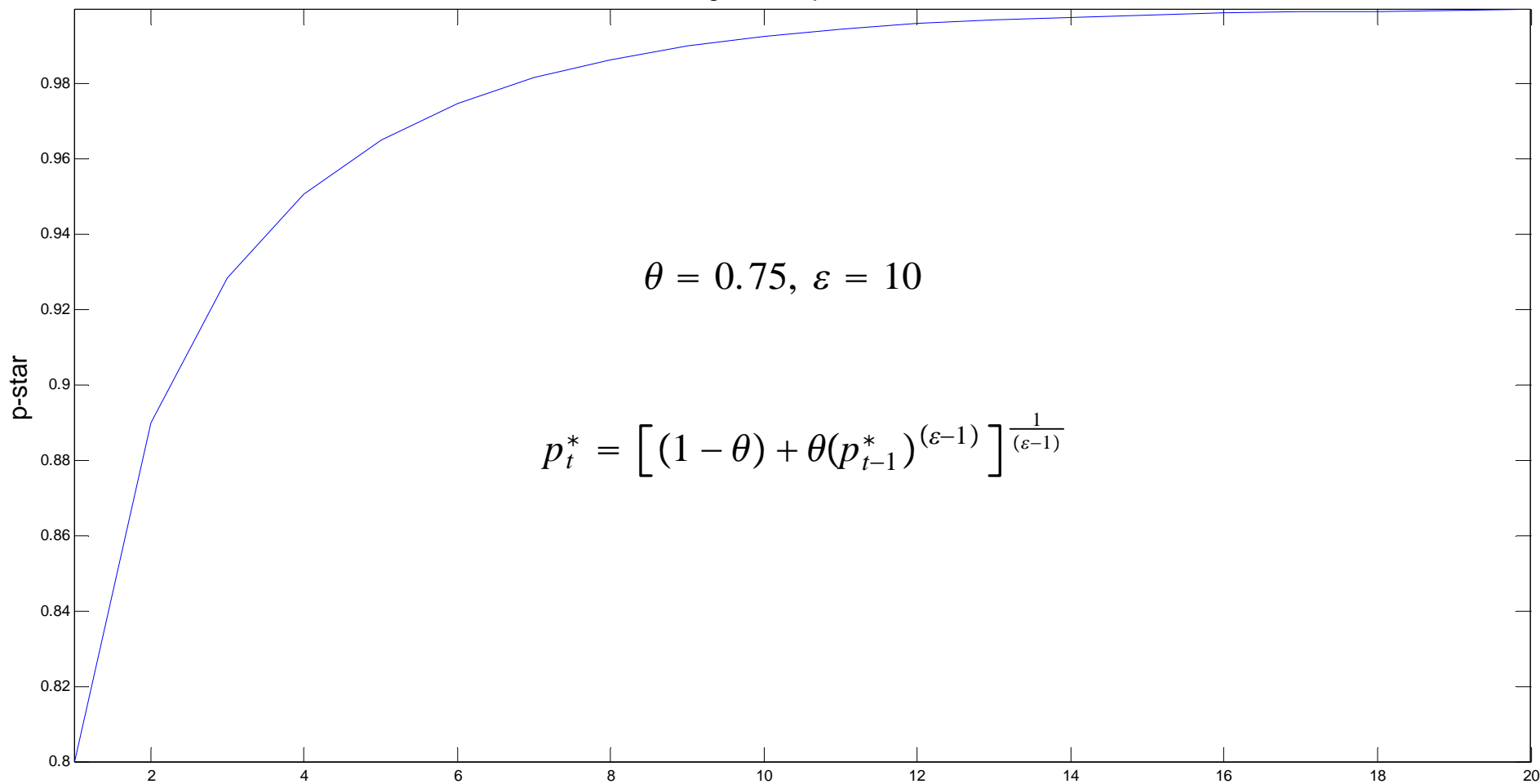
$$1 - v = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = p_t^* e^{a_t} N_t.$$

Consumption corresponds to efficient allocations in real economy, eventually when price distortions gone

# Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide

Convergence of price distortion



- The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., ‘first best allocations’)
- Refer to the Ramsey allocations as the ‘natural allocations’....
  - Natural consumption, natural rate of interest, etc.