1. In the handout, the following learning mechanism was assumed:

\[
\tau_t^e - \tau_{t-1}^e = \left( \frac{g}{1 - \tau_{t-1}^e} \right) \frac{g}{w^2} - \tau_{t-1}^e
\]

Suppose instead that the following is assumed:

\[
\tau_t^e - \tau_{t-1}^e = \lambda \left( \frac{g}{1 - \tau_{t-1}^e} \right) \frac{g}{w^2} - \tau_{t-1}^e
\]

Note that \( \tau_t^e = 1/4, 3/4 \) remain the only rational expectations equilibria.
Show that the equilibrium on the left of the Laffer curve is stable under learning and the equilibrium on the right of the Laffer curve is unstable under learning, for all \( 0 < \lambda \leq 1 \).

2. Suppose a variable, \( y_t \), is determined as follows:

\[
y_t = \alpha + \delta y_{t-1} + \beta_0 \tau_{t-1}^e y_t + \beta_1 \tau_{t-1}^e y_{t+1} + \beta_2 \tau_{t-1}^e y_{t+2} + \nu_t.
\]

where \( \nu_t \) denotes an iid shock that is uncorrelated with past \( y_t \)'s and \( 0 < \delta < 1 \). Also, \( \tau_{t-1}^e \) is an expectation formed as of time \( t-1 \). This is a model in which \( y_t \) is determined in part as a function of its value in the previous period, as well as the value anticipated for \( y_t \) from the perspective of \( t-1 \), in addition to the values anticipated for \( y_{t+1} \) and \( y_{t+2} \). The equation resembles the (linearized) equilibrium condition of an economic model. For example, it resembles the linearized Phillips curve in a New Keynesian model. To form expectations, agents must have in mind a law of motion for \( y_t \). Consider the following one:

\[
y_t = a + by_{t-1} + \nu_t.
\]

(a) Suppose that agents believe the above law of motion for specific values of \( a \) and \( b \) so that

\[
E_{t-1} \tau_t^e y_t = a + by_{t-1}, \quad E_{t-1} \tau_t^e y_{t+1} = a + b a + b^2 y_{t-1}.
\]

Derive an expression for \( E_{t-1} \tau_t^e y_{t+2} \).

(b) Show that given the belief in 2(a), the actual law of motion for \( y_t \) is:

\[
y_t = T(a, b) + J(b) y_{t-1} + \nu_t,
\]

where

\[
J(b) = \delta + \beta_0 b + \beta_1 b^2 + \beta_2 b^3
\]

\[
T(a, b) = \alpha + \beta_0 a + \beta_1 a (1 + b) + \beta_2 a (1 + b + b^2).
\]
(c) Let a rational expectations equilibrium be an \( a \) and \( b \) such that

\[
a = T(a, b), \quad b = J(b).
\]

Suppose

\[
\delta = -0.0216, \quad \beta_0 = 0.234, \quad \beta_1 = -0.81, \quad \beta_2 = 0.729.
\]

How many rational expectations equilibria are there? Report the values of \( a \) and \( b \) in each rational expectations equilibrium.

(d) Consider the following (slightly artificial) learning scheme. Suppose people believe particular values for \( a \) and \( b \). They act based on these beliefs for a long time, during which much data on \( y_t \) are generated by the model, (1). With a long data set on \( y_t \) in hand, agents update their beliefs by running a first order autoregression. Agents’ new beliefs correspond to a new constant and slope term for (2), i.e., \( T(a, b) \) and \( J(b) \), respectively. We say that a particular rational expectations equilibrium is (locally) stable under learning if when people’s beliefs about \( a \) and \( b \) start a little away from that equilibrium, the learning process converges to that equilibrium. A rational expectations equilibrium is not stable under learning if it is the case that no matter how close to that equilibrium agents’ beliefs start, the learning system diverges. Suppose \( a^* \) and \( b^* \) is one rational expectations equilibrium. Explain why the concept of stability of rational expectations equilibrium corresponds to the requirement:

\[
0 < J'(b^*) < 1,
\]

where \( J' \) denotes the derivative of \( J \) with respect to its argument. If we consider the rational expectations equilibria in 2(c) above, how many are stable under learning, in the sense just defined? (The concept of stability just defined is called \( E \)-stability. However, I have given it a real-time interpretation here while in practice \( E \)-stability is defined using a concept of notional time. See, for example, Evans and Honkapohja, European Economic Review, 1994, pp. 1071-1098).