Solving and Analyzing a Model with Two Lucas Trees

This note explores an analysis of an economy with two Lucas trees, which is studied in Cochrane, Longstaff and Santa-Clara ('Two Trees', The Review of Financial Studies, vol. 21 no. 1, 2008) (CLS) and Ian Martin, 'The Lucas Orchard,' Econometrica, January 2013 (see especially Figure 3).

Consider an economy with two trees, tree number 1 and tree number 2. Corresponding to these two trees there are the following two dividend processes,

$$D_{1t}, D_{2t}.$$

The time series representations are:

$$\frac{D_{1,t+1}}{D_{1,t}} = \varepsilon_{1,t+1},\ \frac{D_{2,t+1}}{D_{2,t}} = \varepsilon_{2,t+1},$$

where the two shocks are iid over time and independent of each other. We suppose that consumption is given by:

$$C_t = \alpha D_{1,t} + (1-\alpha) D_{2,t}.$$

The price of tree 1 is $p_{2,t}$

$$p_{2,t} = \beta E \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} [D_{2,t+1} + p_{2,t+1}],$$

or, in terms of price dividend ratio, $P_{2,t} \equiv p_{2,t}/D_{2,t}$:

$$P_{2,t} = \beta E \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} [1 + P_{2,t+1}] \varepsilon_{2,t+1}.$$

Also,

$$
\begin{aligned}
\frac{C_{t+1}}{C_t} &= \frac{\alpha D_{1,t+1} + (1-\alpha) D_{2,t+1}}{\alpha D_{1,t} + (1-\alpha) D_{2,t}} \\
&= \frac{\alpha \varepsilon_{1,t+1} D_{1,t} + (1-\alpha) \varepsilon_{2,t+1} D_{2,t}}{\alpha D_{1,t} + (1-\alpha) D_{2,t}} \\
&= \frac{\alpha D_{1,t}}{\alpha D_{1,t} + (1-\alpha) D_{2,t}} \varepsilon_{1,t+1} + \frac{(1-\alpha) D_{2,t}}{\alpha D_{1,t} + (1-\alpha) D_{2,t}} \varepsilon_{2,t+1} \\
&= x_t \varepsilon_{1,t+1} + (1-x_t) \varepsilon_{2,t+1}.
\end{aligned}
$$

It is convenient to derive an expression for x_{t+1} :

$$
\begin{aligned}
x_{t+1} &= \frac{\alpha D_{1,t+1}}{\alpha D_{1,t+1} + (1-\alpha) D_{2,t+1}} \\
&= \frac{\alpha D_{1,t+1}}{\alpha D_{1,t}} \frac{\alpha D_{1,t}}{\alpha D_{1,t} + (1-\alpha) D_{2,t}} \frac{\alpha D_{1,t} + (1-\alpha) D_{2,t}}{\alpha D_{1,t+1} + (1-\alpha) D_{2,t+1}} \\
&= \frac{\varepsilon_{1,t+1} x_t}{x_t \varepsilon_{1,t+1} + (1-x_t) \varepsilon_{2,t+1}}.
\end{aligned}
$$

It is interesting to think about what the ergodic distribution of x_t is. Surprisingly, perhaps, the distribution of x_t is not a function of α.

We posit that the solution is a function, $P_2(x)$, that satisfies the following fixed point:

$$
P_2(x) = \beta E \left[x \varepsilon_1' + (1-x) \varepsilon_2' \right]^{-\gamma} \left[1 + P_2(x') \right] \varepsilon_2'.
$$

Suppose

$$
\varepsilon_1 - 1, \varepsilon_2 - 1 \in (-\sigma, 0, \sigma).
$$

Let the 9 states be given by the 9 by 1 vector, s :

$$s = \begin{pmatrix} l, l \\ l, m \\ l, h \\ m, l \\ m, m \\ m, h \\ h, l \\ h, m \\ h, h \end{pmatrix}$$

Let π_1 and π_2 denote the Markov transition matrices for ε_1 and ε_2, respectively. The iid assumption implies that the rows of π_1 are all equal. Similarly for the rows of π_2. Let π denote the Markov transition matrix for s :

$$\begin{aligned} \pi &= \pi_1 \otimes \pi_2 \\ & \begin{bmatrix} \pi_1^{11}\pi_2 & \pi_1^{12}\pi_2 & \pi_1^{13}\pi_2 \\ \pi_1^{21}\pi_2 & \pi_1^{22}\pi_2 & \pi_1^{23}\pi_2 \\ \pi_1^{31}\pi_2 & \pi_1^{32}\pi_2 & \pi_1^{33}\pi_2 \end{bmatrix} \end{aligned}$$

Using this notation our functional equation can be written:

$$P_2(x) - \beta \sum_{j=1}^{N} \pi_{ij} \left[x\varepsilon_1(j) + (1-x)\varepsilon_2(j) \right]^{-\gamma} \left[1 + P_2(x'(j)) \right] \varepsilon_2(j) = 0,$$

for all $0 \leq x \leq 1$, where

$$x'(j) = \frac{\varepsilon_1'(j)\, x}{x\varepsilon_1'(j) + (1-x)\varepsilon_2'(j)}$$

We now construct a Chebyshev polynomial approximation to P_1 and P_2. The domain of these functions is $[0,1]$, but the domain of the Chebyshev polynomial is $[-1,1]$. Thus, we require a mapping,

$$\varphi : [0,1] \rightarrow [-1,1],$$

and the following serves our purposes:

$$\varphi(x) = 2x - 1.$$

We approximate P_1 and P_2 with $M-1^{th}$ ordered Chebyshev polynomials, with basis functions, $T_i(\varphi(x))$, for $i = 0, 1, ..., M-1$.* In particular, let

$$T(x) = [T_0(\varphi(x)), T_1(\varphi(x)), ..., T_{M-1}(\varphi(x))]'.$$

Let a and b denote two $M \times 1$ vectors of parameters. Then, one strategy for approximating the solutions is:

$$\hat{P}_1(x;a) = a'T(x), \;\; \hat{P}_2(x;b) = b'T(x).$$

The M zeros of the M^{th} order Chebyshev polynomial, T_M, are

$$r_j = \cos\left(\frac{\pi(j-0.5)}{M}\right), \;\; j = 1, \ldots, M,$$

and let

$$x_j = \varphi^{-1}(r_j) = \frac{r_j + 1}{2}, \;\; j = 1, ..., M. \tag{1}$$

The calculations reported below are based on a finite element approach to approximating the equilibrium price-dividend functions. We fixed a set of grid points for x and then the parameters, a and b, represent the values of the functions, $\hat{P}_1$ and $\hat{P}_2$, at the grid points. The functions were made continuous by spline interpolation using the MATLAB function, interp1.†

Define the error functions, E_1 and E_2 :

$$\begin{aligned}
E_1(x;a) &= \hat{P}_1(x;a) - \beta \sum_{j=1}^{N} \pi_{ij} \left[x\varepsilon_1(j) + (1-x)\varepsilon_2(j)\right]^{-\gamma} \left[1 + \hat{P}_1(x'(j);a)\right] \varepsilon_1(j) \\
E_2(x;b) &= \hat{P}_2(x;b) - \beta \sum_{j=1}^{N} \pi_{ij} \left[x\varepsilon_1(j) + (1-x)\varepsilon_2(j)\right]^{-\gamma} \left[1 + \hat{P}_2(x'(j);b)\right] \varepsilon_2(j).
\end{aligned}$$

Given the grid points for x and the parameters of the parametric functions, a collocation approach solves $2M$ unknowns and $2M$ equations.

Given approximate solutions for the pricing functions, returns are given by:

$$\begin{aligned}
\hat{R}_1(x,j;a) &= \frac{\left[1 + \hat{P}_1(x'(j);a)\right] \varepsilon_1(j)}{\hat{P}_1(x;a)} \\
\hat{R}_2(x,j;a) &= \frac{\left[1 + \hat{P}_2(x'(j);a)\right] \varepsilon_2(j)}{\hat{P}_2(x;a)}.
\end{aligned}$$

Define the mean returns (conditional on the state, x) as follows:

$$\begin{aligned}
M_1(x;a) &= \sum_{j=1}^{N} \pi_{ij} \hat{R}_1(x,j;a) \\
M_2(x;b) &= \sum_{j=1}^{N} \pi_{ij} \hat{R}_2(x,j;b),
\end{aligned}$$

where the value of i can be anything between 1 and N because of the independence assumption. Finally,

$$
\begin{aligned}
Cov(x) &= \sum_{j=1}^{N} \pi_{i,j} \left[\hat{R}_1(x,j;a) - M_1(x;a)\right]\left[\hat{R}_2(x,j;a) - M_2(x;a)\right] \\
V_1(x) &= \sum_{j=1}^{N} \pi_{i,j} \left[\hat{R}_1(x,j;a) - M_1(x;a)\right]^2 \\
V_2(x) &= \sum_{j=1}^{N} \pi_{i,j} \left[\hat{R}_2(x,j;b) - M_2(x;b)\right]^2 \\
\rho(x) &= \frac{Cov(x)}{\sqrt{V_1(x)V_2(x)}}.
\end{aligned}
$$

We studied two parameterizations of the model. In each case,

$$\beta = 1/1.05,\ \gamma = 1.$$

The parameter, α, plays no role in the analysis. Also,

$$\pi_1 = \pi_2 = \begin{bmatrix} \frac{1}{8} & \frac{6}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{6}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{6}{8} & \frac{1}{8} \end{bmatrix}.$$

In Figure 1, results are reported for the case, $\sigma = 0.1$. Figures 2 and 3 report results for the case, $\sigma = 0.3$. In each case, the policy functions were solved by the spline-based finite element method described above. Relatively few grid points, 22, were required to obtain a reasonable solution in the small σ case. Obviously, the error functions, E_1 and E_2, are essentially zero on the grid of values used to compute the equilibrium price-dividend functions. By a 'reasonable solution' we mean a couple of approximate price-dividend functions that drive the error functions close to zero between the grid points used in the construction of the functions. For this, we constructed a very fine grid of 999 equally spaced values of x on the unit interval. Figure 1 displays the graph of the error functions over this very fine grid. Note that the errors are quite small, on the order of 10^{-7}. One metric of 'small' compares the size of the errors with the size of the price-dividend ratio itself. The latter is in the neighborhood of 20, so the errors relative to that are extremely small.

Figure 1 shows that the price dividend ratios vary by only a small amount, between the values of 20 and 21. Because of this small amount of variation, it is not surprising that the correlation between the two rates of return are essentially zero for all x (see the Figure). From the expressions for the rates of return on the two trees given above, we see that if the two pricing functions are literally constant over all x, then the correlation, conditional on x, is zero because of the independence of ε_1 and ε_2.

The findings in Figure 1 contrast sharply with the results reported by CSL and Martin. Those papers consider (among other things) two-tree models that are cast in continuous time where innovations are Normal. In those papers, the price-dividend function for a particular tree rises sharply towards infinity when the share in total consumption of the dividends from that tree approaches zero (see, for example, CLS's equation (22) and their Figure 1). With this sharp variation in the price-dividend ratio, it is perhaps then not surprising that substantial correlation between the returns on the two trees is found by CLS and Martin.

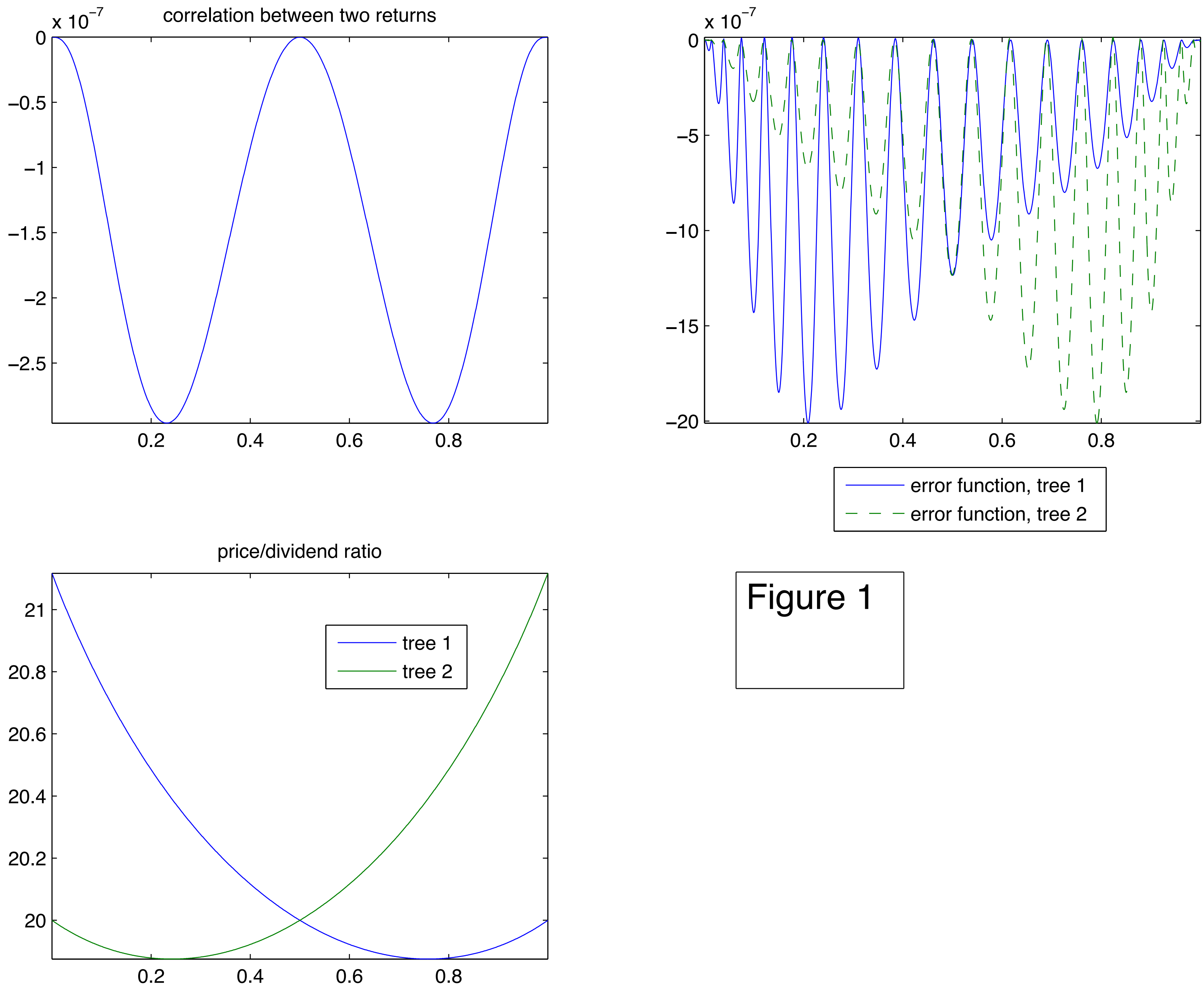
The second computational experiment, based on $\sigma = 0.3$ was more challenging, computationally. Without a lot of grid points for x close to unity and zero, it was difficult to make the error functions, E_1 and E_2, close to zero there. We constructed grids in two different ways, and report results for each. The grids are differentiated by the interval in which additional grid points were concentrated. In each case, we started by constructing 60 grid points for x using the Chebyshev zero approach described above. In the first case, we added additional grid points very close to unity and zero.[‡] In this case, the total number of grid points was 141. In the second case, additional grid points were also placed close to zero and unity, but they were spread out a little more.[§] The total number of grid points in the second case was 172.

Figure 2 displays results based on the first way of constructing the grid. Notice that now the price-dividend ratio for the first tree rises more sharply in states when the share of income from that tree, x, is small. It is our impression that this is consistent with the findings in Martin and CLS. With a larger value of σ our distribution of the dividend growth rates resembles more closely (than in the $\sigma = 0.1$ case) the unbounded distributions that they use.

In Figure 2, the variation in the price dividend ratio is between 0.2 and about 0.38. The shape of the price function as a function of x roughly resembles the one in CLS in that the rise becomes relatively sharp for values of x below 0.2. Despite the great variation in the price-dividend function, however, it is still the case that the correlation between the two asset returns is roughly zero. It is ever-so-slightly bigger in Figure 2 than it is in Figure 1, however. This is consistent with the notion that for larger values of σ perhaps the correlation would turn positive, as it does in the model in which the dividend growth rates are Normally distributed (see Ian Martin's Figure 3).

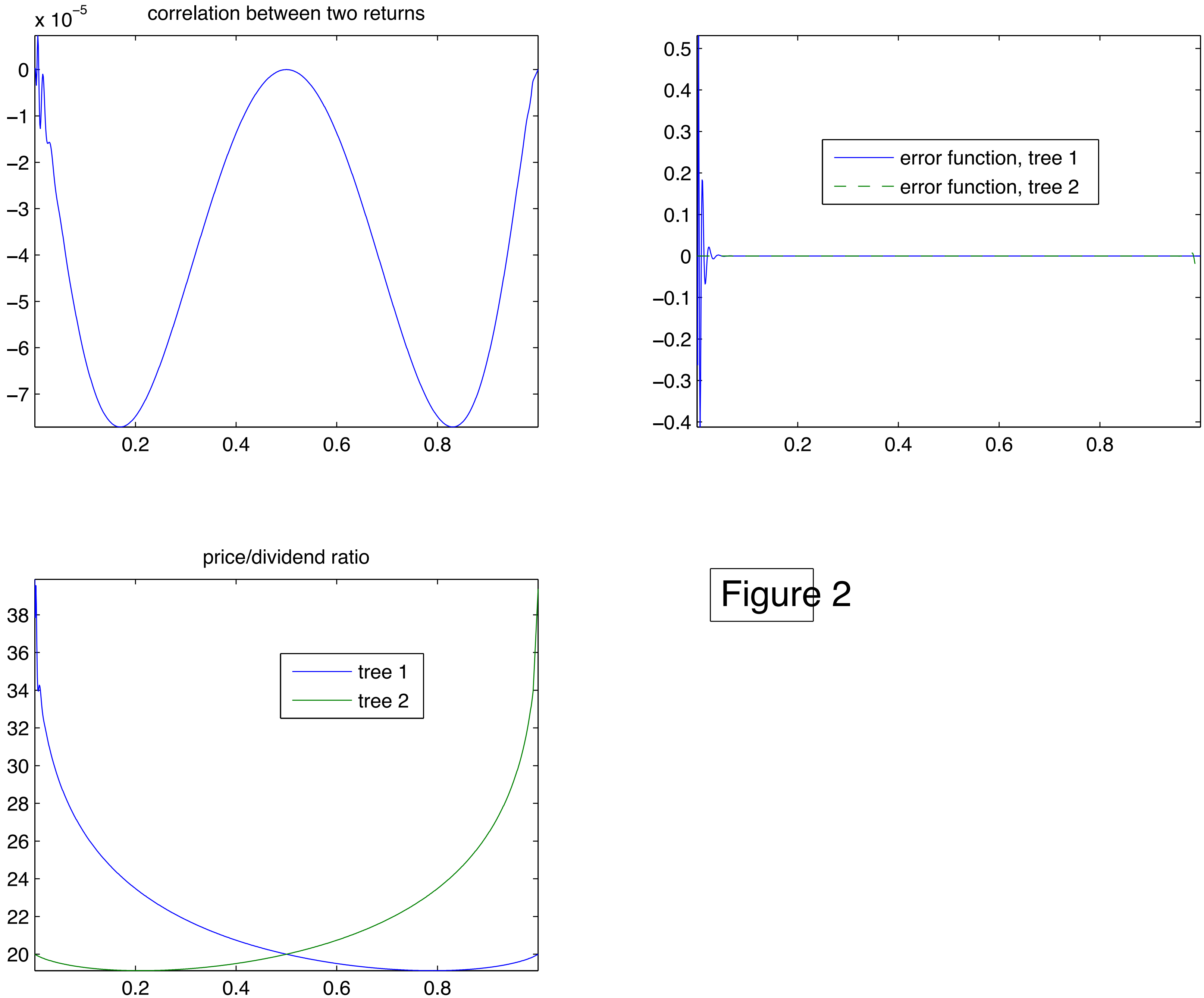
The error function in Figure 2 does appear to be relatively far from zero for x in a neighborhood of 0. This motivated our second method for choosing grid points, the results for which are displayed in Figure 3. Note that in Figure 3 the error function is substantially smaller, although the errors are still somewhat large for x's close to the boundaries. Interestingly, however, the price-dividend functions are very similar to what was observed in Figure 2. We may perhaps infer from this evidence of robustness that our computed price-dividend functions are reasonably accurate.

These results suggest that the findings in Martin and CLS are very sensitive to assumptions made about the support of the distribution of growth rates in regions that have very low probability. This resembles other classic findings in Finance, in which it is shown that the equity premium is very sensitive to whether the underlying shocks are Normal or have even fatter tails (see Geweke?).



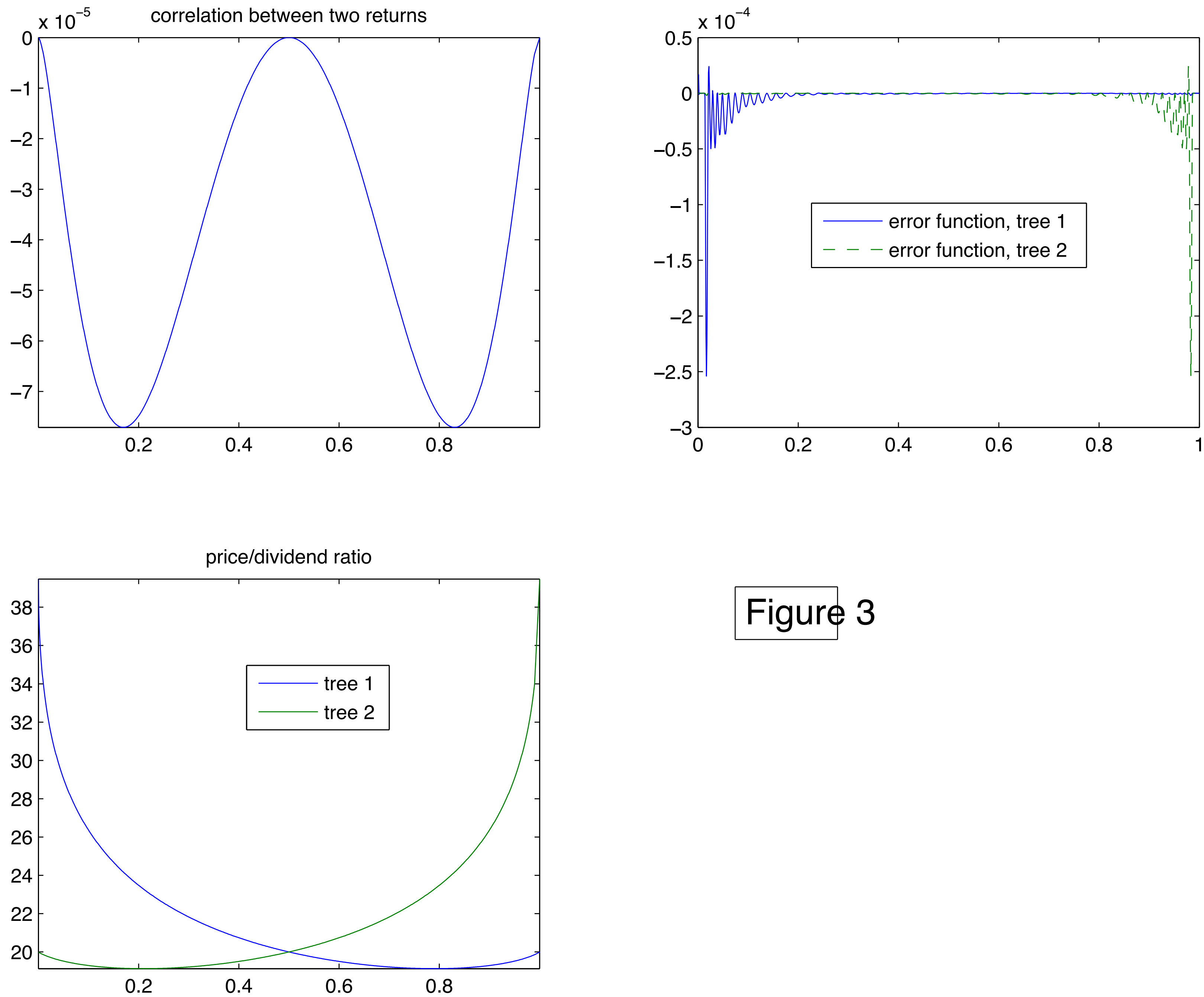
correlation between two returns
x 10^-7
0
−0.5
−1
−1.5
−2
−2.5
0.2
0.4
0.6
0.8
x 10^-7
0
−5
−10
−15
−20
0.2
0.4
0.6
0.8
error function, tree 1
error function, tree 2
price/dividend ratio
21
20.8
20.6
20.4
20.2
20
0.2
0.4
0.6
0.8
tree 1
tree 2

Figure 1



correlation between two returns
x 10^−5
0
−1
−2
−3
−4
−5
−6
−7
0.2
0.4
0.6
0.8
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
0.2
0.4
0.6
0.8
error function, tree 1
error function, tree 2
price/dividend ratio
38
36
34
32
30
28
26
24
22
20
0.2
0.4
0.6
0.8
tree 1
tree 2

Figure 2



correlation between two returns
x 10^-5
0
−1
−2
−3
−4
−5
−6
−7
0.2
0.4
0.6
0.8
x 10^-4
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
0
0.2
0.4
0.6
0.8
1
error function, tree 1
error function, tree 2
price/dividend ratio
38
36
34
32
30
28
26
24
22
20
0.2
0.4
0.6
0.8
tree 1
tree 2

Figure 3