Mixed Monthly/Quarterly Observations

- Different data arrive at different frequencies: daily, monthly, quarterly, etc.
- This feature can be easily handled in state space-observer system.
- Example:
  - suppose inflation and hours are monthly, $t = 0,\ 1/3,\ 2/3,\ 1,\ 4/3,\ 5/3,\ 2, ...$
  - suppose gdp is quarterly, $t = 0, 1, 2, 3, ...$

$$Y_t^{data} = \begin{pmatrix} GDP_t \\ \text{monthly inflation}_t \\ \text{monthly inflation}_{t-1/3} \\ \text{monthly inflation}_{t-2/3} \\ \text{hours}_t \\ \text{hours}_{t-1/3} \\ \text{hours}_{t-2/3} \end{pmatrix},\ t = 0, 1, 2, ...\ .$$

  that is, we can think of our data set as actually being quarterly, with quarterly observations on the first month's inflation, quarterly observations on the second month's inflation, etc.

Mixed Monthly/Quarterly Observations

- Problem: find state-space observer system in which observed data are:

$$Y_t^{data} = \begin{pmatrix} GDP_t \\ \text{monthly inflation}_t \\ \text{monthly inflation}_{t-1/3} \\ \text{monthly inflation}_{t-2/3} \\ \text{hours}_t \\ \text{hours}_{t-1/3} \\ \text{hours}_{t-2/3} \end{pmatrix}, \; t = 0, 1, 2, \ldots .$$

- Solution: easy!

Mixed Monthly/Quarterly Observations

- Model timing: $t = 0, 1/3, 2/3, ...$

$$\begin{aligned} z_t &= A z_{t-1/3} + B s_t, \\ s_t &= P s_{t-1/3} + \epsilon_t,\ E\epsilon_t\epsilon_t' = D. \end{aligned}$$

- Monthly state-space observer system, $t = 0, 1/3, 2/3, ...$

$$\begin{aligned} \xi_t &= F\xi_{t-1/3} + u_t,\ E u_t u_t' = Q,\ u_t \tilde{}\, iid\ t = 0, 1/3, 2/3, ... \\ \\ Y_t &= H\xi_t,\ Y_t = \begin{pmatrix} y_t \\ \pi_t \\ h_t \end{pmatrix}. \end{aligned}$$

- Note:

$$\overbrace{\xi_t = F^3\xi_{t-1} + u_t + F u_{t-1/3} + F^2 u_{t-2/3}}^{\text{first order vector autoregressive representation for quarterly state}} \quad ,$$

$$u_t + F u_{t-1/3} + F^2 u_{t-2/3}\ \tilde{}\, \underline{iid\ for\ t = 0, 1, 2, ...}!!$$

Mixed Monthly/Quarterly Observations

Consider the following system:

$$\begin{pmatrix} \xi_t \\ \xi_{t-\frac{1}{3}} \\ \xi_{t-\frac{2}{3}} \end{pmatrix} = \begin{bmatrix} F^3 & 0 & 0 \\ F^2 & 0 & 0 \\ F & 0 & 0 \end{bmatrix} \begin{pmatrix} \xi_{t-1} \\ \xi_{t-\frac{4}{3}} \\ \xi_{t-\frac{5}{3}} \end{pmatrix} + \begin{bmatrix} I & F & F^2 \\ 0 & I & F \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} u_t \\ u_{t-\frac{1}{3}} \\ u_{t-\frac{2}{3}} \end{pmatrix}.$$

Define

$$\tilde{\xi}_t = \begin{pmatrix} \xi_t \\ \xi_{t-\frac{1}{3}} \\ \xi_{t-\frac{2}{3}} \end{pmatrix}, \tilde{F} = \begin{bmatrix} F^3 & 0 & 0 \\ F^2 & 0 & 0 \\ F & 0 & 0 \end{bmatrix}, \ \tilde{u}_t = \begin{bmatrix} I & F & F^2 \\ 0 & I & F \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} u_t \\ u_{t-\frac{1}{3}} \\ u_{t-\frac{2}{3}} \end{pmatrix},$$

so that

$$\tilde{\xi}_t = \tilde{F}\tilde{\xi}_{t-1} + \tilde{u}_t, \ \tilde{u}_t \tilde{} iid \text{ in quarterly data, } t = 0, 1, 2, ...$$

$$E\tilde{u}_t\tilde{u}_t' = \tilde{Q} = \begin{bmatrix} I & F & F^2 \\ 0 & I & F \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix} \begin{bmatrix} I & F & F^2 \\ 0 & I & F \\ 0 & 0 & I \end{bmatrix}'$$

Mixed Monthly/Quarterly Observations

- Conclude: state space-observer system for mixed monthly/quarterly data, for $t = 0, 1, 2, ...$

$$\tilde{\xi}_t = \tilde{F}\tilde{\xi}_{t-1} + \tilde{u}_t, \ \tilde{u}_t \tilde{\ } iid, \ E\tilde{u}_t\tilde{u}_t' = \tilde{Q},$$

$$Y_t^{data} = \tilde{H}\tilde{\xi}_t + w_t, \ w_t \tilde{\ } iid, \ Ew_t w_t' = R.$$

- Here, $\tilde{H}$ selects elements of $\tilde{\xi}_t$ needed to construct Y_t^{data}
  - can easily handle distinction between whether quarterly data represent monthly averages (as in flow variables), or point-in-time observations on one month in the quarter (as in stock variables).
- Can use Kalman filter to forecast ('nowcast') current quarter data based on first month's (day's, week's) observations.

Connection Between DSGE's and VAR's

- Fernandez-Villaverde, Rubio-Ramirez, Sargent, Watson Result
- Vector Autoregression

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + ... + u_t,$$

  where u_t is iid.
- 'Matching impulse response functions' strategy for building DSGE models fits VARs and assumes u_t are a rotation of economic shocks (for details, see later notes).
- Can use the state space, observer representation to assess this assumption from the perspective of a DSGE.

Connection Between DSGE's and VAR's

- System (ignoring constant terms and measurement error):

$$\text{('State equation')}\ \xi_t = F\xi_{t-1} + D\epsilon_t,\ D = \begin{pmatrix} B \\ 0 \\ I \end{pmatrix},$$

$$\text{('Observer equation')}\ Y_t = H\xi_t.$$

- Substituting:

$$Y_t = HF\xi_{t-1} + HD\epsilon_t$$

- Suppose HD is square and invertible. Then

$$\epsilon_t = (HD)^{-1} Y_t - (HD)^{-1} HF\xi_{t-1}\ (**)$$

Substitute latter into the state equation:

$$\begin{aligned} \xi_t &= F\xi_{t-1} + D(HD)^{-1} Y_t - D(HD)^{-1} HF\xi_{t-1} \\ &= \left[I - D(HD)^{-1} H\right] F\xi_{t-1} + D(HD)^{-1} Y_t. \end{aligned}$$

Connection Between DSGE's and VAR's

We have:

$$\xi_t = M\xi_{t-1} + D\left(HD\right)^{-1} Y_t,\ M = \left[I - D\left(HD\right)^{-1} H\right] F.$$

If eigenvalues of M are less than unity,

$$\xi_t = D\left(HD\right)^{-1} Y_t + MD\left(HD\right)^{-1} Y_{t-1} + M^2 D\left(HD\right)^{-1} Y_{t-2} + ...$$

Substituting into $(**)$

$$\epsilon_t = \left(HD\right)^{-1} Y_t - \left(HD\right)^{-1} HF$$

$$\times \left[D\left(HD\right)^{-1} Y_{t-1} + MD\left(HD\right)^{-1} Y_{t-2} + M^2 D\left(HD\right)^{-1} Y_{t-3} + ...\right].$$

or,

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + ... \ + u_t,$$

where

$$u_t = HD\epsilon_t,\ B_j = HFM^{j-1} D\left(HD\right)^{-1},\ j = 1, 2, ...$$

- The latter is the VAR representation.

Connection Between DSGE's and VAR's

- The VAR repersentation is:

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + \dots + u_t,$$

  where

$$u_t = HD\epsilon_t,\ B_j = HFM^{j-1}D\left(HD\right)^{-1},\ j = 1, 2, \dots$$

- Notes:
  - ϵ_t is 'invertible' because it lies in space of current and past Y_t's.
  - VAR is *infinite*-ordered.
  - assumed system is 'square' (same number of elements in ϵ_t and Y_t). Sims-Zha (Macroeconomic Dynamics) show how to recover ϵ_t from current and past Y_t when the dimension of ϵ_t is greater than the dimension of Y_t.