Introducing Financial Frictions and Unemployment into a Small Open Economy Model

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1. Benchmark Small Open Economy ALLV Model

This manuscript describes the benchmark Adolfson, Laséen, Lindé, Villani, (2007) (AALV) model, and presents a way to introduce financial frictions into the accumulation and management of capital, and search and matching in the labor market. Our benchmark model makes the following small changes on the AALV model

- The price of investment goods is treated as a random variable with a unit root. Thus, growth in the model is driven by two independent unit root processes, one for neutral technology shocks and the other for technology shocks in the production of investment goods.
- Capital maintenance costs are deducted from capital income taxes, and physical depreciation is deducted at historic cost.
- The capital income tax rate is realized at the time the investment decision is made, not at the time when the payoff on investment is realized.
- Wages are indexed to the steady state growth rate of the economy, rather than to the current realization of technology shocks.
- All producers of specialized goods are assumed to require working capital loans.

1.1. Scaling of Variables

We adopt the following scaling of variables. The nominal exchange rate is denoted by S_t and its growth rate is s_t :

$$s_t = \frac{S_t}{S_{t-1}}.$$

The neutral shock to technology is z_t and its growth rate is $\mu_{z,t}$:

$$\frac{z_t}{z_{t-1}} = \mu_{z,t}.$$

The variable, Ψ_t, is an embodied shock to technology and it is convenient to define the following combination of embodied and neutral technology:

$$z_t^+ = \Psi_t^{\frac{\alpha}{1-\alpha}} z_t,\ \mu_{z^+,t} = \mu_{\Psi,t}^{\frac{\alpha}{1-\alpha}} \mu_{z,t}.$$

Capital is scaled by $z_t^+ \Psi_t$. Final investment goods, I_t, and domestically produced intermediate investment goods, I_t^d are also scaled by $z_t^+ \Psi_t$. For reasons explained below, imported investment goods, I_t^m, are scaled by a different factor, z_t^+. Consumption goods (C_t^m are imported intermediate consumption goods, C_t^d are domestically produced intermediate consumption goods and C_t are final consumption goods) are scaled by z_t^+. Government consumption, the real wage and real foreign assets are scaled by z_t^+. Exports (X_t^m are imported intermediate goods for use in producing exports and X_t are final export goods) are scaled z_t^+. Also, υ_t is the shadow value in utility terms to the household of domestic currency and $\upsilon_t P_t$ is the shadow value of one consumption good (i.e., the marginal utility of consumption). The latter must be multiplied by z_t^+ to induce stationarity. Thus,

$$\begin{aligned}
k_{t+1} &= \frac{K_{t+1}}{z_t^+ \Psi_t},\ \bar{k}_{t+1} = \frac{\bar{K}_{t+1}}{z_t^+ \Psi_t},\ i_t^d = \frac{I_t^d}{\Psi_t z_t^+},\ i_t = \frac{I_t}{z_t^+ \Psi_t},\ i_t^m = \frac{I_t^m}{z_t^+} \\
c_t^m &= \frac{C_t^m}{z_t^+}, c_t^d = \frac{C_t^d}{z_t^+},\ c_t = \frac{C_t}{z_t^+},\ g_t = \frac{G_t}{z_t^+},\ \bar{w}_t = \frac{W_t}{z_t^+ P_t},\ a_t \equiv \frac{S_t B_{t+1}^*}{P_t z_t^+}, \\
x_t^m &= \frac{X_t^m}{z_t^+},\ x_t = \frac{X_t}{z_t^+},\ \psi_{z^+,t} = \upsilon_t P_t z_t^+.
\end{aligned}$$

We define the scaled date t price of new installed physical capital for the start of period $t+1$ as $p_{k',t}$ and we defined the scaled real rental rate of capital as $\bar{r}_t^k$:

$$p_{k',t} = \Psi_t P_{k',t},\ \bar{r}_t^k = \Psi_t r_t^k.$$

where $P_{k',t}$ is in units of the domestic homogeneous good. We define the following inflation rates:

$$\begin{aligned}
\pi_t &= \frac{P_t}{P_{t-1}},\ \pi_t^c = \frac{P_t^c}{P_{t-1}^c},\ \pi_t^* = \frac{P_t^*}{P_{t-1}^*}, \\
\pi_t^i &= \frac{P_t^i}{P_{t-1}^i},\ \pi_t^x = \frac{P_t^x}{P_{t-1}^x},\ \pi_t^{m,j} = \frac{P_t^{m,j}}{P_{t-1}^{m,j}},
\end{aligned}$$

for $j = c, x, i$. Here, P_t is the price of a domestic homogeneous output good, P_t^c is the price of the domestic final consumption goods (i.e., the 'CPI'), P_t^* is the price of a foreign homogeneous good, P_t^i is the price of the domestic final investment good and P_t^x is the price (in foreign currency units) of a final export good.

With one exception, we define a lower case price as the corresponding uppercase price divided by the price of the homogeneous good. When the price is denominated in domestic currency units, we divide by the price of the domestic homogeneous good, P_t. When the price is denominated in foreign currency units, we divide by P_t^*, the price of the foreign homogeneous good. The exceptional case has to do with out handling of the price of investment goods, P_t^i. This grows at a rate slower than P_t, and we therefore scale it by P_t/Ψ_t. Thus,

$$\begin{aligned}
p_t^{m,x} &= \frac{P_t^{m,x}}{P_t},\ p_t^{m,c} = \frac{P_t^{m,c}}{P_t},\ p_t^{m,i} = \frac{P_t^{m,i}}{P_t}, \qquad (1.1) \\
p_t^x &= \frac{P_t^x}{P_t^*},\ p_t^c = \frac{P_t^c}{P_t},\ p_t^i = \frac{\Psi_t P_t^i}{P_t}.
\end{aligned}$$

Here, m, j means the price of an imported good which is subsequently used in the production of exports in the case $j = x$, in the production of the final consumption good in the case of $j = c$, and in the production of final investment goods in the case of $j = i$. When there is just a single superscript the underlying good is a final good, with $j = x, c, i$ corresponding to exports, consumption and investment, respectively.

We denote the real exchange rate by q_t :

$$q_t = \frac{S_t P_t^*}{P_t^c} \tag{1.2}$$

1.2. Firms

A homogeneous domestic good, Y_t, is produced using

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{1}{\lambda_{d,t}}} di \right]^{\lambda_{d,t}}, \ 1 \leq \lambda_{d,t} < \infty. \tag{1.3}$$

The domestic good is produced by a competitive, representative firm which takes the price of output, P_t, and the price of inputs, $P_{i,t}$, as given.

The i^{th} intermediate good producer has the following production function:

$$Y_{i,t} = (z_t H_{i,t})^{1-\alpha} \epsilon_t K_{i,t}^{\alpha} - z_t^+ \phi,$$

where $K_{i,t}$ denotes the labor services rented by the i^{th} intermediate good producer. Firms must borrow a fraction of the wage bill, so that one unit of labor costs

$$W_t R_t^f,$$

where

$$R_t^f = \nu_t^f R_t + 1 - \nu_t^f, \tag{1.4}$$

where W_t is the aggregate wage rate, R_t is the interest rate on working capital loans, and ν_t^f corresponds to the fraction that must be financed in advance.

The firm's marginal cost, divided by the price of the homogeneous good is denoted by mc_t :

$$\begin{aligned} mc_t &= \frac{\left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \left(r_t^k P_t\right)^{\alpha} \left(W_t R_t^f\right)^{1-\alpha} \frac{1}{\epsilon_t}}{z_t^{1-\alpha} P_t} \\ &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \left(\bar{r}_t^k\right)^{\alpha} \left(\bar{w}_t R_t^f\right)^{1-\alpha} \frac{1}{\epsilon_t} \end{aligned} \tag{1.5}$$

where r_t^k is the nominal rental rate of capital scaled by P_t. Productive efficiency dictates that another expression for marginal cost must also be satisfied:

$$
\begin{aligned}
mc_t &= \frac{1}{P_t}\frac{W_t R_t^f}{MP_{l,t}} \\
&= \frac{1}{P_t}\frac{W_t R_t^f}{(1-\alpha)\, z_t^{1-\alpha}\left(k_{i,t} z_{t-1}^{+}\Psi_{t-1}/H_{i,t}\right)^{\alpha}} \\
&= \frac{\left(\mu_{\Psi,t}\right)^{\alpha}\bar{w}_t R_t^f}{(1-\alpha)\left(\frac{k_{i,t}}{\mu_{z^+,t}}/H_{i,t}\right)^{\alpha}} \qquad (1.6)
\end{aligned}
$$

The i^{th} firm is a monopolist in the production of the i^{th} good and so it sets its price. Price setting is subject Calvo frictions. With probability ξ_d the intermediate good firm cannot change its price, in which case,

$$P_{i,t} = \left(\pi_{t-1}\right)^{\kappa_d}\left(\bar{\pi}_t^c\right)^{1-\kappa_d} P_{i,t-1},$$

where κ_d is a parameter, π_{t-1} is the lagged inflation rate and $\bar{\pi}_t^c$ is the central bank's target inflation rate. With probability $1-\xi_d$ the firm can change its price. When we combine the optimization conditions of the $1-\xi_d$ intermediate good firms which can optimize their price with the usual cross-firm consistency condition on price, we obtain (after linearizing about steady state):

$$
\begin{aligned}
\hat{\pi}_t - \widehat{\bar{\pi}}_t^c &= \frac{\beta}{1+\kappa_d\beta}E_t\left(\hat{\pi}_{t+1} - \widehat{\bar{\pi}}_{t+1}^c\right) + \frac{\kappa_d}{1+\kappa_d\beta}\left(\hat{\pi}_{t-1} - \widehat{\bar{\pi}}_t^c\right) \qquad (1.7) \\
&\quad - \frac{\kappa_d\beta\left(1-\rho_\pi\right)}{1+\kappa_d\beta}\widehat{\bar{\pi}}_t^c \\
&\quad + \frac{1}{1+\kappa_d\beta}\frac{\left(1-\beta\xi_d\right)\left(1-\xi_d\right)}{\xi_d}\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right).
\end{aligned}
$$

The domestic intermediate output good is allocated among alternative uses as follows:

$$Y_t = G_t + C_t^d + \frac{1}{\Psi_t}\left[I_t^d + a\left(u_t\right)\bar{K}_t\right] + \int_0^1 X_{i,t}^d. \qquad (1.8)$$

Here, C_t^d denotes intermediate goods used (together with foreign consumption goods) to produce final household consumption goods. The term in square brackets corresponds to domestic investment expenditures. These are allocated to two activities. One, I_t^d, is used in combination with imported foreign investment goods to produce a final investment good which can be used to add to the physical stock of capital, $\bar{K}_t$. The other is $a\left(u_t\right)\bar{K}_t$ which is used for maintenance costs arising from the utilization of physical capital. Here, u_t denotes the utilization rate of capital, with capital services being defined by:

$$K_t = u_t \bar{K}_t.$$

We adopt the following functional form for a :

$$a(u) = 0.5\sigma_b\sigma_a u^2 + \sigma_b(1-\sigma_a)u + \sigma_b((\sigma_a/2)-1), \tag{1.9}$$

where σ_a and σ_b are the parameters of this function. Finally, the integral in (1.8) denotes domestic resources allocated to exports. The determination of consumption, investment and export demand is discussed below.

1.3. Exports and Imports

This section reviews the structure of imports and exports. Both activities involve Calvo price setting frictions, and so require the presence of market power. In each case, we follow the Dixit-Stiglitz strategy of introducing a range of specialized goods. This allows there to be market power without the counterfactual implication that there is a small number of firms in the export and import sector. Thus, exports involve a continuum of exporters, each of which is a monopolist which converts a homogeneous domestically produced good and a homogeneous good derived from imports into specialized exports. The exports are sold to foreign, competitive retailers which create a homogenous export good that is sold to foreign citizens.

In the case of imports, specialized domestic importers purchase a homogeneous foreign good, which they turn into a specialized input and sell to domestic retailers. There are three types of domestic retailers. One uses the specialized import goods to create the homogeneous good used as an input into the production of specialized exports. Another uses the specialized import goods to create an input used in the production of investment goods. The third type uses specialized imports to produce a homogeneous input used in the production of consumption goods.

We emphasize two features of this setup. First, before passing to final domestic users, imported goods must first be combined with domestic inputs. This is consistent with the view emphasized by Burstein and Rebelo and Burstein, Eichenbaum and Rebelo, that there are substantial distribution costs associated with imports. Second, say something about "pricing to market versus xx".

1.3.1. Exports

We assume there is a total demand by foreigners for domestic goods, which takes on the following form:

$$X_t = \left(\frac{P_t^x}{P_t^*}\right)^{-\eta_f} Y_t^*.$$

In scaled form, this is

$$x_t = (p_t^x)^{-\eta_f} y_t^* \tag{1.10}$$

Here, Y_t^* is foreign GDP and P_t^* is the foreign currency price of foreign homogeneous goods. Also, P_t^x is an index of export prices, whose determination is discussed below. The goods, X_t, are produced by a representative, competitive foreign retailer firm using specialized inputs as follows:

$$X_t = \left[\int_0^1 X_{i,t}^{\frac{1}{\lambda_{x,t}}} di \right]^{\lambda_{x,t}}. \tag{1.11}$$

Here, $X_{i,t}$, $i \in (0,1)$, are exports of specialized goods. The retailer that produces X_t takes its output price, P_t^x, and its input prices, $P_{i,t}^x$, as given. Optimization leads to the following demand for specialized exports:

$$X_{i,t} = \left(\frac{P_{i,t}^x}{P_t^x} \right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} X_t. \tag{1.12}$$

Combining (1.11) and (1.12), we obtain:

$$P_t^x = \left[\int_0^1 \left(P_{i,t}^x\right)^{\frac{1}{1-\lambda_{x,t}}} di \right]^{1-\lambda_{x,t}}.$$

The i^{th} specialized export is produced by a monopolist using the following technology:

$$X_{i,t} = \left[\omega_x^{\frac{1}{\eta_x}} \left(X_{i,t}^m\right)^{\frac{\eta_x-1}{\eta_x}} + (1-\omega_x)^{\frac{1}{\eta_x}} \left(X_{i,t}^d\right)^{\frac{\eta_x-1}{\eta_x}} \right]^{\frac{\eta_x}{\eta_x-1}},$$

where $X_{i,t}^m$ and $X_{i,t}^d$ are the i^{th} exporter's use of the imported and domestically produced goods, respectively. We derive the marginal cost associated with the CES production function from the multiplier associated with the Lagrangian representation of the cost minimization problem:

$$C = \min P_t^{m,x} R_t^x X_{i,t}^m + P_t R_t^x X_{i,t}^d + \lambda \left\{ X_{i,t} - \left[\omega_x^{\frac{1}{\eta_x}} \left(X_{i,t}^m\right)^{\frac{\eta_x-1}{\eta_x}} + (1-\omega_x)^{\frac{1}{\eta_x}} \left(X_{i,t}^d\right)^{\frac{\eta_x-1}{\eta_x}} \right]^{\frac{\eta_x}{\eta_x-1}} \right\},$$

where $P_t^{m,x}$ is the price of the homogeneous import good and P_t is the price of the homogenous domestic output good. The first order conditions are:

$$\begin{aligned} R_t^x P_t^{m,x} &= \lambda X_{i,t}^{\frac{1}{\eta_x}} \omega_x^{\frac{1}{\eta_x}} \left(X_{i,t}^m\right)^{\frac{-1}{\eta_x}} \\ R_t^x P_t &= \lambda X_{i,t}^{\frac{1}{\eta_x}} (1-\omega_x)^{\frac{1}{\eta_x}} \left(X_{i,t}^d\right)^{\frac{-1}{\eta_x}} \\ X_{i,t} &= \left[\omega_x^{\frac{1}{\eta_x}} \left(X_{i,t}^m\right)^{\frac{\eta_x-1}{\eta_x}} + (1-\omega_x)^{\frac{1}{\eta_x}} \left(X_{i,t}^d\right)^{\frac{\eta_x-1}{\eta_x}} \right]^{\frac{\eta_x}{\eta_x-1}} \end{aligned}$$

Use the first two conditions to solve for the inputs as a function of the exogenous variables and the multiplier:

$$
\begin{aligned}
\left(X_{i,t}^{m}\right)^{\frac{\eta_x-1}{\eta_x}} &= \frac{\lambda^{\eta_x-1} X_{i,t}^{\frac{\eta_x-1}{\eta_x}} \omega_x^{\frac{\eta_x-1}{\eta_x}}}{\left(R_t^x P_t^{m,x}\right)^{\eta_x-1}} \\
\left(X_{i,t}^{d}\right)^{\frac{\eta_x-1}{\eta_x}} &= \frac{\lambda^{\eta_x-1} X_{i,t}^{\frac{\eta_x-1}{\eta_x}} \left(1-\omega_x\right)^{\frac{\eta_x-1}{\eta_x}}}{\left(R_t^x P_t\right)^{\eta_x-1}}. \qquad (1.13)
\end{aligned}
$$

Substitute these into the production function, to get:

$$
X_{i,t} = \lambda^{\eta_x} X_{i,t} \left(\frac{\omega_x}{\left(R_t^x P_t^{m,x}\right)^{\eta_x-1}} + \frac{(1-\omega_x)}{\left(R_t^x P_t\right)^{\eta_x-1}} \right)^{\frac{\eta_x}{\eta_x-1}}.
$$

Nominal marginal cost is λ, so that real (in terms of the homogeneous final export good) marginal cost, mc_t^x, is

$$
mc_t^x = \frac{\lambda}{S_t P_t^x} = \frac{R_t^x}{S_t P_t^x} \left[\omega_x \left(P_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\left(P_t\right)^{1-\eta_x} \right]^{\frac{1}{1-\eta_x}},
$$

where

$$
R_t^x = \nu_t^x R_t + 1 - \nu_t^x. \qquad (1.14)
$$

We rewrite the expression for marginal cost to get it in terms of stationary variables

$$
mc_t^x = \frac{\lambda}{S_t P_t^x} = \frac{R_t^x}{q_t p_t^c p_t^x} \left[\omega_x \left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x) \right]^{\frac{1}{1-\eta_x}}, \qquad (1.15)
$$

where we have used

$$
\frac{S_t P_t^x}{P_t} = \frac{S_t P_t^*}{P_t^c} \frac{P_t^c}{P_t} \frac{P_t^x}{P_t^*} = q_t p_t^c p_t^x. \qquad (1.16)
$$

The i^{th}, $i \in (0,1)$, domestic exporting firm takes (1.12) as its demand curve. This producer sets prices subject to a Calvo sticky-price mechanism. In a given period, $1-\xi_x$ producers can reoptimize their price and ξ_x cannot. The firms that cannot optimize price, do so as follows:

$$
P_{i,t}^x = \left(\pi_{t-1}^x\right)^{\kappa_x} \left(\pi^x\right)^{1-\kappa_x} P_{i,t-1}^x.
$$

This leads to the following Phillips curve for export prices:

$$
\begin{aligned}
\hat{\pi}_t^x &= \frac{\beta}{1+\kappa_x\beta} E_t \hat{\pi}_{t+1}^x + \frac{\kappa_x}{1+\kappa_x\beta} \hat{\pi}_{t-1}^x \qquad (1.17) \\
&\quad + \frac{1}{1+\kappa_x\beta} \frac{(1-\beta\xi_x)(1-\xi_x)}{\xi_x} \left(\widehat{mc}_t^x + \hat{\lambda}_{x,t} \right).
\end{aligned}
$$

The domestic resources used by specialized exporters are equal to:

$$
\int_0^1 X_{i,t}^d di,
$$

and this needs to be expressed in terms of aggregates. Rewriting (1.13), one of the first order conditions of the foreign retailer who purchases the specialized export goods:

$$X_{i,t}^{d} = \frac{\lambda^{\eta_x} X_{i,t} (1-\omega_x)}{(P_t)^{\eta_x}}.$$

Integrating this expression:

$$\begin{aligned}\int_0^1 X_{i,t}^d di &= \left(\frac{\lambda}{P_t}\right)^{\eta_x} (1-\omega_x) \int_0^1 X_{i,t} di \\ &= \left(\frac{\lambda}{P_t}\right)^{\eta_x} (1-\omega_x) X_t \frac{\int_0^1 \left(P_{i,t}^x\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} di}{\left(P_t^x\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}}}.\end{aligned}$$

Define $\bar{P}_t$, a linear homogeneous function of $P_{i,t}^x$:

$$\bar{P}_t = \left[\int_0^1 \left(P_{i,t}^x\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} di\right]^{\frac{\lambda_{x,t}-1}{-\lambda_{x,t}}}.$$

Then,

$$\bar{P}_t^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} = \int_0^1 \left(P_{i,t}^x\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} di,$$

and

$$\int_0^1 X_{i,t}^d di = \left(\frac{\lambda}{P_t}\right)^{\eta_x} (1-\omega_x) X_t \left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}}. \tag{1.18}$$

Recall the definition of the price index of exports,

$$P_t^x = \left[\int_0^1 \left(P_{i,t}^x\right)^{\frac{1}{1-\lambda_{x,t}}} di\right]^{1-\lambda_{x,t}},$$

so that $\bar{P}_t/P_t^x$ is the ratio of two linear homogeneous functions of $P_{i,t}^x$, where each weights $P_{i,t}^x$ for different $i \in (0,1)$ in different ways. As argued in a similar context by Yun ( ), $\bar{P}_t/P_t^x$ can be replaced by unity when studying the first order properties of this model about its steady state. This is guaranteed by our assumption about the form of the price updating equation, which implies that there are no price distortions in steady state, that it, that $\bar{P}_t/P_t^x = 1$ in steady state.

Substituting out for λ in (1.18), we obtain:

$$\begin{aligned}\int_0^1 X_{i,t}^d di &= \left(\frac{\lambda}{P_t}\right)^{\eta_x} (1-\omega_x) X_t \left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} \\ &= \left(mc_t^x q_t p_t^c p_t^x\right)^{\eta_x} (1-\omega_x) X_t \left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}}. \end{aligned} \tag{1.19}$$

Using (1.15), we obtain:

Substituting the latter into (1.19):

$$\begin{aligned} \int_0^1 X_{i,t}^d di &= \left(mc_t^x q_t p_t^c p_t^x\right)^{\eta_x} (1-\omega_x) X_t \left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} \\ &= \left(R_t^x\right)^{\eta_x} \left[\omega_x \left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}} (1-\omega_x) \left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t}-1}} \left(p_t^x\right)^{-\eta_f} Y_t^* \end{aligned} \tag{1.20}$$

1.3.2. Imports

We now turn to a discussion of imports. Foreign firms sell a homogeneous good to domestic importers. The importers convert the homogeneous good into a specialized input and supply that input monopolistically to domestic retailers. These importers are subject to Calvo price setting frictions. There are three types of importing firms: (i) one produces goods used to produce an intermediate good for the production of consumption, (ii) one produces goods used to produce an intermediate good for the production of investment goods, and (iii) one produces an intermediate good used for the production of an input into the production of export goods.

Consider (i) first. The production function is:

$$C_t^m = \left[\int_0^1 \left(C_{i,t}^m\right)^{\frac{1}{\lambda_t^{m,C}}} di\right]^{\lambda_t^{m,C}},$$

where $C_{i,t}^m$ is the output of the specialized producer and C_t^m is an intermediate good used in the production of consumption goods. Let $P_t^{m,c}$ denote the price index of C_t^m and let $P_{i,t}^{m,c}$ denote the price of the i^{th} intermediate input. The marginal cost, in domestic currency units, of the firm that produces $C_{i,t}^m$ is

$$S_t P_t^* R_t^{\nu,*}, \tag{1.21}$$

where

$$R_t^{\nu,*} = \nu_t^* R_t^* + 1 - \nu_t^*, \tag{1.22}$$

and R_t^* is the foreign nominal, intratemporal rate of interest. The notion here is that the firm must pay the inputs with foreign currency and because they have no resources themselves at the beginning of the period, they must borrow those resources if they are to buy the foreign inputs needed to produce $C_{i,t}^m$. There is no risk to this firm, because all shocks are realized at the beginning of the period, and so there is no uncertainty within the duration of the working capital loan about the realization of prices and exchanges rates. We are somewhat uncomfortable with this feature of the model. The fact that interest is due and matters indicates that some time evolves over the duration of the loan. Our assumption that no uncertainty is realized over a period of significant duration of time seems implausible. We suspect that a more realistic representation would involve some risk. Our timing assumptions in effect abstract away from this risk, and we conjecture that this does not affect the first order properties of the model.

Now consider (ii). The production function for, I_t^m, the intermediate good used in the production of investment goods is:

$$I_t^m = \left[\int_0^1 \left(I_{i,t}^m \right)^{\frac{1}{\lambda_t^{m,I}}} di \right]^{\lambda_t^{m,I}},$$

where $I_{i,t}^m$ is the output of the specialized producer. The marginal cost of the specialized producer is also (1.21). Note that we implicitly assume the importing firm's cost is P_t^* (before borrowing costs and exchange rate conversion), which is the same cost for the specialized inputs used to produce C_t^m. This may seem inconsistent with the property of the domestic economy that domestically produced consumption and investment goods have different relative prices. We assume that (1.21) applies to both types of producer in order to simplify notation. Below, we suppose that the efficiency of imported investment goods grows over time, in a way that makes our assumptions about the relative costs of consumption and investment, whether imported or domestically produced.

Now consider (iii). The production function for, X_t^m, the intermediate good used in the production of exports goods is:

$$X_t^m = \left[\int_0^1 \left(X_{i,t}^m \right)^{\frac{1}{\lambda_t^{m,X}}} di \right]^{\lambda_t^{m,X}}.$$

This importer is competitive, and takes the prices of X_t^m and $X_{i,t}^m$ as given. This importer's marginal cost is (1.21).

Each of the above three types of intermediate good firm is subject to Calvo price-setting frictions. With probability $1 - \xi_{m,j}$, the j^{th} type of firm can reoptimize its price and with probability $\xi_{m,j}$ it sets price according to the following relation:

$$P_{i,t}^{m,j} = \left(\pi_{t-1}^{m,j} \right)^{\kappa_{m,j}} \left(\widehat{\bar{\pi}}_t^c \right)^{1-\kappa_{m,j}} P_{i,t-1}^{m,j},$$

for $j = c, i, x$.

The usual Phillips curve argument applies to each of the above producers, so that,

$$\begin{aligned} \hat{\pi}_t^{m,j} - \widehat{\bar{\pi}}_t^c &= \frac{\beta}{1+\kappa_{m,j}\beta} E_t \left(\hat{\pi}_{t+1}^{m,j} - \widehat{\bar{\pi}}_{t+1}^c \right) + \frac{\kappa_{m,j}}{1+\kappa_{m,j}\beta} \left(\hat{\pi}_{t-1}^{m,j} - \widehat{\bar{\pi}}_t^c \right) \\ &\quad - \frac{\kappa_{m,j}\beta (1-\rho_\pi)}{1+\kappa_{m,j}\beta} \widehat{\bar{\pi}}_t^c \\ &\quad + \frac{1}{1+\kappa_{m,j}\beta} \frac{\left(1-\beta\xi_{m,j}\right)\left(1-\xi_{m,j}\right)}{\xi_{m,j}} \left(\widehat{mc}_t^{m,j} + \hat{\lambda}_t^{m,j} \right), \end{aligned} \tag{1.23}$$

for $j = c, i, x$. Real marginal cost is

$$mc_t^{m,j} = \frac{S_t P_t^*}{P_t^{m,j}} R_t^{\nu,*} = \frac{S_t P_t^* P_t^c P_t}{P_t^c P_t^{m,j} P_t} R_t^{\nu,*} = \frac{q_t p_t^c}{p_t^{m,j}} R_t^{\nu,*} \tag{1.24}$$

for $j = c, i, x$.

1.4. Households

Household preferences are given by:

$$E_0^j \sum_{t=0}^{\infty} \beta^t \left[\zeta_t^c \ln\left(C_t - bC_{t-1}\right) - \zeta_t^h A_L \frac{\left(h_{j,t}\right)^{1+\sigma_L}}{1+\sigma_L} \right], \tag{1.25}$$

where

$$C_t = \left[(1-\omega_c)^{\frac{1}{\eta_c}} \left(C_t^d\right)^{\frac{(\eta_c-1)}{\eta_c}} + \omega_c^{\frac{1}{\eta_c}} \left(C_t^m\right)^{\frac{(\eta_c-1)}{\eta_c}} \right]^{\frac{\eta_c}{\eta_c-1}}, \tag{1.26}$$

which (under competition) results in demand equations

$$C_t^d = (1-\omega_c)\left(p_t^c\right)^{\eta_c} C_t$$
$$C_t^m = \omega_m \left(p_t^{m,c}\right)^{\eta c} C_t.$$

The price of C_t is

$$P_t^c = \left[(1-\omega_c)\left(P_t\right)^{1-\eta_c} + \omega_c \left(P_t^{m,c}\right)^{1-\eta_c} \right]^{\frac{1}{1-\eta_c}}.$$

After dividing by P_t, this becomes

$$p_t^c = \left[(1-\omega_c) + \omega_c \left(p_t^{m,c}\right)^{1-\eta_c} \right]^{\frac{1}{1-\eta_c}}. \tag{1.27}$$

The rate of inflation of the consumption good is:

$$\pi_t^c = \frac{P_t^c}{P_{t-1}^c} = \pi_t \left[\frac{(1-\omega_c) + \omega_c \left(p_t^{m,c}\right)^{1-\eta_c}}{(1-\omega_c) + \omega_c \left(p_{t-1}^{m,c}\right)^{1-\eta_c}} \right]^{\frac{1}{1-\eta_c}}. \tag{1.28}$$

Households do the economy's investment, using the following technology:

$$I_t = \left[(1-\omega_i)^{\frac{1}{\eta_i}} \left(I_t^d\right)^{\frac{\eta_i-1}{\eta_i}} + \omega_i^{\frac{1}{\eta_i}} \left(\Psi_t I_t^m\right)^{\frac{\eta_i-1}{\eta_i}} \right]^{\frac{\eta_i}{\eta_i-1}}.$$

To obtain the demand for the two inputs we use the fact that this technology is operated by a representative, competitive firm which takes the output price, P_t^i, and the prices of I_t^d and

I_t^m, (P_t/Ψ_t) and $P_t^{m,i}$ respectively, as given. Profit maximization implies:

$$\begin{aligned}
P_t^i \left(\frac{I_t}{I_t^d}\right)^{\frac{1}{\eta_i}} (1-\omega_i)^{\frac{1}{\eta_i}} &= \frac{P_t}{\Psi_t} \\
P_t^i \left(\frac{I_t}{I_t^m}\right)^{\frac{1}{\eta_i}} \omega_i^{\frac{1}{\eta_i}} (\Psi_t)^{\frac{\eta_i-1}{\eta_i}} &= P_t^{m,i},
\end{aligned}$$

or,

$$\begin{aligned}
I_t^{\frac{\eta_i-1}{\eta_i}} \frac{(1-\omega_i)^{\frac{\eta_i-1}{\eta_i}}}{\left(\frac{P_t}{P_t^i\Psi_t}\right)^{\eta_i-1}} &= \left(I_t^d\right)^{\frac{\eta_i-1}{\eta_i}} \\
(\omega_i I_t)^{\frac{\eta_i-1}{\eta_i}} \left(\frac{\Psi_t}{P_t^{m,i}/P_t^i}\right)^{(\eta_i-1)} &= (\Psi_t I_t^m)^{\frac{\eta_i-1}{\eta_i}}.
\end{aligned}$$

Substituting these into the production function:

$$\begin{aligned}
I_t &= \left[(1-\omega_i)^{\frac{1}{\eta_i}} I_t^{\frac{\eta_i-1}{\eta_i}} (1-\omega_i)^{\frac{\eta_i-1}{\eta_i}} \left(\frac{P_t^i\Psi_t}{P_t}\right)^{\eta_i-1} + \omega_i^{\frac{1}{\eta_i}} (\omega_i I_t)^{\frac{\eta_i-1}{\eta_i}} \left(\frac{P_t^i\Psi_t}{P_t^{m,i}}\right)^{(\eta_i-1)}\right]^{\frac{\eta_i}{\eta_i-1}} \\
&= I_t \left(P_t^i\Psi_t\right)^{\eta_i} \left[(1-\omega_i)\left(\frac{1}{P_t}\right)^{\eta_i-1} + \omega_i \left(\frac{1}{P_t^{m,i}}\right)^{(\eta_i-1)}\right]^{\frac{\eta_i}{\eta_i-1}},
\end{aligned}$$

so that

$$\left(P_t^i\right)^{-\eta_i} = (\Psi_t)^{\eta_i} \left[(1-\omega_i)\left(\frac{1}{P_t}\right)^{\eta_i-1} + \omega_i \left(\frac{1}{P_t^{m,i}}\right)^{(\eta_i-1)}\right]^{\frac{\eta_i}{\eta_i-1}}$$

Dividing the last equation by P_t and rearranging:

$$p_t^i = \left[(1-\omega_i) + \omega_i \left(p_t^{m,i}\right)^{1-\eta_i}\right]^{\frac{1}{1-\eta_i}}. \tag{1.29}$$

Then, the inflation in the price of investment goods is:

$$\pi_t^i = \frac{\pi_t}{\mu_{\Psi,t}} \left[\frac{(1-\omega_i) + \omega_i \left(p_t^{m,i}\right)^{1-\eta_i}}{(1-\omega_i) + \omega_i \left(p_{t-1}^{m,i}\right)^{1-\eta_i}}\right]^{\frac{1}{1-\eta_i}}. \tag{1.30}$$

The law of motion of the physical stock of capital is:

$$\bar{K}_{t+1} = (1-\delta)\bar{K}_t + \Upsilon_t F(I_t, I_{t-1}),$$

where

$$F(I_t, I_{t-1}) = \left(1 - \tilde{S}\left(\frac{I_t}{I_{t-1}}\right)\right) I_t,$$

and

$$\begin{aligned}\tilde{S}(x) &= \frac{1}{2}\left\{\exp\left[\sqrt{\tilde{S}''}(x-\mu_{z^+}\mu_\Psi)\right]+\exp\left[-\sqrt{\tilde{S}''}(x-\mu_{z^+}\mu_\Psi)\right]-2\right\} \\ &= 0,\ x=\mu_{z^+}\mu_\Psi.\end{aligned}$$

Also,

$$\begin{aligned}\tilde{S}'(x) &= \frac{1}{2}\sqrt{\tilde{S}''}\left\{\exp\left[\sqrt{\tilde{S}''}(x-\mu_{z^+}\mu_\Psi)\right]-\exp\left[-\sqrt{\tilde{S}''}(x-\mu_{z^+}\mu_\Psi)\right]\right\} \\ &= 0,\ x=\mu_{z^+}\mu_\Psi.\end{aligned}$$

and

$$\begin{aligned}\tilde{S}''(x) &= \frac{1}{2}\tilde{S}''\left\{\exp\left[\sqrt{\tilde{S}''}(x-\mu_{z^+}\mu_\Psi)\right]+\exp\left[-\sqrt{\tilde{S}''}(x-\mu_{z^+}\mu_\Psi)\right]\right\} \\ &= \tilde{S}'',\ x=\mu_{z^+}\mu_\Psi.\end{aligned}$$

Also,

$$\begin{aligned}F_1(I_t,I_{t-1}) &= \left(1-\tilde{S}\left(\frac{I_t}{I_{t-1}}\right)\right)-\tilde{S}'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}} \\ &= 1,\ \frac{I_t}{I_{t-1}}=\mu_{z^+}\mu_\Psi,\end{aligned}$$

and,

$$\begin{aligned}F_2(I_t,I_{t-1}) &= \tilde{S}'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)^2 \\ &= 0,\ \frac{I_t}{I_{t-1}}=\mu_{z^+}\mu_\Psi.\end{aligned}$$

Scaling,

$$\begin{aligned}F(I_t,I_{t-1}) &= \left(1-\tilde{S}\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)\right)z_t^+\Psi_t i_t \\ F_1(I_t,I_{t-1}) &= \left(1-\tilde{S}\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)\right)-\tilde{S}'\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}} \\ F_2(I_t,I_{t-1}) &= \tilde{S}'\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)^2\end{aligned}$$

In this notation, the law of motion of capital is written,

$$\bar{k}_{t+1}z_t^+\Psi_t=(1-\delta)\bar{K}_t z_{t-1}^+\Psi_{t-1}+\Upsilon_t\left(1-\tilde{S}\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)\right)z_t^+\Psi_t i_t,$$

or,

$$\bar{k}_{t+1}=\frac{1-\delta}{\mu_{z^+,t}\mu_{\Psi,t}}\bar{k}_t+\Upsilon_t\left(1-\tilde{S}\left(\frac{\mu_{z^+,t}\mu_{\Psi,t}i_t}{i_{t-1}}\right)\right)i_t. \tag{1.31}$$

The household's first order conditions are as follows. The first order condition for consumption is:

$$\frac{\zeta_t^c}{c_t - bc_{t-1}\frac{1}{\mu_{z^+,t}}} - \beta b E_t \frac{\zeta_{t+1}^c}{c_{t+1}\mu_{z^+,t+1} - bc_t} - \psi_{z^+,t} p_t^c \left(1 + \tau_t^c\right) = 0. \tag{1.32}$$

To define the intertemporal Euler equation associated with the household's capital accumulation decision, we need to define the rate of return on a period t investment in a unit of physical capital, R_{t+1}^k :

$$R_{t+1}^k = \frac{\left(1-\tau_t^k\right)\left[u_{t+1} r_{t+1}^k - \frac{1}{\Upsilon_{t+1}} a(u_{t+1})\right] P_{t+1} + (1-\delta) P_{t+1} P_{k',t+1} + \tau_t^k \delta P_t P_{k',t}}{P_t P_{k',t}} \tag{1.33}$$

Here, $P_{k',t}$ denotes the price of a unit of newly installed physical capital, which operates in period $t+1$. This price is expressed in units of the homogeneous good, so that $P_t P_{k',t}$ is the domestic currency price of physical capital. The numerator in the expression for R_t^k represents the period $t+1$ payoff from a unit of additional physical capital. The timing of the capital tax rate reflects the assumption that the relevant tax rate is known at the time the investment decision is made. The expression in square brackets in (1.33) captures the idea that maintenance expenses associated with the operation of capital are deductible from taxes. The last expression in the numerator expresses the idea that physical depreciation is deductible at historical cost. It is convenient to express R_t^k in terms of scaled variables:

$$\begin{aligned} R_t^k &= \frac{P_{t+1}\Psi_{t+1}}{P_t \Psi_{t+1}} \frac{\left(1-\tau_t^k\right)\left[u_{t+1} r_{t+1}^k - \frac{1}{\Upsilon_{t+1}} a(u_{t+1})\right] + (1-\delta) P_{k',t+1} + \tau_t^k \delta \frac{P_t}{P_{t+1}} P_{k',t}}{P_{k',t}} \\ &= \pi_{t+1} \frac{\left(1-\tau_t^k\right)\left[u_{t+1} \bar{r}_{t+1}^k - a(u_{t+1})\right] + (1-\delta)\Psi_{t+1} P_{k',t+1} + \tau_t^k \delta \frac{P_t}{P_{t+1}} \Psi_{t+1} P_{k',t}}{\Psi_{t+1} P_{k',t}}. \end{aligned}$$

so that

$$R_{t+1}^k = \frac{\pi_{t+1}}{\mu_{\Psi,t+1}} \frac{\left(1-\tau_t^k\right)\left[u_{t+1} \bar{r}_{t+1}^k - a(u_{t+1})\right] + (1-\delta) p_{k',t+1} + \tau_t^k \delta \frac{\mu_{\Psi,t+1}}{\pi_{t+1}} p_{k',t}}{p_{k',t}}. \tag{1.34}$$

Capital is a good hedge against inflation, except for the way depreciation is treated. A rise in inflation effectively raises the tax rate on capital because of the practice of valuing depreciation at historical cost. The first order condition for capital implies:

$$\psi_{z^+,t} = \beta E_t \psi_{z^+,t+1} \frac{R_{t+1}^k}{\pi_{t+1}\mu_{z^+,t+1}},$$

or, after substituting out for R_{t+1}^k from (1.34):

$$\psi_{z^+,t} = \beta E_t \frac{\psi_{z^+,t+1}}{\mu_{z^+,t+1}\mu_{\Psi,t+1}} \frac{\left(1-\tau_t^k\right)\left[u_{t+1} \bar{r}_{t+1}^k - a(u_{t+1})\right] + (1-\delta) p_{k',t+1} + \tau_t^k \delta \frac{\mu_{\Psi,t+1}}{\pi_{t+1}} p_{k',t}}{p_{k',t}}. \tag{1.35}$$

We differentiate the Lagrangian representation of the household's problem as displayed in ALLV, p. 15 with respect to I_t :

$$-\upsilon_t P_t^i + \omega_t \Upsilon_t F_1(I_t, I_{t-1}) + \beta \omega_{t+1} \Upsilon_{t+1} F_2(I_{t+1}, I_t) = 0,$$

where υ_t denotes the multiplier on the household's nominal budget constraint and ω_t denotes the multiplier on the capital accumulation technology. In addition, the price of capital is the ratio of these multipliers:

$$P_t P_{k',t} = \frac{\omega_t}{\upsilon_t}.$$

Expressing the investment first order condition in terms of scaled variables,

$$\begin{aligned} -\frac{\psi_{z^+,t}}{z_t^+}\frac{p_t^i}{\Psi_t} + \upsilon_t P_t P_{k',t} \Upsilon_t \left[1 - \tilde{S}\left(\frac{\mu_{z^+,t}\mu_{\Psi,t} i_t}{i_{t-1}}\right) - \tilde{S}'\left(\frac{\mu_{z^+,t}\mu_{\Psi,t} i_t}{i_{t-1}}\right)\frac{\mu_{z^+,t}\mu_{\Psi,t} i_t}{i_{t-1}}\right] & \\ +\beta \upsilon_{t+1} P_{t+1} P_{k',t+1} \Upsilon_{t+1} \tilde{S}'\left(\frac{\mu_{z^+,t+1}\mu_{\Psi,t+1} i_{t+1}}{i_t}\right)\left(\frac{\mu_{z^+,t+1}\mu_{\Psi,t+1} i_{t+1}}{i_t}\right)^2 &= 0. \end{aligned}$$

Now multiply by $z_t^+ \Psi_t$

$$\begin{aligned} -\psi_{z^+,t} p_t^i + \psi_{z^+,t} p_{k',t} \Upsilon_t \left[1 - \tilde{S}\left(\frac{\mu_{z^+,t}\mu_{\Psi,t} i_t}{i_{t-1}}\right) - \tilde{S}'\left(\frac{\mu_{z^+,t}\mu_{\Psi,t} i_t}{i_{t-1}}\right)\frac{\mu_{z^+,t}\mu_{\Psi,t} i_t}{i_{t-1}}\right] & \\ +\beta \psi_{z^+,t+1} p_{k',t+1} \Upsilon_{t+1} \tilde{S}'\left(\frac{\mu_{z^+,t+1}\mu_{\Psi,t+1} i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2 \mu_{\Psi,t+1}\mu_{z^+,t+1} &= 0. \end{aligned} \tag{1.36}$$

Our first order condition for I_t appears to differ slightly from the first order condition in ALLV, equation (2.55), but the two actually coincide when we take into account the definition of f.

The first order condition associated with capital utilization is:

$$\Psi_t r_t^k = a'(u_t),$$

or, in scaled terms,

$$\bar{r}_t^k = a'(u_t). \tag{1.37}$$

The tax rate on capital income does not enter here because of the deductibility of maintenance costs. The first order condition associated with foreign bond holdings is:

$$\begin{aligned} -\psi_{z^+,t} + \beta E_t[\tfrac{\psi_{z^+,t+1}}{\mu_{z^+,t+1}\pi_{t+1}}(s_{t+1} R_t^* \Phi\left(a_t, E_t s_{t+1} s_t, \tilde{\phi}_t\right) \\ -\tau_{t+1}^k s_{t+1}\left(R_t^* \Phi\left(a_t, E_t s_{t+1} s_t, \tilde{\phi}_t\right) - 1\right) - \tau_{t+1}^k (s_{t+1} - 1))] = 0, \end{aligned} \tag{1.38}$$

where

$$\Phi\left(a_t, E_t s_{t+1} s_t, \tilde{\phi}_t\right) = \exp\left(-\tilde{\phi}_a (a_t - \bar{a}) - \tilde{\phi}_s (E_t s_{t+1} s_t - 1) + \tilde{\phi}_t\right). \tag{1.39}$$

This expression is zero in steady state. This reflects that in the model, S_t is constant in steady state and our assumption that $\tilde{\phi}_t$ is zero in steady state.

Note that the interest rate on foreign bonds acquired in period t is:

$$R_t^* \Phi\left(a_t, E_t s_{t+1} s_t, \tilde{\phi}_t\right).$$

Recall that R_t^* is the intratemporal rate of interest. The intertemporal rate of interest on bonds is different from R_t^* because there is uncertainty between the time that bonds are purchased and the time that they pay off. In the case of intratemporal loans (i.e., the working capital loans) there is no risk because no uncertainty is realized during the duration of the loan.

The Fisher equation is:

$$-\psi_{z^+,t} + \beta E_t\left[\frac{\psi_{z^+,t+1}}{\mu_{z^+,t+1}} \frac{R_t - \tau_{t+1}^k \left(R_t - 1\right)}{\pi_{t+1}}\right] = 0, \tag{1.40}$$

where R_t is the state non-contingent return on a domestic bond acquired in period t, which pays off in period $t+1$. Finally, we consider wage setting. We suppose that the specialized labor supplied by households is combined by labor contractors into a homogeneous labor service as follows:

$$H_t = \left[\int_0^1 \left(h_{j,t}\right)^{\frac{1}{\lambda_w}} dj\right]^{\lambda_w}, \ 1 \leq \lambda_w < \infty,$$

where h_j denotes the j^{th} household supply of labor services. Households are subject to Calvo wage setting frictions as in EHL. With probability $1-\xi_w$ the j^{th} household is able to reoptimize its wage and with probability ξ_w it sets its wage according to:

$$W_{j,t+1} = \left(\pi_t^c\right)^{\kappa_w} \left(\bar{\pi}_{t+1}^c\right)^{(1-\kappa_w)} \mu_{z^+} W_{j,t}. \tag{1.41}$$

If we combine the first order optimality condition of optimizing households with the cross household wage restriction, we obtain the familiar dynamic expression for the scaled wage rate:

$$\mathrm{E}_t \left[\begin{array}{c} \eta_0 \widehat{\bar{w}}_{t-1} + \eta_1 \widehat{\bar{w}}_t + \eta_2 \widehat{\bar{w}}_{t+1} + \eta_3\left(\hat{\pi}_t - \widehat{\bar{\pi}}_t^c\right) + \eta_4\left(\hat{\pi}_{t+1} - \rho_{\hat{\bar{\pi}}^c} \widehat{\bar{\pi}}_t^c\right) \\ +\eta_5\left(\hat{\pi}_{t-1}^c - \widehat{\bar{\pi}}_t^c\right) + \eta_6\left(\hat{\pi}_t^c - \rho_{\hat{\bar{\pi}}^c} \widehat{\bar{\pi}}_t^c\right) \\ +\eta_7 \hat{\psi}_{z^+,t} + \eta_8 \hat{H}_t + \eta_9 \hat{\tau}_t^y + \eta_{10} \hat{\tau}_t^w + \eta_{11} \hat{\zeta}_t^h \\ +\eta_{12} \hat{\mu}_{z^+,t} + \eta_{13} \hat{\mu}_{z^+,t+1} \end{array}\right] = 0, \tag{1.42}$$

where

$$b_w = \frac{\left[\lambda_w \sigma_L - \left(1 - \lambda_w\right)\right]}{\left[\left(1 - \beta \xi_w\right)\left(1 - \xi_w\right)\right]}$$

and

$$\begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \\ \eta_{10} \\ \eta_{11} \\ \eta_{12} \\ \eta_{13} \end{pmatrix} = \begin{pmatrix} b_w\xi_w \\ \left(\sigma_L\lambda_w - b_w\left(1+\beta\xi_w^2\right)\right) \\ b_w\beta\xi_w \\ -b_w\xi_w \\ b_w\beta\xi_w \\ b_w\xi_w\kappa_w \\ -b_w\beta\xi_w\kappa_w \\ (1-\lambda_w) \\ -(1-\lambda_w)\sigma_L \\ -(1-\lambda_w)\frac{\tau^y}{(1-\tau^y)} \\ -(1-\lambda_w)\frac{\tau^w}{(1+\tau^w)} \\ -(1-\lambda_w) \\ -b_w\xi_w \\ b_w\beta\xi_w \end{pmatrix}.$$

With one exception, this reduced form expression was obtained from ALLV, equation B.5. The exception stems from the fact that ALLV index the wage to current realized technology growth, while in our specification the wage is indexed to steady state technology growth. Our indexation strategy necessitates adding technology growth to the dynamic wage equation. In doing this, we followed the formula in the technical appendix of ACEL, which shows that technology growth appears in the manner indicated when the wage is not indexed to realized technology (and which also shows that technology growth does not appear in the event that there is full indexation). We did not re-derive this dynamic equation because both ALLV and ACEL address essentially the same environment.

1.5. Fiscal and Monetary Authorities

We suppose that the central bank pursues the following Taylor rule:

$$\begin{aligned} \widehat{R}_t &= \rho_R\widehat{R}_{t-1} + (1-\rho_R)\left[\widehat{\bar{\pi}}_t^c + \frac{r_\pi}{n}E_t\left[\left(\hat{\pi}_{t+1}^c - \widehat{\bar{\pi}}_{t+1}^c\right) + ... + \left(\hat{\pi}_{t+n}^c - \widehat{\bar{\pi}}_{t+n}^c\right)\right]\right. \\ &\quad \left. + r_y\hat{y}_{t-1} + r_q\hat{q}_{t-1}\right] + r_{\Delta\pi}\Delta\hat{\pi}_t^c + r_{\Delta y}\Delta\hat{y}_t + \varepsilon_{R,t}. \end{aligned} \tag{1.43}$$

The fiscal authorities have access to lump sum taxes which are used to redistribute the revenues from distortionary taxes, $\tau^c, \tau^y, \tau^w, \tau^k$, and raise revenues to cover government consumption, g_t.

1.6. Resource Constraints

The market clearing condition for the homogeneous domestic output good is, using (1.20),

$$\begin{aligned} Y_t &= G_t + C_t^d + \frac{1}{\Psi_t}\left[I_t^d + a(u_t)\bar{K}_t\right] \\ &+ \left(R_t^x\right)^{\eta_x}\left[\omega_x\left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}(1-\omega_x)\left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t-1}}}\left(p_t^x\right)^{-\eta_f} Y_t^*. \end{aligned}$$

and we need to express the last variable in terms of an observable. Applying the production function:

$$\begin{aligned} \left[\vartheta(\cdot)\left(z_t H_t\right)^{1-\alpha}\epsilon_t K_t^{\alpha} - z_t^+\phi\right] &= G_t + C_t^d + \frac{1}{\Psi_t}\left[I_t^d + a(u_t)\bar{K}_t\right] \\ + \left(R_t^x\right)^{\eta_x}\left[\omega_x\left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}&(1-\omega_x)\left(\frac{\bar{P}_t}{P_t^x}\right)^{\frac{-\lambda_{x,t}}{\lambda_{x,t-1}}}\left(p_t^x\right)^{-\eta_f} Y_t^*. \end{aligned}$$

where $\vartheta(\cdot)$ is a function of wage and price dispersion. Following the logic of Tak Yun, we ignore these in our analysis of the first order approximation of the model. So, the resource constraint that we work with is:

$$\begin{aligned} \left(z_t H_t\right)^{1-\alpha}\epsilon_t K_t^{\alpha} - z_t^+\phi &= G_t + C_t^d + \frac{1}{\Psi_t}\left[I_t^d + a(u_t)\bar{K}_t\right] \\ + \left(R_t^x\right)^{\eta_x}\left[\omega_x\left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}&(1-\omega_x)\left(p_t^x\right)^{-\eta_f} Y_t^*. \end{aligned}$$

We now express this in terms of stationary variables:

$$\begin{aligned} \left(z_t H_t\right)^{1-\alpha}\epsilon_t\left(k_t z_{t-1}^+\Psi_{t-1}\right)^{\alpha} - z_t^+\phi &= z_t^+ g_t + z_t^+ c_t^d + \frac{1}{\Psi_t}\left[z_t^+\Psi_t i_t^d + z_{t-1}^+\Psi_{t-1}a(u_t)\bar{k}_t\right] \\ &+ \left(R_t^x\right)^{\eta_x}\left[\omega_x\left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}(1-\omega_x)\left(p_t^x\right)^{-\eta_f} z_t^+ y_t^*, \end{aligned}$$

or, after division by z_t^+ :

$$\begin{aligned} y_t &= g_t + c_t^d + i_t^d \\ + \left(R_t^x\right)^{\eta_x}\left[\omega_x\left(p_t^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}&(1-\omega_x)\left(p_t^x\right)^{-\eta_f} y_t^*. \end{aligned} \tag{1.44}$$

where

$$y_t = \left(H_t\right)^{1-\alpha}\epsilon_t\left(\frac{k_t}{\mu_{z^+,t}\mu_{\Psi,t}}\right)^{\alpha} - \phi - \frac{1}{\mu_{z^+,t}\mu_{\Psi,t}}a(u_t)\bar{k}_t \tag{1.45}$$

$$k_t = \bar{k}_t u_t \tag{1.46}$$

We obtain the current account by combining the resource constraint, the government budget constraint (which says that $G = \mathit{taxes} + \mathit{seignorage}$) and the household's budget constraint. According to the resource constraint and the government budget constraint, household accumulation of foreign assets plus acquisition of foreign goods must equal foreign acquisition of domestic output:

$$S_tB^*_{t+1}-R^*_{t-1}\Phi\left(a_{t-1},\frac{E_{t-1}S_t}{S_{t-2}},\tilde{\phi}_{t-1}\right)S_tB^*_t+\exp enses \text{ on } imports_t = receipts \text{ from } \exp orts_t.$$

The expenses on imports is the amount spent by the three types of domestic importers of specialized import goods. These three types make zero profits, so that the amount that they spend on imports equals the receipts from their sales. Their sales are the sum of sales to households of consumption goods, C_t^m, sales to businesses of investment goods, I_t^m, and sales to domestic exporters, X_t^m :

$$\exp enses\ on\ imports_t = S_tP_t^*R_t^{\nu,*}\left(C_t^m+I_t^m+X_t^m\right).$$

A similar reasoning based on zero profits implies that

$$receipts \text{ from } \exp orts_t = S_tP_t^xX_t.$$

We conclude that the current account can be written as follows:

$$S_tB^*_{t+1}-R^*_{t-1}\Phi\left(a_{t-1},\frac{E_{t-1}S_t}{S_{t-2}},\tilde{\phi}_{t-1}\right)S_tB^*_t+S_tP_t^*R_t^{\nu,*}\left(C_t^m+I_t^m+X_t^m\right)=S_tP_t^xX_t.$$

Expressing this in scaled form,

$$a_tP_tz_t^+-R^*_{t-1}\Phi\left(a_{t-1},\frac{E_{t-1}S_t}{S_{t-2}},\tilde{\phi}_{t-1}\right)S_t\frac{P_{t-1}z_{t-1}^+a_{t-1}}{S_{t-1}}+S_tP_t^*z_t^+R_t^{\nu,*}\left(c_t^m+i_t^m+x_t^m\right)=z_t^+S_tP_t^xx_t.$$

Dividing by $P_tz_t^+$, we obtain

$$a_t-R^*_{t-1}\Phi\left(a_{t-1},E_{t-1}s_ts_{t-1},\tilde{\phi}_{t-1}\right)s_t\frac{a_{t-1}}{\pi_t\mu_{z^+,t}}+q_tp_t^cR_t^{\nu,*}\left(c_t^m+i_t^m+x_t^m\right)=q_tp_t^cp_t^xx_t, \quad (1.47)$$

using (1.16).

We define GDP as the sum of final consumption, investment, government consumption and net exports:

$$\text{nominal } GDP_t = P_t^cC_t+P_t^iI_t+P_tG_t+NX_t,$$

where NX_t denotes net exports and

$$NX_t = S_tP_t^xX_t - S_tP_t^*\left(C_t^m+I_t^m+X_t^m\right).$$

Gross exports is $S_t P_t^x X_t$ and gross imports is $S_t P_t^* \left(C_t^m + I_t^m + X_t^m \right)$. We model real GDP as nominal GDP divided by P_t. After scaling real GDP by z_t^+, we obtain:

$$c_t^m + i_t^m + x_t^m = \frac{p_t^c c_t + p_t^i i_t + g_t + q_t p_t^c p_t^x x_t - y_t}{q_t p_t^c},$$

after rearranging. We use this equation to substitute out for $c_t^m + i_t^m + x_t^m$ in (1.47),

$$\begin{aligned} & a_t - R_{t-1}^* \Phi \left(a_{t-1}, s_{t-1} E_{t-1} s_t, \tilde{\phi}_{t-1} \right) s_t \frac{a_{t-1}}{\pi_t \mu_{z^+,t}} + R_t^{\nu,*} \left[p_t^c c_t + p_t^i i_t + g_t + q_t p_t^c p_t^x x_t - y_t \right] \\ = \; & q_t p_t^c p_t^x x_t, \end{aligned} \tag{1.48}$$

In addition to the above equations we also include the restrictions across inflation rates implied by our relative price formulas. In terms of the expressions in (1.1) there are the restrictions implied by $p_t^{m,j}/p_{t-1}^{m,j}$, $j = x, c, i$, and p_t^x. The restrictions implied by the other two relative prices in (1.1), p_t^i and p_t^c, have already been exploited in (1.30) and (1.31), respectively. Finally, we also exploit the restriction across inflation rates implied by q_t/q_{t-1} and (1.2). Thus,

$$\frac{p_t^{m,x}}{p_{t-1}^{m,x}} = \frac{\pi_t^{m,x}}{\pi_t} \tag{1.49}$$

$$\frac{p_t^{m,c}}{p_{t-1}^{m,c}} = \frac{\pi_t^{m,c}}{\pi_t} \tag{1.50}$$

$$\frac{p_t^{m,i}}{p_{t-1}^{m,i}} = \frac{\pi_t^{m,i}}{\pi_t} \tag{1.51}$$

$$\frac{p_t^x}{p_{t-1}^x} = \frac{\pi_t^x}{\pi_t^*} \tag{1.52}$$

$$\frac{q_t}{q_{t-1}} = \frac{s_t \pi_t^*}{\pi_t^c}. \tag{1.53}$$

1.7. Solving the Model

In the previous section we derived 37 equations, which can be used to solve for the following 37 unknowns:

$$\begin{gathered} \bar{r}_t^k, \bar{w}_t, R_t^{\nu,*}, R_t^f, R_t^x, R_t, mc_t, mc_t^x, mc_t^{m,c}, mc_t^{m,i}, mc_t^{m,x}, \pi_t, \pi_t^x, \pi_t^c, \pi_t^i, \pi_t^{m,c}, \pi_t^{m,i}, \pi_t^{m,x}, \\ p_t^c, p_t^x, p_t^i, p_t^{m,x}, p_t^{m,c}, p_t^{m,i}, p_{k',t}, k_{t+1}, \bar{k}_{t+1}, u_t, H_t, q_t, i_t, c_t, x_t, a_t, s_t, \psi_{z^+,t}, y_t. \end{gathered}$$

1.7.1. Steady State

For our steady state calculations, we remove u_t from the list of unknowns and replace it with σ_b. We assign u_t a value of unity in steady state. With this replacement, we still have 37 equations in 37 unknowns.

In steady state, we lose the equations specified in terms of deviations from steady state. This includes the central bank policy rule, (1.43). We replace this equation with the restriction that π^c equals an exogenously specified number. We also lose the six Phillips curve relations, (1.7), (1.23), for $j = c, x, i$, equation (1.17) and equation (1.42). Each of these equations is replaced with the relevant 'price equals markup over marginal cost relation'. Thus, we remain with 37 equations and 37 unknowns in steady state.

In our system, π^* is an exogenous variable, and we suppose foreign policy implements $\pi^* = \pi^c$, so that

$$\pi^c = \pi^*, \ s = 1,$$

using (1.53). Equations (1.28), (1.30) and (1.49)-(1.53) imply:

$$\pi^{m,x} = \pi^{m,c} = \pi^{m,i} = \mu_\Psi \pi^i = \pi = \pi^c, \ \pi^x = \pi^*.$$

We obtain R from the steady state Fisher equation, (1.40) (the intertemporal Euler equation for bonds is the same in steady state):

$$R = \frac{\frac{\mu_{z+}\pi}{\beta} - \tau^k}{(1 - \tau^k)}.$$

Applying equations (1.38) and (1.39) in steady state, we obtain

$$1 = \beta \frac{1}{\mu_{z+}\pi} (R^* - \tau^k (R^* - 1)),$$

so that, in light of the steady state Fisher equation,

$$R^* = R.$$

Also,

$$\begin{aligned} R_t^{\nu,*} &= \nu^* R^* + 1 - \nu^* \\ R^f &= \nu^f R + 1 - \nu^f \\ R^x &= \nu^x R + 1 - \nu^x. \end{aligned}$$

In the case of the specialized domestic producers, we have

$$mc = \frac{1}{\lambda_d}.$$

We now describe our strategy for computing the steady state. We start by computing all the variables down to the equation for mc above. Now, fix a value for φ. In the case of the three types of price-setting monopolist in the import sector, we obtain, after rearranging,

$$p^{m,j} = \lambda^{m,j}\varphi R^{\nu,*},\ j = x, c, i. \tag{1.54}$$

We solve for p^c using the consumption price equation and $p^{m,c}$ obtained from (1.54):

$$p^c = \left[(1-\omega_c) + \omega_c \left(p^{m,c}\right)^{1-\eta_c}\right]^{\frac{1}{1-\eta_c}}. \tag{1.55}$$

We solve for p^i using the investment price equation:

$$p^i = \left[(1-\omega_i) + \omega_i \left(p^{m,i}\right)^{1-\eta_i}\right]^{\frac{1}{1-\eta_i}}, \tag{1.56}$$

and $p^{m,i}$ from (1.54).

Because $F_1 = 1$ and $F_2 = 0$ in steady state, the intertemporal Euler equation for investment reduces to:

$$p_{k'} = \frac{p^i}{\Upsilon},$$

giving us an expression for $p_{k'}$. The household's intertemporal Euler equation in steady state is:

$$p_{k'} = \beta\frac{1}{\mu_{z^+}\mu_\Psi}\left[(1-\tau^k)\bar{r}^k + (1-\delta)p_{k'} + \tau^k\delta\frac{\mu_\Psi}{\pi}p_{k'}\right]$$

which can be solved for $\bar{r}^k$:

$$\bar{r}^k = \frac{\frac{p_{k'}\mu_{z^+}\mu_\Psi}{\beta} - (1-\delta)p_{k'} - \tau^k\delta\frac{\mu_\Psi}{\pi}p_{k'}}{1-\tau^k}$$

The following expression can be used to solve for $\bar{w}$:

$$\frac{1}{\lambda_d} = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\left(\bar{r}^k\right)^{\alpha}\left(\bar{w}R^f\right)^{1-\alpha}\frac{1}{\epsilon}.$$

In particular:

$$\bar{w} = \frac{(1-\alpha)}{R^f}\left[\frac{\epsilon}{\lambda_d}\left(\frac{\bar{r}^k}{\alpha}\right)^{-\alpha}\right]^{\frac{1}{1-\alpha}} \tag{1.57}$$

From (1.6) we can solve for the steady state capital labor ratio, k/H :

$$mc = \frac{1}{\lambda_d} = \frac{\left(\mu_\Psi\right)^{\alpha}\bar{w}R^f}{(1-\alpha)\left(\frac{1}{\mu_{z^+}}\frac{\bar{k}}{H}\right)^{\alpha}},$$

or,

$$\frac{\bar{k}}{H} = \mu_{z^+}\left(\frac{\lambda_d\left(\mu_\Psi\right)^{\alpha}\bar{w}R^f}{1-\alpha}\right)^{1/\alpha} \tag{1.58}$$

Consider the specialized, monopolist exporters. In steady state, they are unconstrained by the Calvo price frictions, and they set their price as a markup over their marginal cost. That is, they set

$$mc^x = \frac{1}{\lambda^x},$$

or, from (1.15),

$$\frac{1}{\lambda^x} = \frac{R^x}{\varphi p^x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{1}{1-\eta_x}}.$$

Rearranging and using (1.54) to determine $p^{m,x}$, we obtain,

$$p^x = \lambda^x \frac{R^x}{\varphi}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{1}{1-\eta_x}}. \tag{1.59}$$

We solve for q using

$$q = \frac{\varphi}{p^c}. \tag{1.60}$$

In steady state,

$$\begin{aligned} y &= (H)^{1-\alpha}\epsilon\left(\frac{\bar{k}}{\mu_{z+}\mu_\Psi}\right)^\alpha - \phi \\ &= \frac{1}{\lambda^d}H\epsilon\left(\frac{\bar{k}/H}{\mu_{z+}\mu_\Psi}\right)^\alpha. \end{aligned} \tag{1.61}$$

Combining this with the steady state expression for the resource constraint:

$$\frac{1}{\lambda^d}H\epsilon\left(\frac{k/H}{\mu_{z+}\mu_\Psi}\right)^\alpha = g + c^d + i^d$$

$$+\left(R^x\right)^{\eta_x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}(1-\omega_x)\left(p^x\right)^{-\eta_f}y^*,$$

or, if

$$g = \eta_g y,$$

where η_g is a given number (danger: if you do an experiment and change taxes or inflation, you have to realize that if η_g constant, the experiment necessarily involves a simultaneous change in g). Then,

$$\frac{1-\eta_g}{\lambda^d}H\epsilon\left(\frac{k/H}{\mu_{z+}\mu_\Psi}\right)^\alpha = c^d + i^d$$

$$+\left(R^x\right)^{\eta_x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}(1-\omega_x)\left(p^x\right)^{-\eta_f}y^*,$$

The steady state demand for the intermediate consumption goods, I_t^d and C_t^d, after scaling is:

$$c^d = (1-\omega_c)\left(p^c\right)^{\eta_c}c \tag{1.62}$$

$$i^d = (1-\omega_i)\left(p^i\right)^{\eta_i}i, \tag{1.63}$$

so that, after substituting into the resource constraint:

$$\frac{1-\eta_g}{\lambda^d} H\epsilon \left(\frac{k/H}{\mu_{z+}\mu_\Psi}\right)^\alpha = (1-\omega_c)(p^c)^{\eta_c} c + (1-\omega_i)\left(p^i\right)^{\eta_i} \frac{\bar{k}}{H} \frac{\left[1-\frac{1-\delta}{\mu_{z+}\mu_\Psi}\right]}{\Upsilon} H \tag{1.64}$$
$$+ (R^x)^{\eta_x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}} (1-\omega_x)(p^x)^{-\eta_f} y^*.$$

As noted above, in the steady state we lose the Phillips curve expressions and we replace them by the condition that price is a markup over marginal cost. In the case of the household, this implies:

$$\frac{W_t(1-\tau^y)}{P_t(1+\tau^w)} = \lambda_w \frac{1}{\psi_t} \zeta_t^h A_L H_t^{\sigma_L},$$

or, after scaling this reduces, in steady state, to:

$$\frac{\bar{w}(1-\tau^y)}{1+\tau^w} = \lambda_w \frac{\zeta^h A_L H^{\sigma_L}}{\psi_{z^+}}, \tag{1.65}$$

The steady state expression for

$$\psi_{z^+} = \frac{1}{c} \frac{\zeta^c(\mu_{z^+} - \beta b)}{p^c(1+\tau^c)(\mu_{z^+} - b)}, \tag{1.66}$$

which after using this to substitute out for ψ_{z^+} in (1.65) yields, after rearranging:

$$c = \frac{H^{-\sigma_L}}{\lambda_w \zeta^h A_L} \frac{\zeta^c(\mu_{z^+} - \beta b)}{p^c(1+\tau^c)(\mu_{z^+} - b)} \frac{\bar{w}(1-\tau^y)}{1+\tau^w}. \tag{1.67}$$

Use this to substitute out for c in the resource constraint:

$$H\left\{\frac{1-\eta_g}{\lambda^d}\epsilon\left(\frac{k/H}{\mu_{z+}\mu_\Psi}\right)^\alpha - (1-\omega_i)\left(p^i\right)^{\eta_i}\frac{\bar{k}}{H}\frac{\left[1-\frac{1-\delta}{\mu_{z+}\mu_\Psi}\right]}{\Upsilon}\right\}$$
$$= \left[\frac{(1-\omega_c)(p^c)^{\eta_c}}{\lambda_w \zeta^h A_L} \frac{\zeta^c(\mu_{z^+} - \beta b)}{p^c(1+\tau^c)(\mu_{z^+} - b)} \frac{\bar{w}(1-\tau^y)}{1+\tau^w}\right] H^{-\sigma_L}$$
$$+ (R^x)^{\eta_x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x} + (1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}} (1-\omega_x)(p^x)^{-\eta_f} y^*.$$

Note that the expression on the left of the equality is linear function of H with positive slope (if the object in braces is not positive, then positive consumption is not feasible in steady state and the model parameterization chosen is not a good one). With $H = 0$ the object is zero, and as $H \to \infty$ the object goes to ∞. The right side goes to $+\infty$ as $H \to 0$ and to a finite number as $H \to \infty$. Because these functions are monotone and continuous, they must have a single crossing. This crossing is easily found by first looking for a sign switch on a grid for H and then assigning the zero-finding algorithm to an interval over with the sign switch occurs.

We consider two alternatives. In the first, we calibrate y^* so that it is some large multiple of y, say

$$y^* = \eta_y y.$$

In the second, we calibrate y^* so that net foreign assets, a, is zero in steady state.

Consider the first alternative. The resource constraint is:

$$\begin{aligned}
&[\frac{\epsilon}{\lambda^d}\left(\frac{k/H}{\mu_{z+}\mu_\Psi}\right)^\alpha \left\{1-\eta_g-(R^x)^{\eta_x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x}+(1-\omega_x)\right]^{\frac{\eta_x}{1-\eta_x}}(1-\omega_x)(p^x)^{-\eta_f}\eta_y\right\} \\
&\qquad -(1-\omega_i)\left(p^i\right)^{\eta_i}\frac{\bar{k}}{H}\frac{\left[1-\frac{1-\delta}{\mu_{z+}\mu_\Psi}\right]}{\Upsilon}]H \\
&= (1-\omega_c)(p^c)^{\eta_c}\frac{H^{-\sigma_L}}{\lambda_w\zeta^h A_L}\frac{\zeta^c(\mu_{z+}-\beta b)}{p^c(1+\tau^c)(\mu_{z+}-b)}\frac{\bar{w}(1-\tau^y)}{1+\tau^w} \\
&= BH^{-\sigma_L},
\end{aligned}$$

in obvious notation. Then,

$$H = \left(\frac{B}{A}\right)^{\frac{1}{1+\sigma_L}},$$

where A is the coefficient on H on the left of the above equality. Note that

$$AH = y - g - \text{homogeneous goods assigned to exports} - i^d \text{ and } BH^{-\sigma_L} = c^d.$$

As a result, by choosing H to solve the previous expression, we are enforcing clearing in the market for the intermediate homogeneous good, y.

We compute y using (1.61) and $\bar{k}$ using

$$\bar{k} = \frac{\bar{k}}{H}H.$$

We can obtain i from

$$i = \bar{k}\frac{\left[1-\frac{1-\delta}{\mu_{z+}\mu_\Psi}\right]}{\Upsilon}.$$

The variable, x, can be obtained from

$$x = (p^x)^{-\eta_f} y^* = (p^x)^{-\eta_f}\eta_y y.$$

We solve for ψ_{z+} from the intratemporal Euler equation:

$$\psi_{z+} = (1+\tau^w)\lambda_w\frac{\zeta^h A_L H^{\sigma_L}}{\bar{w}(1-\tau^y)}.$$

We can then solve for c from the steady state formula for the multiplier:

$$c = \frac{1}{\psi_{z+}}\frac{\zeta^c(\mu_{z+}-\beta b)}{p^c(1+\tau^c)(\mu_{z+}-b)},$$

and c^d, i^d can be obtained from (1.62) and (1.63), respectively. Compute

$$g = \eta_g y.$$

We adjust φ until trade is balanced in steady state:

$$\begin{aligned} R^{\nu,*}\left[p^c c + p^i i + g + qp^c p^x x - y\right] - qp^c p^x x &= 0, \\ a &= 0. \end{aligned} \tag{1.68}$$

We solve for σ_b to ensure (1.37) is satisfied for $u_t = 1$:

$$a'(u) = \sigma_b \sigma_a + \sigma_b (1 - \sigma_a) = \bar{r}^k,$$

so that

$$\sigma_b = \bar{r}^k.$$

2. Introducing Financial Frictions into the Model

A number of the activities in the model of the previous section require financing. Producers of specialized inputs must borrow working capital within the period. The management of capital involves financing because the construction of capital requires a substantial initial outlay of resources, while the return from capital comes in over time as a flow. In the model of the previous section financing requirements affect the allocations, but not very much. This is because none of the messy realities of actual financial markets are present. There is no asymmetric information between borrower and lender, there is no risk to lenders. In the case of capital accumulation, the borrower and lender are actually the same household, who puts up the finances and later reaps the rewards. When real-world financial frictions are introduced into a model, then intermediation becomes distorted by the presence of balance sheet constraints and other factors.

Although the literature shows how to introduce financial frictions much more extensively, here we proceed by assuming that only the accumulation and management of capital involves frictions. We will continue to assume that working capital loans are frictionless. At the end of this introduction, we briefly discuss the idea of introducing financial frictions into working capital loans. Our strategy of introducing frictions in the accumulation and management of capital follows the variant of the Bernanke, Gertler and Gilchrist (1999) model implemented in Christiano, Motto and Rostagno (2003) (CMR). The discussion here borrows heavily from the derivation in Christiano, Motto and Rostagno (2007).

The financial frictions we introduce reflect fundamentally that borrowers and lenders are different people, and that they have different information. Thus, we introduce 'entrepreneurs'. These are agents who have a special skill in the operation and management of capital.

Although these agents have their own financial resources, their skill in operating capital is such that it is optimal for them to operate more capital than their own resources can support, by borrowing additional funds. There is a financial friction because the management of capital is risky. Individual entrepreneurs are subject to idiosyncratic shocks which are observed only by them. The agents that they borrow from, 'banks', can only observe the idiosyncratic shocks by paying a monitoring cost. This type of asymmetric information implies that it is impractical to have an arrangement in which banks and entrepreneurs simply divide up the proceeds of entrepreneurial activity, because entrepreneurs have an incentive to understate their earnings. An alternative arrangement that is more efficient is one in which banks extend entrepreneurs a 'standard debt contract', which specifies a loan amount and a given interest payment. Entrepreneurs who suffer an especially bad idiosyncratic income shock and who therefore cannot afford to pay the required interest, are 'bankrupt'. Banks pay the cost of monitoring these entrepreneurs and take all of their net worth in partial compensation for the interest that they are owed.

The amount that banks are willing to lend to an entrepreneur under the standard debt contract is a function of the entrepreneur's net worth. This is how balance sheet constraints enter the model. When a shock occurs that reduces the value of the entrepreneur's assets, this cuts into their ability to borrow. As a result, they acquire less capital and this translates into a reduction in investment and ultimately into a slowdown in the economy.

The ultimate source of funds for lending to entrepreneurs is the household. The standard debt contracts extended by banks to entrepreneurs are financed by issuing liabilities to households. Although individual entrepreneurs are risky, banks themselves are not. We suppose that banks lend to a sufficiently diverse group of entrepreneurs that the uncertainty that exists in individual entrepreneurial loans washes out across all loans. Extensions of the model that introduce risk into banking have been developed, but it is not clear that the added complexity is justified.

In the model, the interest rate that households receive is nominally non state-contingent. This gives rise to potentially interesting wealth effects of the sort emphasized by Irving Fisher. For example, when a shock occurs which drives the price level down, households receive a wealth transfer. Because this transfer is taken from entrepreneurs, their net worth is reduced. With the tightening in their balance sheets, their ability to invest is reduced, and this produces an economic slowdown.

At the level of abstraction of the model, the capital stock includes both housing and business capital. As a result, the entrepreneurs can also be interpreted as households in their capacity of homeowners. An expanded version of the model would pull apart the household and business sectors to study each individually. Another straightforward expansion of the model would apply the model of financial frictions to working capital loans.

With this model, it is typically the practice to compare the net worth of entrepreneurs with a stock market quantity such as the Dow Jones Industrial Average. Whether this is really appropriate is uncertain. A case can be made that the 'bank loans' of entrepreneurs in the model correspond well with actual bank loans *plus* actual equity. It is well known that dividend payments on equity are very smooth. Firms work hard to accomplish this. For example, during the US Great Depression some firms were willing to sell their own capital in order to avoid cutting dividends. That this is so is perhaps not surprising. The asymmetric information problems with actual equity are surely as severe as they are for the banks in our model. Under these circumstances one might expect equity holders to demand a payment that is not contingent on the realization of uncertainty within the firm (payments could be contingent upon publically observed variables). Under this vision, the net worth in the model would correspond not to a measure of the aggregate stock market, but to the ownership stake of the managers and others who exert most direct control over the firm. The 'bank loans' in this model would, under this view of things, correspond to the actual loans of firms (i.e, bank loans and other loans such as commercial paper) plus the outstanding equity. While this is perhaps too extreme, these observations highlight that there is substantial uncertainty over exactly what variable should be compared with net worth in the model. It is important to emphasize, however, that whatever the right interpretation is of net worth, the model potentially captures balance sheet problems very nicely.

Finally, we make some remarks on the introduction of financial frictions into working capital loans. It is possible to accomplish this with relatively little modification to the model, by following the strategy described in Fisher (). However, with this strategy, the effects of financial frictions are quite modest, because the firms in the model which use working capital do not have assets. As a result, the balance sheet channel does not operate. We conjecture that for financial frictions in working capital to be interesting, the borrowing firms would need to have assets. One way this could be accomplished would be to assume that they use and own capital that is specific to their firm. In this way, fluctuations in the value of that capital induced by changes in asset prices would change their ability to borrow, and hence to produce. This strategy is algebra-intensive because of the fact that these firms also set their prices subject to Calvo frictions.

2.1. Modifying the Benchmark Model

The financial frictions bring a net increase of two equations over the equations in the model of the previous section. In addition, they introduce two new endogenous variables, one related to the interest rate paid by entrepreneurs as well as their net worth. The financial frictions also allow us to introduce two new random variables. We now provide a formal discussion of the model.

As we shall see, entrepreneurs all have different histories, as they experience different idiosyncratic shocks. Thus, in general, solving for the aggregate variables would require also solving for the distribution of entrepreneurs according to their characteristics and for the law of motion for that distribution. However, as emphasized in BGG, the right functional form assumptions have been made in the model, which guarantee the result that the aggregate variables associated with entrepreneurs are not a function of distributions. The loan contract specifies that all entrepreneurs, regardless of their net worth, receive the same interest rate. Also, the loan amount received by a entrepreneur is proportional to his level of net worth. These are enough to guarantee the aggregation result.

2.1.1. The Individual Entrepreneur

At the end of period t each entrepreneur has a level of net worth, N_{t+1}. The entrepreneur's net worth, N_{t+1}, constitutes his state at this time, and nothing else about his history is relevant. We imagine that there are many entrepreneurs for each level of net worth and that for each level of net worth, there is a competitive bank with free entry that offers a loan contract. The contract is defined by a loan amount and by an interest rate, and is derived as the solution to a particular optimization problem.

Consider a type of entrepreneur with a particular level of net worth, N_{t+1}. The entrepreneur combines this net worth with a bank loan, B_{t+1}, to purchase new, installed physical capital, $\bar{K}_{t+1}$, from capital producers. The loan the entrepreneur requires for this is:

$$B_{t+1} = P_t P_{k',t} \bar{K}_{t+1} - N_{t+1}. \tag{2.1}$$

The entrepreneur is required to pay a gross interest rate, Z_{t+1}, on the bank loan at the end of period $t+1$, if it is feasible to do so. After purchasing capital the entrepreneur experiences an idiosyncratic productivity shock which converts the purchased capital, $\bar{K}_{t+1}$, into $\bar{K}_{t+1}\omega$. Here, ω is a unit mean, lognormally and independently distributed random variable across entrepreneurs. The variance of $\log \omega$ is σ_t^2. The t subscript indicates that σ_t is itself the realization of a random variable. This allows us to consider the effects of an increase in the riskiness of individual entrepreneurs. We denote the cumulative distribution function of ω by F_t.

After observing the period $t+1$ shocks, the entrepreneur sets the utilization rate, u_{t+1}, of capital and rents capital out in competitive markets at nominal rental rate, $P_{t+1} r_{t+1}^k$. In choosing the capital utilization rate, the entrepreneur takes into account that operating one unit of physical capital at rate u_{t+1} requires $a(u_{t+1})$ of domestically produced investment goods for maintenance expenditures, where a is defined in (1.9). The entrepreneur then sells the undepreciated part of physical capital to capital producers. Per unit of physical capital purchased, the entrepreneur who draws idiosyncratic shock ω earns a return (after taxes), of

$R^k_{t+1}\omega$, where R^k_{t+1} is defined in (1.34). Because the mean of ω across entrepreneurs is unity, the average return across all entrepreneurs is R^k_{t+1}.

After entrepreneurs sell their capital, they settle their bank loans. At this point, the resources available to an entrepreneur who has purchased $\bar{K}_{t+1}$ units of physical capital in period t and who experiences an idiosyncratic productivity shock ω are $P_{t+1}P_{k',t+1}R^k_{t+1}\omega\bar{K}_{t+1}$. There is a cutoff value of ω, $\bar{\omega}_{t+1}$, such that the entrepreneur has just enough resources to pay interest:

$$\bar{\omega}_{t+1}R^k_{t+1}P_tP_{k',t}\bar{K}_{t+1} = Z_{t+1}B_{t+1}. \tag{2.2}$$

Entrepreneurs with $\omega < \bar{\omega}_{t+1}$ are bankrupt and turn over all their resources,

$$R^k_{t+1}\omega P_tP_{k',t}\bar{K}_{t+1},$$

to the bank, which is less than $Z_{t+1}B_{t+1}$. In this case, the bank monitors the entrepreneur, at cost

$$\mu R^k_{t+1}\omega P_tP_{k',t}\bar{K}_{t+1},$$

where $\mu \geq 0$ is a parameter.

We note briefly that the definition of R^k_{t+1} lacks some realism because it does not take into account the deductibility of interest payments. With the more realistic treatment of interest, the after tax rate of return on capital would be

$$\begin{aligned} R^k_{t+1} &= \frac{(1-\tau^k_t)\left[u_{t+1}r^k_{t+1} - \frac{1}{\Upsilon_{t+1}}a(u_{t+1}) - (Z_{t+1}-1)\frac{B_{t+1}}{P_tP_{k',t}\bar{K}_{t+1}}\right]P_{t+1} + (1-\delta)P_{t+1}P_{k',t+1} + \tau^k_t\delta P_tP_{k',t}}{P_tP_{k',t}} \\ &= \frac{(1-\tau^k_t)\left[u_{t+1}r^k_{t+1} - \frac{1}{\Upsilon_{t+1}}a(u_{t+1}) - \bar{\omega}_{t+1}R^k_{t+1} + \frac{B_{t+1}}{P_tP_{k',t}\bar{K}_{t+1}}\right]P_{t+1} + (1-\delta)P_{t+1}P_{k',t+1} + \tau^k_t\delta P_tP_{k',t}}{P_tP_{k',t}}, \end{aligned}$$

by (2.2). With this representation, R^k_t is a function of features of the loan contract. This will change the choice of optimal contract, discussed below. We plan to explore the implications of this in future work

Banks obtain the funds loaned in period t to entrepreneurs by issuing deposits to households at gross nominal rate of interest, R_t. The subscript on R_t indicates that the payoff to households in $t+1$ is not contingent on the period $t+1$ uncertainty. This feature of the relationship between households and banks is simply assumed. There is no risk in household bank deposits, and the household Euler equation associated with deposits is exactly the same as (1.40).

We suppose that there is competition and free entry among banks, and that banks participate in no financial arrangements other than the liabilities issued to households and the loans issued to entrepreneurs.[1] It follows that the bank's cash flow in each state of period

[1] If banks also had access to state contingent securities, then free entry and competition would imply that banks earn zero profits in an ex ante expected sense from the point of view of period t.

$t+1$ is zero, for each loan amount.[2] For loans in the amount, B_{t+1}, the bank receives gross interest, $Z_{t+1}B_{t+1}$, from the $1-F_t(\bar{\omega}_{t+1})$ entrepreneurs who are not bankrupt. The bank takes all the resources possessed by bankrupt entrepreneurs, net of monitoring costs. Thus, the state-by-state zero profit condition is:

$$[1-F_t(\bar{\omega}_{t+1})]\,Z_{t+1}B_{t+1}+(1-\mu)\int_0^{\bar{\omega}_{t+1}}\omega dF_t(\omega)\,R^k_{t+1}P_tP_{k',t}\bar{K}_{t+1}=R_tB_{t+1},$$

or, after making use of (2.2) and rearranging,

$$[\Gamma_t(\bar{\omega}_{t+1})-\mu G_t(\bar{\omega}_{t+1})]\,\frac{R^k_{t+1}}{R_t}\varrho_t=\varrho_t-1 \tag{2.3}$$

where

$$G_t(\bar{\omega}_{t+1})=\int_0^{\bar{\omega}_{t+1}}\omega dF_t(\omega).$$
$$\Gamma_t(\bar{\omega}_{t+1})=\bar{\omega}_{t+1}\,[1-F_t(\bar{\omega}_{t+1})]+G_t(\bar{\omega}_{t+1})$$
$$\varrho_t=\frac{P_tP_{k',t}\bar{K}_{t+1}}{N_{t+1}}.$$

The expression, $\Gamma_t(\bar{\omega}_{t+1})-\mu G_t(\bar{\omega}_{t+1})$ is the share of revenues earned by entrepreneurs that borrow B_{t+1}, which goes to banks. Note that $\Gamma_t'(\bar{\omega}_{t+1})=1-F_t(\bar{\omega}_{t+1})>0$ and $G_t'(\bar{\omega}_{t+1})=\bar{\omega}_{t+1}F_t'(\bar{\omega}_{t+1})>0$. It is thus not surprising that the share of entrepreneurial revenues accruing to banks is non-monotone with respect to $\bar{\omega}_{t+1}$. BGG argue that the expression on the left of (2.3) has an inverted 'U' shape, achieving a maximum value at $\bar{\omega}_{t+1}=\omega^*$, say. The expression is increasing for $\bar{\omega}_{t+1}<\omega^*$ and decreasing for $\bar{\omega}_{t+1}>\omega^*$. Thus, for any given value of ϱ_t and R^k_{t+1}/R_t, generically there are either no values of $\bar{\omega}_{t+1}$ or two that satisfy (2.3). The value of $\bar{\omega}_{t+1}$ realized in equilibrium must be the one on the left side of the inverted 'U' shape. This is because, according to (2.2), the lower value of $\bar{\omega}_{t+1}$ corresponds to a lower interest rate for entrepreneurs which yields them higher welfare. As discussed below, the equilibrium contract is one that maximizes entrepreneurial welfare subject to the zero profit condition on banks. This reasoning leads to the conclusion that $\bar{\omega}_{t+1}$ falls with a period $t+1$ shock that drives R^k_{t+1} up. The fraction of entrepreneurs that experience bankruptcy is $F_t(\bar{\omega}_{t+1})$, so it follows that a shock which drives up R^k_{t+1} has a negative contemporaneous impact on the bankruptcy rate. According to (1.33), shocks that drive R^k_{t+1} up include anything which raises the value of physical capital and/or the rental rate of capital.

As just noted, we suppose that the equilibrium debt contract maximizes entrepreneurial welfare, subject to the zero profit condition on banks and the specified required return on

[2] Absence of state contingent securities markets guarantee that cash flow is non-negative. Free entry guarantees that ex ante profits are zero. Given that each state of nature receives positive probability, the two assumptions imply the state by state zero profit condition quoted in the text.

household bank liabilities. The date t debt contract specifies a level of debt, B_{t+1} and a state $t+1-$contingent rate of interest, Z_{t+1}. We suppose that entrepreneurial welfare corresponds to the entrepreneur's expected wealth at the end of the contract. It is convenient to express welfare as a ratio to the amount the entrepreneur could receive by depositing his net worth in a bank:

$$\begin{aligned}
&\frac{E_t \int_{\bar{\omega}_{t+1}}^{\infty} \left[R_{t+1}^k \omega P_t P_{k',t} \bar{K}_{t+1} - Z_{t+1} B_{t+1}\right] dF_t(\omega)}{R_t N_{t+1}} \\
&= \frac{E_t \int_{\bar{\omega}_{t+1}}^{\infty} \left[\omega - \bar{\omega}_{t+1}\right] dF_t(\omega) R_{t+1}^k P_t P_{k',t} \bar{K}_{t+1}}{R_t N_{t+1}} \\
&= E_t \left\{ \left[1 - \Gamma_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} \right\} \varrho_t,
\end{aligned}$$

after making use of (2.1), (2.2) and

$$1 = \int_0^{\infty} \omega dF_t(\omega) = \int_{\bar{\omega}_{t+1}}^{\infty} \omega dF_t(\omega) + G_t(\bar{\omega}_{t+1}).$$

We can equivalently characterize the contract by a state-$t+1$ contingent set of values for $\bar{\omega}_{t+1}$ and a value of ϱ_t. The equilibrium contract is the one involving $\bar{\omega}_{t+1}$ and ϱ_t which maximizes entrepreneurial welfare (relative to $R_t N_{t+1}$), subject to the bank zero profits condition. The Lagrangian representation of this problem is:

$$\max_{\varrho_t, \{\bar{\omega}_{t+1}\}} E_t \left\{ \left[1 - \Gamma_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} \varrho_t + \lambda_{t+1} \left(\left[\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} \varrho_t - \varrho_t + 1 \right) \right\},$$

where λ_{t+1} is the Lagrange multiplier which is defined for each period $t+1$ state of nature. The first order conditions for this problem are:

$$\begin{aligned}
E_t \left\{ \left[1 - \Gamma_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} + \lambda_{t+1} \left(\left[\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} - 1 \right) \right\} &= 0 \\
-\Gamma_t'(\bar{\omega}_{t+1}) \frac{R_{t+1}^k}{R_t} + \lambda_{t+1} \left[\Gamma_t'(\bar{\omega}_{t+1}) - \mu G_t'(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} &= 0 \\
\left[\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} \varrho_t - \varrho_t + 1 &= 0,
\end{aligned}$$

where the absence of λ_{t+1} from the complementary slackness condition reflects that we assume $\lambda_{t+1} > 0$ in each period $t+1$ state of nature. Substituting out for λ_{t+1} from the second equation into the first, the first order conditions reduce to:

$$\begin{aligned}
E_t \left\{ \left[1 - \Gamma_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} + \frac{\Gamma_t'(\bar{\omega}_{t+1})}{\Gamma_t'(\bar{\omega}_{t+1}) - \mu G_t'(\bar{\omega}_{t+1})} \left(\left[\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1})\right] \frac{R_{t+1}^k}{R_t} - 1 \right) \right\} &= 0 \quad (2.4) \\
\left[\Gamma_{t-1}(\bar{\omega}_t) - \mu G_{t-1}(\bar{\omega}_t)\right] \frac{R_t^k}{R_{t-1}} \varrho_{t-1} - \varrho_{t-1} + 1 &= 0,
\end{aligned}$$

for $t = 0, 1, 2, \ldots$.

Since N_{t+1} does not appear in the last two equations, we conclude that ϱ_t and $\bar{\omega}_{t+1}$ are the same for all entrepreneurs, regardless of their net worth. The results for ϱ_t implies that

$$\frac{B_{t+1}}{N_{t+1}} = \varrho_t - 1,$$

that an entrepreneur's loan amount is proportional to his net worth. Rewriting (2.1) and (2.2) we see that the rate of interest paid by the entrepreneur is

$$Z_{t+1} = \frac{\bar{\omega}_{t+1} R_{t+1}^k}{1 - \varrho_t}, \tag{2.5}$$

which is the same for all entrepreneurs, regardless of their net worth.

2.1.2. Aggregation Across Entrepreneurs and the Risk Premium

Let $f(N_{t+1})$ denote the density of entrepreneurs with net worth, N_{t+1}. Then, aggregate average net worth, $\bar{N}_{t+1}$, is

$$\bar{N}_{t+1} = \int_{N_{t+1}} N_{t+1} f(N_{t+1})\, dN_{t+1}.$$

We now derive the law of motion of $\bar{N}_{t+1}$. Consider the set of entrepreneurs who in period $t-1$ had net worth N. Their net worth after they have settled with the bank in period t is denoted V_t^N, where

$$V_t^N = R_t^k P_{t-1} P_{k',t-1} \bar{K}_t^N - \Gamma_{t-1}(\bar{\omega}_t) R_t^k P_{t-1} P_{k',t-1} \bar{K}_t^N, \tag{2.6}$$

where $\bar{K}_t^N$ is the amount of physical capital that entrepreneurs with net worth N_t acquired in period $t-1$. Clearing in the market for capital requires:

$$\bar{K}_t = \int_{N_t} \bar{K}_t^N f(N_t)\, dN_t.$$

Integrating (2.6) over all entrepreneurs,

$$\begin{aligned}
V_t &\equiv \int_{N_t} N_t f(N_t)\, dN_t = \int_{N_t} f(N_t) \left\{ R_t^k P_{t-1} P_{k',t-1} \bar{K}_t^N - \Gamma_{t-1}(\bar{\omega}_t) R_t^k P_{t-1} P_{k',t-1} \bar{K}_t^N \right\} dN_t \\
&= \int_{N_t} f(N_t) \left\{ R_t^k \varrho_{t-1} N_t - \Gamma_{t-1}(\bar{\omega}_t) R_t^k \varrho_{t-1} N_t \right\} dN_t \\
&= R_t^k \varrho_{t-1} \bar{N}_t - \Gamma_{t-1}(\bar{\omega}_t) R_t^k \varrho_{t-1} \bar{N}_t.
\end{aligned}$$

Because ϱ_{t-1} is the same for all entrepreneurs, it follows that

$$\varrho_{t-1} = \frac{P_{t-1} P_{k',t-1} \bar{K}_t}{\bar{N}_t}, \tag{2.7}$$

so that

$$V_t = R_t^k P_{t-1} P_{k',t-1} \bar{K}_t - \Gamma_{t-1}(\bar{\omega}_t) R_t^k P_{t-1} P_{k',t-1} \bar{K}_t$$

Writing this out more fully:

$$\begin{aligned} V_t &= R_t^k P_{t-1} P_{k',t-1} \bar{K}_t - \left\{ [1 - F_{t-1}(\bar{\omega}_t)] \bar{\omega}_t + \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega) \right\} R_t^k P_{t-1} P_{k',t-1} \bar{K}_t \\ &= \left(1 + R_t^k\right) P_{t-1} P_{k',t-1} \bar{K}_t \\ &\quad - \left\{ [1 - F_{t-1}(\bar{\omega}_t)] \bar{\omega}_t + (1-\mu) \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega) + \mu \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega) \right\} R_t^k P_{t-1} P_{k',t-1} \bar{K}_t. \end{aligned}$$

Note that the first two terms in braces correspond to the net revenues of the bank, which must equal $R_t(P_{t-1} P_{k',t-1} \bar{K}_t - \bar{N}_t)$. Substituting:

$$V_t = R_t^k P_{t-1} P_{k',t-1} - \left\{ R_t + \frac{\mu \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega) \left(1 + R_t^k\right) Q_{\bar{K}',t-1} \bar{K}_t}{P_{t-1} P_{k',t-1} \bar{K}_t - \bar{N}_t} \right\} (P_{t-1} P_{k',t-1} \bar{K}_t - \bar{N}_t).$$

After V_t is determined, each entrepreneur faces an identical and independent probability $1 - \gamma_t$ of being selected to exit the economy. With the complementary probability, γ_t, each entrepreneur remains. Because the selection is random, the net worth of the entrepreneurs who survive is simply $\gamma_t \bar{V}_t$. A fraction, $1-\gamma_t$, of new entrepreneurs arrive. Entrepreneurs who survive or who are new arrivals receive a transfer, W_t^e. This ensures that all entrepreneurs, whether new arrivals or survivors that experienced bankruptcy, have sufficient funds to obtain at least some amount of loans. The average net worth across all entrepreneurs after the W_t^e transfers have been made and exits and entry have occurred, is $\bar{N}_{t+1} = \gamma_t \bar{V}_t + W_t^e$, or,

$$\bar{N}_{t+1} = \gamma_t \{ R_t^k P_{t-1} P_{k',t-1} \bar{K}_t - \left[R_t + \frac{\mu \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega) R_t^k P_{t-1} P_{k',t-1} \bar{K}_t}{P_{t-1} P_{k',t-1} \bar{K}_t - \bar{N}_t} \right] (P_{t-1} P_{k',t-1} \bar{K}_t - \bar{N}_t) \} + W_t^e. \tag{2.8}$$

2.2. Solving the Financial Frictions Model

In this subsection we indicate how the equilibrium conditions of the benchmark model must be modified to accommodate financial frictions. We then consider the problem of solving for the model's steady state.

2.2.1. Equilibrium Conditions

Consider the households. Households no longer accumulate physical capital, and the first order condition, (1.35), must be dropped. No other changes need to be made to the household first order conditions. Equation (1.40) can be interpreted as applying to the household's decision to make bank deposits. The household equations, (1.31) and (1.36), pertaining to investment can be thought of as reflecting that the household builds and sells physical capital, or it can be interpreted as the first order condition of many identical, competitive firms that build capital (note that each has a state variable in the form of lagged investment). We must add the three equations pertaining to the entrepreneur's loan contract: the law of motion of net worth, the bank's zero profit condition and the optimality condition. Finally, we must adjust the resource constraints to reflect the resources used in bank monitoring and in consumption by entrepreneurs.

We adopt the following scaling of variables:

$$n_{t+1} = \frac{\bar{N}_{t+1}}{P_t z_t^+},\; w_t^e = \frac{W_t^e}{P_t z_t^+}.$$

Dividing both sides of (2.8) by $P_t z_t^+$, we obtain the scaled law of motion for net worth:

$$n_{t+1} = \frac{\gamma_t}{\pi_t \mu_{z^+,t}} \left[R_t^k p_{k',t-1} \bar{k}_t - R_t \left(p_{k',t-1} \bar{k}_t - n_t \right) - \mu G_{t-1} \left(\bar{\omega}_t \right) R_t^k p_{k',t-1} \bar{k}_t \right] + w_t^e, \tag{2.9}$$

for $t = 0, 1, 2, \ldots$. Equation (2.9) has a simple intuitive interpretation. The first object in square brackets is the average gross return across all entrepreneurs in period t. The two negative terms correspond to what the entrepreneurs pay to the bank, including the interest paid by non-bankrupt entrepreneurs and the resources turned over to the bank by the bankrupt entrepreneurs. Since the bank makes zero profits, the payments to the bank by entrepreneurs must equal bank costs. The term involving R_t represents the cost of funds loaned to entrepreneurs by the bank, and the term involving μ represents the bank's total expenditures on monitoring costs.

The zero profit condition on banks, in terms of the scaled variables, is:

$$\Gamma_t(\bar{\omega}_{t+1}) - \mu G_t(\bar{\omega}_{t+1}) = \frac{R_t}{R_{t+1}^k} \left(1 - \frac{n_{t+1}}{p_{k',t} \bar{k}_{t+1}} \right), \tag{2.10}$$

for $t = -1, 0, 1, 2, \ldots$. The optimality condition for bank loans is (2.4).

The household's first order condition associated with the accumulation of capital, (1.35), must be dropped. The output equation, (1.45), does not have to be modified:

$$\begin{aligned} y_t &= g_t + c_t^d + i_t^d \\ &+ \left(R_t^x \right)^{\eta_x} \left[\omega_x \left(p_t^{m,x} \right)^{1-\eta_x} + (1 - \omega_x) \right]^{\frac{\eta_x}{1-\eta_x}} (1 - \omega_x) \left(p_t^x \right)^{-\eta_f} y_t^* + d_t, \end{aligned} \tag{2.11}$$

where

$$d_t = \frac{\mu G_{t-1}(\bar{\omega}_t) R_t^k p_{k',t-1} \bar{k}_t}{\pi_t \mu_{z^+,t}}.$$

Account has to be taken of the consumption by exiting entrepreneurs. The net worth of these entrepreneurs is $(1 - \gamma_t) V_t$ and we assume a fraction, $1 - \Theta$, is taxed and transferred in lump-sum form to households, while the complementary fraction, Θ, is consumed by the exiting entrepreneurs. This consumption can be taken into account by subtracting

$$\Theta\frac{1-\gamma_t}{\gamma_t}(n_{t+1}-w_t^e)z_t^+P_t$$

from the right side of (1.26). In practice we do not make this adjustment because we assume Θ is sufficiently small that the adjustment is negligible.

We now turn to the risk premium on entrepreneurs. The cost to the entrepreneur of internal funds (i.e., his own net worth) is the interest rate, R_t, which he loses by applying it to capital rather than just depositing it in the bank. The average payment by all entrepreneurs to the bank is the entire object in square brackets. So, the term involving μ represents the excess of external funds over the internal cost of funds. As a result, this is one measure of the risk premium in the model. Another is the excess of the interest rate paid by entrepreneurs who are not bankrupt, over R_t :

$$Z_{t+1}-R_t=\frac{\bar{\omega}_{t+1}R_{t+1}^k}{1-\frac{p_{k',t-1}\bar{k}_t}{n_t}},$$

according to (2.5) and (2.7).

The financial frictions brings a net increase of 2 equations (we add (2.4), (2.9) and (2.10), and delete (1.35)) and two variables, n_{t+1} and $\bar{\omega}_{t+1}$. This increases the size of our system to 39 equations in 39 variables. The financial frictions also introduce additional shocks, σ_t and γ_t.

2.2.2. Steady State

To solve for the steady state, start by proceding as in section 1.7.1, up to the point where $p_{k'}$ is calculated. Thus, we have

$$\pi^c=\pi^*,\ s=1,\ \pi^{m,x}=\pi^{m,c}=\pi^{m,i}=\mu_\Psi\pi^i=\pi=\pi^c,\ \pi^x=\pi^*,\ R=\frac{\frac{\mu_z+\pi}{\beta}-\tau^k}{1-\tau^k},\ R^*=R,$$

and

$$R_t^{\nu,*}=\nu^*R^*+1-\nu^*,\ R^f=\nu^fR+1-\nu^f,\ R^x=\nu^xR+1-\nu^x.$$

The household intertemporal Euler equation that appears after that is not part of the model with financial frictions, because households are no longer the ones accumulating capital.

Fix a value for $\varphi>0$. Solve (1.54) for $p^{m,x}$, $p^{m,c}$, $p^{m,i}$. Solve (1.55), (1.56) and (1.59) for p^c, p^i, and p^x, respectively. Solve for $p_{k'}$:

$$p_{k'}=\frac{p^i}{\Upsilon}.$$

Fix a value for $\bar{r}^k \in \left[\bar{r}_l^k, \bar{r}_u^k\right]$. Below, we adjust $\bar{r}^k$ until the expression defining the scaled Lagrange multiplier on the household budget constraint, (1.66), is satisfied as a strict equality. The bounds on $\bar{r}^k$ were selected to ensure that all the variables in the multiplier equation are well defined. We found some values of φ for which the interval, $\left[\bar{r}_l^k, \bar{r}_u^k\right]$, is empty. The lower bound, $\bar{r}_l^k$, is the least quantity greater than

$$\bar{r}^k_{\text{lower bound}} = \frac{\frac{\mu_\Psi}{\pi} p_{k'} R - (1-\delta) p_{k'} - \tau^k \delta \frac{\mu_\Psi}{\pi} p_{k'}}{1-\tau^k},$$

where we were able to find a solution to (2.12) below. The object, $\bar{r}^k_{\text{lower bound}}$ is the value of $\bar{r}^k$ such that $R^k/R = 1$. Our choice of upper bound was based on the numerical finding that n is monotonically increasing in $\bar{r}^k$. When $\bar{r}^k \to \bar{r}_u^k$, then we found that n goes to $+\infty$. For values of $\bar{r}^k > \bar{r}_u^k$ our formula for n below implies $n < 0$.

Given $\bar{r}^k \in \left[\bar{r}_l^k, \bar{r}_u^k\right]$ we compute R^k using the steady state version of (1.34):

$$R^k = \frac{\pi}{\mu_\Psi} \frac{(1-\tau^k)\bar{r}^k + (1-\delta)p_{k'} + \tau^k \delta \frac{\mu_\Psi}{\pi} p_{k'}}{p_{k'}}.$$

Consider the steady state version of equation (2.4):

$$[1-\Gamma(\bar{\omega})] \frac{R^k}{R} + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} \left([\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] \frac{R^k}{R} - 1 \right) = 0. \tag{2.12}$$

Note that in the limiting case, $\mu \to 0$, the only equilibrium is one in which $R^k = R$, and there is no wedge between these two variables. Interestingly, this model does not converge to the benchmark model in the limiting case. In the benchmark model, it is after tax R that is equated to R^k.

The log normal cdf, F, has two parameters, the mean and variance of the normal distribution of $\log \omega$. We solve for the two parameters and for $\bar{\omega}$ by imposing: (i) $E\omega = 1$, (ii) $F(\bar{\omega})$ (the bankruptcy rate) is equal to some specified calibrated value, and (iii) (2.12) holds. In practice, we conduct a nonlinear search in $\bar{\omega}$. For a given $\bar{\omega}$ we first compute the two parameters of F to satisfy (i) and (ii). We then evaluate (2.12). We adjust $\bar{\omega}$ until (2.12) is satisfied. Technical notes on these computations are provided at the end of this section.

Next, solve for the steady state value of $n/\bar{k}$ using the steady state version of (2.10):

$$\frac{n}{\bar{k}} = p_{k'} \left\{ 1 - \frac{R^k}{R} [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] \right\}.$$

Then, use the steady state law of motion of net worth, (2.9):

$$n = \frac{\gamma}{\pi \mu_{z^+}} \left[\left(R^k - R - \mu G(\bar{\omega}) R^k \right) p_{k'} \frac{\bar{k}}{n} + R \right] n + w^e,$$

to solve for n and $\bar{k}$:

$$\begin{aligned} n &= \frac{w^e}{1-\frac{\gamma}{\pi\mu_{z+}}\left[\left(R^k-R-\mu G\left(\bar{\omega}\right)R^k\right)p_{k'}\frac{\bar{k}}{n}+R\right]} \qquad (2.13) \\ \bar{k} &= \frac{\bar{k}}{n}n. \end{aligned}$$

Next, solve (1.57) for $\bar{w}$, (1.58) for H, (1.61) for y, and (1.65) for ψ_{z+}.

Consider the resource constraint, after g and y^* have been solved in terms of y :

$$\begin{aligned} & y\left[1-\eta_g-\left(R^x\right)^{\eta_x}\left[\omega_x\left(p^{m,x}\right)^{1-\eta_x}+\left(1-\omega_x\right)\right]^{\frac{\eta_x}{1-\eta_x}}\left(1-\omega_x\right)\left(p^x\right)^{-\eta_f}\eta_y\right] \\ = \; & d+\left(1-\omega_c\right)\left(p^c\right)^{\eta_c}c+\left(1-\omega_i\right)\left(p^i\right)^{\eta_i}\bar{k}\frac{\left[1-\frac{1-\delta}{\mu_{z+}\mu_\Psi}\right]}{\Upsilon}, \end{aligned}$$

and (1.62) and (1.63) have been used. Here,

$$d=\frac{\mu G(\bar{\omega})R^k p_{k'}\bar{k}}{\pi\mu_{z+}}.$$

This expression can be solved for c. Adjust $\bar{r}^k$ until (1.66) is satisfied. Adjust φ until the trade balance, (1.68) is satisfied.

We now provide some technical observations on the computations surrounding (2.4) and (2.12) and conditions (i)-(iii) above. We first develop straightforward expressions for computing

$$G(\bar{\omega})=\int_0^{\bar{\omega}}\omega dF(\omega),\ \Gamma\left(\bar{\omega}\right)=\bar{\omega}\left[1-F(\bar{\omega})\right]+G(\bar{\omega}),$$

and their derivatives for a given value of $\bar{\omega}$. Consider $G\left(\bar{\omega}\right)$ first. Let $x=\log\left(\omega\right)$, so that

$$\int_0^{\bar{\omega}}\omega dF(\omega)=\int_{-\infty}^{\log\bar{\omega}}e^x f\left(x\right)dx,$$

where f is the Normal density function. Writing this explicitly:

$$\begin{aligned} \int_0^{\bar{\omega}}\omega dF(\omega) &= \int_{-\infty}^{\log\bar{\omega}}e^x f\left(x\right)dx \\ &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}}e^x\exp^{\frac{-(x-Ex)^2}{2\sigma_x^2}}dx, \end{aligned}$$

where σ_x^2 is the variance of x. Now, $E\omega=1$ implies $Ex=-\left(1/2\right)\sigma_x^2$, so that

$$\begin{aligned} \int_0^{\bar{\omega}}\omega dF(\omega) &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}}e^x\exp^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}dx \\ &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}}\exp^{\frac{x2\sigma_x^2-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}dx \\ &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\log\bar{\omega}}\exp^{\frac{-\left(x-\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}}dx. \end{aligned}$$

Now, make the change of variable,

$$\begin{aligned} v &= \frac{x - \frac{1}{2}\sigma_x^2}{\sigma_x} = \frac{x + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x \\ \bar{v} &= \frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x \\ dv &= \frac{1}{\sigma_x} dx \end{aligned}$$

so that

$$\begin{aligned} \int_0^{\bar{\omega}} \omega dF(\omega) &= \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x} \exp^{\frac{-v^2}{2}} \sigma_x dv \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x} \exp^{\frac{-v^2}{2}} dv, \end{aligned}$$

or,

$$G(\bar{\omega}) = prob \left[v < \frac{\log(\bar{\omega}) + \frac{1}{2}\sigma_x^2}{\sigma_x} - \sigma_x \right],$$

for a standard normal variable, v. The expression on the right of the equality can be evaluated using the cdf of a standard normal distribution. Similarly, $\bar{\omega}[1 - F(\bar{\omega})]$ can be evaluated using the cdf of a log normal distribution, with log mean $-\sigma_x^2/2$ and log standard deviation, σ_x. Both these cdf's are available as part of standard computer packages.

By Leibniz's rule, $G'(\bar{\omega}) = \bar{\omega}F'(\bar{\omega})$, where F' is the pdf of a log normal. This pdf is available in standard computer packages. Regarding the derivative of Γ, note that

$$\begin{aligned} \Gamma'(\bar{\omega}) &= 1 - F(\bar{\omega}) - \bar{\omega}F'(\bar{\omega}) + G'(\bar{\omega}) \\ &= 1 - F(\bar{\omega}), \end{aligned}$$

which again is evaluated using the cdf of a log normal. We find the value of σ_x that ensures $F(\bar{\omega})$ equals some calibrated value by trying different values of σ_x and using a lognormal cdf to evaluate $F(\bar{\omega})$.

We also require

$$\frac{R^k}{R} [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] = 1 - \frac{n}{p_{k'}\bar{k}},$$

where

$$\Gamma(\bar{\omega}) = \bar{\omega}[1 - F(\bar{\omega})] + G(\bar{\omega}),$$

so that

$$\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) = \bar{\omega}[1 - F(\bar{\omega})] + (1 - \mu) G(\bar{\omega}).$$

To obtain the derivative, $F'(\bar{\omega})$, note

$$F(\bar{\omega}) = \int_0^{\bar{\omega}} dF(\omega) = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\log\bar{\omega}} \exp^{\frac{-\left(x+\frac{1}{2}\sigma_x^2\right)^2}{2\sigma_x^2}} dx.$$

We make a change of variable similar to the one above:

$$\begin{aligned} v &= \frac{x+\frac{1}{2}\sigma_x^2}{\sigma_x} \\ \bar{v} &= \frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x} \\ dv &= \frac{1}{\sigma_x}dx \end{aligned}$$

so that

$$\begin{aligned} F(\bar{\omega}) &= \int_0^{\bar{\omega}} dF(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x}} \exp^{\frac{-v^2}{2}} dv \\ &= prob\left[v < \frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x}\right] \end{aligned}$$

Differentiate with respect to $\bar{\omega}$:

$$\begin{aligned} F'(\omega) &= \frac{1}{\bar{\omega}\sigma_x}\frac{1}{\sqrt{2\pi}}\exp^{\frac{-\left[\frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x}\right]^2}{2}} \\ &= \frac{1}{\bar{\omega}\sigma_x}\text{Standard Normal pdf}\left(\frac{\log(\bar{\omega})+\frac{1}{2}\sigma_x^2}{\sigma_x}\right), \end{aligned}$$

where the first equality uses Leibniz's rule.

3. Introducing Unemployment into the Model

This section replaces the model of the labor market in our benchmark model with the search and matching framework of Mortensen and Pissarides (1994) and, more recently, Hall (2005a,b,c) and Shimer (2005a,b). We integrate the framework into our specific framework - which includes capital and monetary factors - following the version of the Gertler, Sala and Trigari (2006) (GST) strategy implemented in Christiano, Ilut, Motto, and Rostagno (2007). A key feature of the GST model is that there are wage-setting frictions, but they do have a direct impact on on-going worker employer relations. In this sense, the setup is not vulnerable to the Barro (1977) critique of sticky wages. The model is also attractive because of the richness of its labor market implications: the model differentiates between hours worked and the quantity of people employment, it has unemployment and vacancies.

The labor market in our alternative labor market model is a slightly modified version of the GST model. GST assume wage-setting frictions of the Calvo type, while we instead work with Taylor-type frictions. In addition, we adopt a slightly different representation of the production sector in order to maximize comparability with our benchmark model. In what follows, we first provide an overview and after that we present the detailed decision problems of agents in the labor market.

3.1. Sketch of the Model

As in the discussion of section 1.2, we adopt the Dixit-Stiglitz specification of homogeneous goods production. A representative, competitive retail firm aggregates differentiated intermediate goods into a homogeneous good. Intermediate goods are supplied by monopolists, who hire labor and capital services in competitive factor markets. The intermediate good firms are assumed to be subject to the same Calvo price setting frictions in the benchmark model.

In the benchmark model, the homogeneous labor services supplied to the competitive labor market by labor retailers (contractors) who combine the labor services supplied to them by households who monopolistically supply specialized labor services (see section 1.2). The modified model dispenses with the specialized labor services abstraction. Labor services are instead supplied to the homogeneous labor market by 'employment agencies'. The change leaves the equilibrium conditions associated with the production of the homogeneous good unaffected.[3]

Each employment agency retains a large number of workers. At the beginning of the period a fraction, ρ, of workers is randomly selected to separate from the firm and go into unemployment.[4] Also, a number of new workers arrive from unemployment in proportion to the number of vacancies posted by the agency in the previous period. After separation and new arrivals occur, the nominal wage rate is set.

The nominal wage paid to an individual worker is determined by Nash bargaining, which occurs once every N periods. Each employment agency is permanently allocated to one of N different cohorts. Cohorts are differentiated according to the period in which they renegotiate their wage. Since there is an equal number of agencies in each cohort, $1/N$ of the agencies bargain in each period. The wage in agencies that do not bargain in the current period is updated from the previous period according to the same rule used in our simple

[3] An alternative (perhaps more natural) formulation would be for the intermediate good firms to do their own employment search. We instead separate the task of finding workers from production of intermediate goods in order to avoid adding a state variable to the intermediate good firm, which would complicate the solution of their price-setting problem.

[4] We thus specify that the job separation rate is constant. This is consistent with the findings reported in Hall (2005b,c) and Shimer (2005a,b), who report that the job separation rate is relatively acyclical.

monetary model.

Once a wage rate is determined - whether by Nash bargaining or not - we assume that each matched worker-firm pair finds it optimal to proceed with the relationship in that period. In our calculations, we verify that this assumption is correct, by confirming that the wage rate in each worker-agency relationship lies inside the bargaining set associated with that relationship.

Next, the intensity of labor effort is determined according to a particular efficiency criterion. To explain this, we discuss the implications of increased intensity for the worker and for the employment agency. The utility function of the household in the present labor market model is a modified version of (1.25):

$$E_t \sum_{l=0}^{\infty} \beta^{l-t} \{ \zeta_{t+l}^c \log(C_{t+l} - bC_{t+l-1}) - \zeta_{t+l}^h A_L \frac{\varsigma_{t+l}^{1+\sigma_L}}{1+\sigma_L} L_{t+l} \}, \ \gamma, \omega > 0, \tag{3.1}$$

where L_t is the fraction of members of the household that are working and ς_t is the intensity with with each worker works. As in GST, we follow the family household construct of Merz (1995) in supposing that each household has a large number of workers. Although the individual worker's labor market experience - whether employed or unemployed - is determined in part by idiosyncratic shocks, the household has sufficiently many workers that the total fraction of workers employed, L_t, as well as the fractions allocated among the different cohorts, l_t^i, $i = 0, ..., N-1$, is the same for each household. We suppose that all the household's workers are supplied inelastically to the labor market (i.e., labor force participation is constant). Each worker passes randomly from employment with a particular agency to unemployment and back to employment according to exogenous probabilities described below.

The household's currency receipts arising from the labor market are:

$$(1 - L_t) P_t^c b^u z_t^+ + \sum_{i=0}^{N-1} W_t^i l_t^i \varsigma_t \tag{3.2}$$

where W_t^i is the nominal wage rate earned by workers in cohort $i = 0, ..., N-1$. The index, i, indicates the number of periods in the past when bargaining occurred most recently. Note that we implicitly assume that labor intensity is the same in each employment agency, regardless of cohort. This is explained below. The presence of the term involving b^u indicates the assumption that unemployed workers receive a payment of $b^u z_t^+$ final consumption goods. The unemployment benefits are financed by lump sum taxes.

Let the price of labor services, W_t, denote the marginal gain to the employment agency that occurs when an individual worker raises labor intensity by one unit. Because the employment agency is competitive in the supply of labor services, W_t is taken as given and is the same for all agencies, regardless of which cohort it is in. Labor intensity equates the worker's marginal cost to the agency's marginal benefit:

$$W_t = \zeta_t^h A_L \varsigma_t^{\sigma_L} \frac{1}{\upsilon_t}. \tag{3.3}$$

To understand the expression on the right of the equality, note that the marginal cost, in utility terms, to an individual worker who increases labor intensity by one unit is $\zeta_t^h A_L \varsigma_t^{\sigma_L}$. This is converted to currency units by dividing by the multiplier, υ_t, on the household's nominal budget constraint.

Labor intensity is the same for all cohorts because none of the variables in (3.3) is indexed by cohort. When the wage rate is determined by Nash bargaining, it is taken into account that labor intensity is determined according to (3.3).

Finally, the employment agency in the i^{th} cohort determines how many employees it will have in period $t+1$ by choosing vacancies, v_t^i. The vacancy posting costs associated with v_t^i are:

$$\frac{\kappa z_t^+}{2} \left(\frac{Q_t^\iota v_t^i}{l_t^i} \right)^2 l_t^i,$$

units of the domestic homogeneous good. Here, l_t^i denotes the number of employees in the i^{th} cohort and $\kappa z_t^+/2$ is a cost parameter which is assumed to grow at the same rate as the overall economic growth rate. Also, Q_t is the probability that a posted vacancy is filled. The functional form of our cost function nests GT and GST when $\iota = 1$. With this paramerization the cost function is in terms of the number of people hired, not the number of vacancies per se. We interpret this as reflecting that the GT and GST specifications emphasize internal costs (such as training and other) of adjusting the work force, and not search costs. In models used in the search literature (see, e.g., Shimer (AER)), vacancy posting costs are independent of Q_t, i.e., $\iota = 0$. We also plan to investigate this latter case. We suspect that the model implies less amplification in response to expansionary shock in the case, $\iota = 0$. In a boom, Q_t can be expected to fall, so that with $\iota = 1$, costs of posting vacancies decrease in the GT specification.

3.2. Model Details

An employment agency in the i^{th} cohort which does not renegotiate its wage in period t sets the period t wage, $W_{i,t}$, as in (1.41):

$$W_{i,t} = \tilde{\pi}_{w,t} \mu_{z^+} W_{i-1,t-1},\ \tilde{\pi}_{w,t} \equiv \left(\pi_{t-1}^c\right)^{\kappa_w} \left(\bar{\pi}_t^c\right)^{(1-\kappa_w)}, \tag{3.4}$$

for $i = 1, ..., N-1$ (note that an agency that was in the i^{th} cohort in period t was in cohort $i-1$ in period $t-1$). After wages are set, employment agencies in cohort i supply labor services, $l_t^i \varsigma_t$, into competitive labor markets. In addition, they post vacancies to attract new workers in the next period.

3.2.1. The Agency Problem

To understand how agencies bargain and how they make their employment decisions, it is useful to consider $F\left(l_t^0, \omega_t\right)$, the value function of the representative employment agency in the cohort that negotiates its wage in the current period. The arguments of F are the agency's workforce after beginning-of-period separations and new arrivals, l_t^0, and an arbitrary value for the nominal wage rate, ω_t. We are thus interested in the firm's problem after the wage rate has been set, when vacancy decisions remain to be made. To simplify notation, we leave out arguments of F that correspond to economy-wide variables. We find it convenient to adopt a change of variables. We suppose that the firm chooses a particular monotone transform of vacancy postings, which we denote by $\tilde{v}_t^i$:

$$\tilde{v}_t^i \equiv \frac{Q_t^\iota v_t^i}{l_t^i}.$$

The agency's hiring rate is related to $\tilde{v}_t^i$ by:

$$\chi_t^i = Q_t^{1-\iota} \tilde{v}_t^i. \tag{3.5}$$

In this notation, the agency's objective is to solve:

$$\begin{aligned} F\left(l_t^0, \omega_t\right) &= \sum_{j=0}^{N-1} \beta^j E_t \frac{\upsilon_{t+j}}{\upsilon_t} \max_{\tilde{v}_{t+j}^j} \left[\left(W_{t+j} - \Gamma_{t,j}\omega_t\right) \varsigma_{t+j} - P_{t+j} \frac{\kappa z_{t+j}^+}{2} \left(\tilde{v}_t^i\right)^2 \right] l_{t+j}^j \qquad (3.6) \\ &\quad + \beta^N E_t \frac{\upsilon_{t+N}}{\upsilon_t} F\left(l_{t+N}^0, \tilde{W}_{t+N}\right), \end{aligned}$$

where ς_t is assumed to satisfy (3.3). Here,

$$\Gamma_{t,j} = \begin{cases} \tilde{\pi}_{w,t+j} \cdots \tilde{\pi}_{w,t+1} \mu_{z^+}^j, & j > 0 \\ 1 & j = 0 \end{cases}. \tag{3.7}$$

Also, $\tilde{W}_{t+N}$ denotes the Nash bargaining wage rate that will be negotiated when the agency next has an opportunity to do so. At time t, the agency takes $\tilde{W}_{t+N}$ as given. The law of motion of an agency's work force is:

$$l_{t+1}^{i+1} = \left(\chi_t^i + \rho\right) l_t^i, \tag{3.8}$$

for $i = 0, 1, ..., N-1$, with the understanding here and throughout that $i = N$ is to be interpreted as $i = 0$. Expression (3.8) is deterministic, reflecting the assumption that the agency employs a large number of workers.

The firm chooses vacancies to solve the problem in (3.6). It is easy to verify:

$$F\left(l_t^0, \omega_t\right) = J\left(\omega_t\right) l_t^0, \tag{3.9}$$

where $J(\omega_t)$ is not a function of l_t^0. The function, $J(\omega_t)$, is the surplus that a firm bargaining in the current period enjoys from a match with an individual worker, when the current wage is ω_t. Let

$$\begin{aligned}
J(\omega_t) &= \max_{\{v_{t+j}^j\}_{j=0}^{N-1}} \{(W_t-\omega_t)\varsigma_t - P_t z_t^+ \frac{\kappa}{2}\left(\tilde{v}_t^0\right)^2 \\
&+\beta\frac{\upsilon_{t+1}}{\upsilon_t}\left[(W_{t+1}-\Gamma_{t,1}\omega_t)\varsigma_{t+1} - P_{t+1}z_{t+1}^+\frac{\kappa}{2}\left(\tilde{v}_{t+1}^1\right)^2\right]\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right) \\
&+\beta^2\frac{\upsilon_{t+2}}{\upsilon_t}\left[(W_{t+2}-\Gamma_{t,2}\omega_t)\varsigma_{t+2} - P_{t+2}z_{t+2}^+\frac{\kappa}{2}\left(\tilde{v}_{t+2}^2\right)^2\right]\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota}+\rho\right) \\
&+\dots+ \\
&+\beta^N\frac{\upsilon_{t+N}}{\upsilon_t}J\left(\tilde{W}_{t+N}\right)\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_t^{1-\iota}+\rho\right)\cdots\left(\tilde{v}_{t+N-1}^{N-1} Q_{t+N-1}^{1-\iota}+\rho\right)\}
\end{aligned}$$

Differentiate with respect to $\tilde{v}_t^0$ and multiply the result by $\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)/Q_t^{1-\iota}$, to obtain:

$$\begin{aligned}
0 &= -P_t z_t^+\kappa\tilde{v}_t^0\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)/Q_t^{1-\iota} \\
&+\beta\frac{\upsilon_{t+1}}{\upsilon_t}\left[(W_{t+1}-\Gamma_{t,1}\omega_t)\varsigma_{t+1} - P_{t+1}z_{t+1}^+\frac{\kappa}{2}\left(\tilde{v}_{t+1}^1\right)^2\right]\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right) \\
&+\beta^2\frac{\upsilon_{t+2}}{\upsilon_t}\left[(W_{t+2}-\Gamma_{t,2}\omega_t)\varsigma_{t+2} - P_{t+2}z_{t+2}^+\frac{\kappa}{2}\left(\tilde{v}_{t+2}^2\right)^2\right]\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota}+\rho\right) \\
&+\dots+ \\
&+\beta^N\frac{\upsilon_{t+N}}{\upsilon_t}J\left(\tilde{W}_{t+N}\right)\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_t^{1-\iota}+\rho\right)\cdots\left(\tilde{v}_{t+N-1}^{N-1} Q_{t+N-1}^{1-\iota}+\rho\right)\} \\
&= J(\omega_t) - (W_t-\omega_t)\varsigma_t + P_t z_t^+\frac{\kappa}{2}\left(\tilde{v}_t^0\right)^2 - P_t z_t^+\kappa\tilde{v}_t^0\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)/Q_t^{1-\iota}
\end{aligned}$$

Since the latter expression must be zero, we conclude:

$$\begin{aligned}
J(\omega_t) &= (W_t-\omega_t)\varsigma_t - P_t z_t^+\frac{\kappa}{2}\left(\tilde{v}_t^0\right)^2 + P_t z_t^+\kappa\tilde{v}_t^0\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)/Q_t^{1-\iota} \\
&= (W_t-\omega_t)\varsigma_t + P_t z_t^+\frac{\kappa}{2}\left(\tilde{v}_t^0\right)^2 + P_t z_t^+\kappa\tilde{v}_t^0\frac{\rho}{Q_t^{1-\iota}}
\end{aligned}$$

Next, we obtain simple expressions for the first order conditions associated with the vacancy decisions. Differentiate J with respect to $\tilde{v}_{t+1}^1$ and then multiply the result by

$$\left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota}+\rho\right)\frac{1}{Q_{t+1}^{1-\iota}}$$

to obtain:

$$\begin{aligned}
0 &= -\beta\frac{\upsilon_{t+1}}{\upsilon_t}P_{t+1}z_{t+1}^+\kappa\tilde{v}_{t+1}^1\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota}+\rho\right)\frac{1}{Q_{t+1}^{1-\iota}} \\
&+\beta^2\frac{\upsilon_{t+2}}{\upsilon_t}\left[(W_{t+2}-\Gamma_{t,2}\omega_t)\varsigma_{t+2} - P_{t+2}z_{t+2}^+\frac{\kappa}{2}\left(\tilde{v}_{t+2}^2\right)^2\right]\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota}+\rho\right) \\
&+\dots+ \\
&+\beta^N\frac{\upsilon_{t+N}}{\upsilon_t}J\left(\tilde{W}_{t+N}\right)\left(\tilde{v}_t^0 Q_t^{1-\iota}+\rho\right)\left(\tilde{v}_{t+1}^1 Q_t^{1-\iota}+\rho\right)\cdots\left(\tilde{v}_{t+N-1}^{N-1} Q_{t+N-1}^{1-\iota}+\rho\right)
\end{aligned}$$

Substituting this into the first order condition for v_t^0, we obtain:

$$\frac{P_t z_t^+ \kappa \tilde{v}_t^0}{Q_t^{1-\iota}} = \beta \frac{\upsilon_{t+1}}{\upsilon_t} \left[\left(W_{t+1} - \Gamma_{t,1} \omega_t \right) \varsigma_{t+1} + P_{t+1} z_{t+1}^+ \kappa \left(\frac{\left(\tilde{v}_{t+1}^1 \right)^2}{2} + \frac{\tilde{v}_{t+1}^1 \rho}{Q_{t+1}^{1-\iota}} \right) \right].$$

Differentiate J with respect to $\tilde{v}_{t+2}^2$ and then multiply the result by

$$\left(\tilde{v}_{t+2}^2 Q_{t+2}^{1-\iota} + \rho \right) \frac{1}{Q_{t+2}^{1-\iota}}.$$

$$\begin{aligned} 0 \;\; = \;\; & -\beta^2 \frac{\upsilon_{t+2}}{\upsilon_t} P_{t+2} z_{t+2}^+ \kappa \tilde{v}_{t+2}^2 \left(\tilde{v}_t^0 Q_t^{1-\iota} + \rho \right) \left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota} + \rho \right) \left(\tilde{v}_{t+2}^2 Q_{t+2}^{1-\iota} + \rho \right) \frac{1}{Q_{t+2}^{1-\iota}} \\ & +\beta^3 \frac{\upsilon_{t+3}}{\upsilon_t} \left[\left(W_{t+3} - \Gamma_{t,3} \omega_t \right) \varsigma_{t+3} - P_{t+3} z_{t+3}^+ \frac{\kappa}{2} \left(\tilde{v}_{t+3}^3 \right)^2 \right] \left(\tilde{v}_t^0 Q_t^{1-\iota} + \rho \right) \left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota} + \rho \right) \left(\tilde{v}_{t+2}^2 Q_{t+2}^{1-\iota} + \rho \right) \\ & +... + \\ & +\beta^N \frac{\upsilon_{t+N}}{\upsilon_t} J \left(\tilde{W}_{t+N} \right) \left(\tilde{v}_t^0 Q_t^{1-\iota} + \rho \right) \left(\tilde{v}_{t+1}^1 Q_t^{1-\iota} + \rho \right) \cdots \left(\tilde{v}_{t+N-1}^{N-1} Q_{t+N-1}^{1-\iota} + \rho \right) \} \end{aligned}$$

Using this to simplify the first order condition for $\tilde{v}_{t+1}^1$, we obtain

$$P_{t+1} z_{t+1}^+ \kappa \tilde{v}_{t+1}^1 \frac{1}{Q_{t+1}^{1-\iota}} = \beta \frac{\upsilon_{t+2}}{\upsilon_{t+1}} \left[\left(W_{t+2} - \Gamma_{t,2} \omega_t \right) \varsigma_{t+2} + P_{t+2} z_{t+2}^+ \kappa \left(\frac{\left(\tilde{v}_{t+2}^2 \right)^2}{2} + \frac{\tilde{v}_{t+2}^2 \rho}{Q_{t+2}^{1-\iota}} \right) \right]$$

We continue in this way and derive,

$$P_{t+j} z_{t+j}^+ \kappa \tilde{v}_{t+j}^j \frac{1}{Q_{t+j}^{1-\iota}} = \beta \frac{\upsilon_{t+j+1}}{\upsilon_{t+j}} \left[\left(W_{t+j+1} - \Gamma_{t,j+1} \omega_t \right) \varsigma_{t+j+1} + P_{t+j+1} z_{t+j+1}^+ \kappa \left(\frac{\left(\tilde{v}_{t+j+1}^{j+1} \right)^2}{2} + \frac{\tilde{v}_{t+j+1}^{j+1} \rho}{Q_{t+j+1}^{1-\iota}} \right) \right],$$

for $j = 0, 1, ..., N-2$. Now consider the derivative of J with respect to $\tilde{v}_{t+N-1}^{N-1}$, after multiplication by

$$\left(\tilde{v}_{t+N-1}^{N-1} Q_{t+N-1}^{1-\iota} + \rho \right) \frac{1}{Q_{t+N-1}^{1-\iota}}.$$

is,

$$\begin{aligned} 0 \;\; = \;\; & -\beta^{N-1} \frac{\upsilon_{t+N-1}}{\upsilon_t} P_{t+N-1} z_{t+N-1}^+ \kappa \tilde{v}_{t+N-1}^{N-1} \left(\tilde{v}_t^0 Q_t^{1-\iota} + \rho \right) \left(\tilde{v}_{t+1}^1 Q_{t+1}^{1-\iota} + \rho \right) \cdots \left(\tilde{v}_{t+N-2}^{N-2} Q_{t+N-2}^{1-\iota} + \rho \right) \left(\tilde{v}_{t+N-1}^{N-1} Q \right. \\ & +\beta^N \frac{\upsilon_{t+N}}{\upsilon_t} J \left(\tilde{W}_{t+N} \right) \left(\tilde{v}_t^0 Q_t^{1-\iota} + \rho \right) \left(\tilde{v}_{t+1}^1 Q_t^{1-\iota} + \rho \right) \cdots \left(\tilde{v}_{t+N-1}^{N-1} Q_{t+N-1}^{1-\iota} + \rho \right) \} \end{aligned}$$

or,

$$P_{t+N-1} z_{t+N-1}^+ \kappa \tilde{v}_{t+N-1}^{N-1} \frac{1}{Q_{t+N-1}^{1-\iota}} = \beta \frac{\upsilon_{t+N}}{\upsilon_{t+N-1}} J \left(\tilde{W}_{t+N} \right).$$

Making use of our expression for J, we obtain:

$$P_{t+N-1} z_{t+N-1}^+ \kappa \tilde{v}_{t+N-1}^{N-1} \frac{1}{Q_{t+N-1}^{1-\iota}} = \beta \frac{\upsilon_{t+N}}{\upsilon_{t+N-1}} \left[\left(W_{t+N} - \tilde{W}_{t+N} \right) \varsigma_{t+N} + P_{t+N} z_{t+N}^+ \kappa \left(\frac{\left(\tilde{v}_{t+N}^0 \right)^2}{2} + \frac{\tilde{v}_{t+N}^0 \rho}{Q_{t+N}^{1-\iota}} \right) \right].$$

The above first order conditions apply over time to a group of agencies that bargain at date t. We now express the first order conditions for a fixed date and different cohorts:

$$P_t z_t^+ \kappa \tilde{v}_t^j \frac{1}{Q_t^{1-\iota}} = \beta \frac{\upsilon_{t+1}}{\upsilon_t} \left[\left(W_{t+1} - \Gamma_{t-j,j+1} \tilde{W}_{t-j} \right) \varsigma_{t+1} + P_{t+1} z_{t+1}^+ \kappa \left(\frac{\left(\tilde{v}_{t+1}^{j+1}\right)^2}{2} + \frac{\tilde{v}_{t+1}^{j+1} \rho}{Q_{t+1}^{1-\iota}} \right) \right],$$

Divide both sides by $P_t z_t^+$ and express the result in terms of scaled variables:

$$\kappa \tilde{v}_t^j \frac{1}{Q_t^{1-\iota}} = \beta \frac{\psi_{z^+,t+1}}{\psi_{z^+,t}} \left[\left(\bar{w}_{t+1} - G_{t-j,j+1} w_{t-j} \bar{w}_{t-j} \right) \varsigma_{t+1} + \kappa \left(\frac{\left(\tilde{v}_{t+1}^{j+1}\right)^2}{2} + \frac{\tilde{v}_{t+1}^{j+1} \rho}{Q_{t+1}^{1-\iota}} \right) \right], \; j = 0, ..., N-2 \tag{3.10}$$

where

$$\begin{aligned} G_{t-i,i+1} &= \frac{\tilde{\pi}_{w,t+1} \cdots \tilde{\pi}_{w,t-i+1}}{\pi_{t+1} \cdots \pi_{t-i+1}} \left(\frac{\mu_{z^+}}{\mu_{z^+,t-i+1}} \right) \cdots \left(\frac{\mu_{z^+}}{\mu_{z^+,t+1}} \right), \; i \geq 0, \\ w_t &= \frac{\tilde{W}_t}{W_t}, \; \bar{w}_t = \frac{W_t}{z_t^+ P_t}. \end{aligned}$$

Also,

$$G_{t,j} = \begin{cases} \frac{\tilde{\pi}_{w,t+j} \cdots \tilde{\pi}_{w,t+1}}{\pi_{t+j} \cdots \pi_{t+1}} \left(\frac{\mu_{z^+}}{\mu_{z^+,t+1}} \right) \cdots \left(\frac{\mu_{z^+}}{\mu_{z^+,t+j}} \right) & j > 0 \\ 1 & j = 0 \end{cases}. \tag{3.11}$$

The scaled vacancy first order condition of agencies that are in the last period of their contract is:

$$\kappa \tilde{v}_t^{N-1} \frac{1}{Q_t^{1-\iota}} = \beta \frac{\psi_{z^+,t+1}}{\psi_{z^+,t}} \left[\left(\bar{w}_{t+1} - w_{t+1} \bar{w}_{t+1} \right) \varsigma_{t+1} + \kappa \left(\frac{\left(\tilde{v}_{t+1}^{0}\right)^2}{2} + \frac{\tilde{v}_{t+1}^{0} \rho}{Q_{t+1}^{1-\iota}} \right) \right]. \tag{3.12}$$

We require the derivative of J with respect to ω_t. By the envelope condition, we can ignore the impact of a change in ω_t on the vacancy decisions and only be concerned with the direct impact of ω_t on J :

$$\begin{aligned} J_{w,t} = & -\varsigma_t \\ & -\beta \frac{\upsilon_{t+1}}{\upsilon_t} \Gamma_{t,1} \varsigma_{t+1} \left(\chi_t^0 + \rho \right) \\ & -\beta^2 \frac{\upsilon_{t+2}}{\upsilon_t} \Gamma_{t,2} \varsigma_{t+2} \left(\chi_t^0 + \rho \right) \left(\chi_{t+1}^1 + \rho \right) \\ & -... - \beta^{N-1} \frac{\upsilon_{t+N-1}}{\upsilon_t} \Gamma_{t,N-1} \varsigma_{t+N-1} \left(\chi_t^0 + \rho \right) \left(\chi_{t+1}^1 + \rho \right) \cdots \left(\chi_{t+1}^{N-2} + \rho \right). \end{aligned}$$

Let,

$$\Omega_{t,j} = \begin{cases} \prod_{l=0}^{j-1} \left(\chi_{t+l}^l + \rho \right) & j > 0 \\ 1 & j = 0 \end{cases}. \tag{3.13}$$

so that

$$
\begin{aligned}
J_{w,t} &= -\varsigma_t \Omega_{t,0} \\
&\quad -\beta \frac{\upsilon_{t+1}}{\upsilon_t} \Gamma_{t,1} \varsigma_{t+1} \Omega_{t,1} \\
&\quad -\beta^2 \frac{\upsilon_{t+2}}{\upsilon_t} \Gamma_{t,2} \varsigma_{t+2} \Omega_{t,2} \\
&\quad -... - \beta^{N-1} \frac{\upsilon_{t+N-1}}{\upsilon_t} \Gamma_{t,N-1} \varsigma_{t+N-1} \Omega_{t,N-1} \left(\chi_t^0 + \rho\right) \left(\chi_{t+1}^1 + \rho\right) \cdots \left(\chi_{t+1}^{N-2} + \rho\right) \\
&= -\sum_{j=0}^{N-1} \beta^j \frac{\upsilon_{t+j}}{\upsilon_t} \Gamma_{t,j} \Omega_{t,j} \varsigma_{t+j}.
\end{aligned}
$$

Then, in terms of scaled variables,

$$
J_{w,t} = -\sum_{j=0}^{N-1} \beta^j \frac{\psi_{z^+,t+j}}{\psi_{z^+,t}} G_{t,j} \Omega_{t,j} \varsigma_{t+j}. \tag{3.14}
$$

Following is an expression for J_t evaluated at $\omega_t = \tilde{W}_t$, in terms of scaled variables. Dividing by $P_t z_t^+$:

$$
\begin{aligned}
J_{z^+,t} &= \frac{J\left(\tilde{W}_t\right)}{P_t z_t^+} = \frac{\left(W_t - \tilde{W}_t\right)}{P_t z_t^+} \varsigma_t - \frac{\kappa}{2} \left(\tilde{v}_t^0\right)^2 \\
&\quad + \beta \frac{\psi_{z^+,t+1}}{\psi_{z^+,t}} \left[\frac{W_{t+1} - \Gamma_{t,1} \tilde{W}_t}{P_{t+1} z_{t+1}^+} \varsigma_{t+1} - \frac{\kappa}{2} \left(\tilde{v}_{t+1}^1\right)^2 \right] \left(\chi_t^0 + \rho\right) \\
&\quad + \beta^2 \frac{\psi_{z^+,t+2}}{\psi_{z^+,t}} \left[\frac{W_{t+2} - \Gamma_{t,2} \tilde{W}_t}{P_{t+2} z_{t+2}^+} \varsigma_{t+2} - \frac{\kappa}{2} \left(\tilde{v}_{t+2}^2\right)^2 \right] \left(\chi_t^0 + \rho\right) \left(\chi_{t+1}^1 + \rho\right) \\
&\quad + ... + \\
&\quad + \beta^N \frac{\psi_{z^+,t+N}}{\psi_{z^+,t}} J_{z^+,t+N} \left(\chi_t^0 + \rho\right) \left(\chi_{t+1}^1 + \rho\right) \cdots \left(\chi_{t+N-1}^{N-1} + \rho\right)
\end{aligned}
$$

or,

$$
J_{z^+,t} = \sum_{j=0}^{N-1} \beta^j \frac{\psi_{z^+,t+j}}{\psi_{z^+,t}} \left[\left(\bar{w}_{t+j} - G_{t,j} w_t \bar{w}_t\right) \varsigma_{t+j} - \frac{\kappa}{2} \left(\tilde{v}_{t+j}^j\right)^2 \right] \Omega_{t,j}. \tag{3.15}
$$

3.2.2. The Worker Problem

We now turn to the worker. The period t value of being a worker in an agency in cohort i is V_t^i :

$$
V_t^i = \Gamma_{t-i,i} \tilde{W}_{t-i} \varsigma_t - \zeta_t^h A_L \frac{\varsigma_t^{1+\sigma_L}}{\left(1 + \sigma_L\right) \upsilon_t} + \beta E_t \frac{\upsilon_{t+1}}{\upsilon_t} [\rho V_{t+1}^{i+1} + (1 - \rho) U_{t+1}], \tag{3.16}
$$

for $i = 0, 1, ..., N-1$. Here, ρ is the probability of remaining with the firm in the next period and U_t is the value of being unemployed in period t. The values, V_t^i and U_t, pertain to the beginning of period t, after job separation and job finding has occurred. Scaling V_t^i by $P_t z_t^+$, we obtain:

$$V_{z^+,t}^i = G_{t-i,i} w_{t-i} \bar{w}_{t-i} \varsigma_t - \zeta_t^h A_L \frac{\varsigma_t^{1+\sigma_L}}{(1+\sigma_L)\psi_{z^+,t}} + \beta E_t \frac{\psi_{z^+,t+1}}{\psi_{z^+,t}} [\rho V_{z^+,t+1}^{i+1} + (1-\rho) U_{z^+,t+1}], \quad (3.17)$$

for $i = 0, 1, ..., N-1$, where

$$\frac{V_t^i}{P_t z_t^+} = V_{z^+,t}^i, \; U_{z^+,t+1} = \frac{U_{z^+,t+1}}{P_{t+1} z_{t+1}^+}.$$

For workers employed by agencies in cohort $i = 0$, the value function is $V^0(\omega_t)$, where ω_t is an arbitrary value for the current period wage rate,

$$V^0(\omega_t) = \omega_t \varsigma_t - \zeta_t^h A_L \frac{\varsigma_t^{1+\sigma_L}}{(1+\sigma_L)\upsilon_t} + \beta E_t \frac{\upsilon_{t+1}}{\upsilon_t} [\rho V_{t+1}^1 + (1-\rho) U_{t+1}]. \quad (3.18)$$

The notation makes the dependence of V^0 on ω_t explicit to simplify the discussion of the Nash bargaining problem below. Below, we require the derivative of $V^0(\omega_t)$ with respect to ω_t, evaluated at $\omega_t = \tilde{W}_t$:

$$\begin{aligned} V_w^0(\omega_t) &= \sum_{j=0}^{N-1} (\beta\rho)^j E_t \varsigma_{t+j} \Gamma_{t,j} \frac{\upsilon_{t+j}}{\upsilon_t} \\ &= \sum_{j=0}^{N-1} (\beta\rho)^j E_t \varsigma_{t+j} \Gamma_{t,j} \frac{\psi_{z^+,t+j} P_t z_t^+}{\psi_{z^+,t} P_{t+j} z_{t+j}^+} \\ &= \sum_{j=0}^{N-1} (\beta\rho)^j E_t \varsigma_{t+j} G_{t,j} \frac{\psi_{z^+,t+j}}{\psi_{z^+,t}}. \end{aligned} \quad (3.19)$$

The value of being an unemployed worker is U_t :

$$U_t = P_t z_t^+ b^u + \beta E_t \frac{\upsilon_{t+1}}{\upsilon_t} [f_t V_{t+1}^x + (1-f_t) U_{t+1}], \quad (3.20)$$

where f_t is the probability that an unemployed worker will land a job in period $t+1$. Also, V_t^x is the period $t+1$ value function of a worker who finds a job, before is is known which agency it is found with:

$$V_{z^+,t}^x = \sum_{i=0}^{N-1} \frac{\chi_{t-1}^i l_{t-1}^i}{m_{t-1}} V_{z^+t}^{i+1}, \quad (3.21)$$

where the total number of new matches is given by:

$$m_t = \sum_{j=0}^{N-1} \chi_t^j l_t^j. \quad (3.22)$$

In (**??**),

$$\frac{\chi_{t-1}^{i} l_{t-1}^{i}}{m_{t-1}}$$

is the probability of finding a job in an agency which was of type i in the previous period, conditional on being a worker who finds a job in t.

Scaling (3.20),

$$U_{z^{+},t} = b^{u} + \beta E_{t} \frac{\psi_{z^{+},t+1}}{\psi_{z^{+},t}} \left[f_{t} V_{z^{+},t+1}^{x} + (1 - f_{t}) U_{z^{+},t+1} \right] \tag{3.23}$$

Total job matches must also satisfy the following matching function:

$$m_{t} = \sigma_{m} (1 - L_{t})^{\sigma} v_{t}^{1-\sigma}, \tag{3.24}$$

where

$$L_{t} = \sum_{j=0}^{N-1} l_{t}^{j}, \tag{3.25}$$

and v_t, total vacancies, are related to vacancies posted by the individual cohorts as follows:

$$v_{t} = \frac{1}{Q_{t}^{\iota}} \sum_{j=0}^{N-1} \tilde{v}_{t}^{i} l_{t}^{i}, . \tag{3.26}$$

Total hours worked is:

$$H_{t} = \varsigma_{t} \sum_{j=0}^{N-1} l_{t}^{j}. \tag{3.27}$$

The job finding rate is:

$$f_{t} = \frac{m_{t}}{1 - L_{t}}. \tag{3.28}$$

The probability of filling a vacancy is:

$$Q_{t} = \frac{m_{t}}{v_{t}}. \tag{3.29}$$

The $i = 0$ cohort of agencies in period t solve the following Nash bargaining problem:

$$\max_{\omega_{t}} \left(V^{0} (\omega_{t}) - U_{t} \right)^{\eta} J (\omega_{t})^{(1-\eta)}, \tag{3.30}$$

where $V^{0}(\omega_t) - U_t$ is the match surplus enjoyed by a worker. We denote the wage that solves this problem by $\tilde{W}_t$. Note that $\tilde{W}_t$ takes into account that intensity will be chosen according to (3.3) as well as (3.4). The first order condition associated with this problem is:

$$\eta V_{w,t} J_{z^{+},t} + (1 - \eta) \left[V_{z^{+},t} - U_{z^{+},t} \right] J_{w,t} = 0, \tag{3.31}$$

after division by $z_t^{+} P_t$.

We assume that the posting of vacancies uses the homogeneous domestic good. We leave the production technology equation, (1.45), unchanged, and we alter the resource constraint:

$$
\begin{aligned}
y_t &= g_t + c_t^d + i_t^d \\
&\quad + \left(R_t^x\right)^{\eta_x} \left[\omega_x \left(p_t^{m,x}\right)^{1-\eta_x} + \left(1-\omega_x\right)\right]^{\frac{\eta_x}{1-\eta_x}} \left(1-\omega_x\right) \left(p_t^x\right)^{-\eta_f} y_t^* + \frac{\kappa}{2} \sum_{j=0}^{N-1} \left(\tilde{v}_t^j\right)^2
\end{aligned}
\tag{3.32}
$$

3.3. Solving the Model

The endogenous variables in this system are (apart from (**??**)):

$$
G_{t,j}, \Omega_{t,j}, \tilde{v}_t^j, l_t^j, \chi_t^j, V_{z^+,t}^j, V_{w,t}, J_{z^+,t}, V_{z^+,t}^x, \psi_{z^+,t}, \pi_t, U_{z^+,t}, J_{w,t}, L_t, m_t, \varsigma_t, H_t, v_t, f_t, w_t, \bar{w}_t, Q_t
$$

These are 16 variables not indexed by j, plus 6 variables indexed by j. Equations (3.5), (3.10), (3.12), (3.17), (3.8), (3.11), (3.13) are the ones associated with the variables indexed by j. Not counting the resource constraint, (**??**), there are 14 additional equations, (3.14), (3.15), (3.25), (3.24), (3.19), (3.21), (3.22), (3.23), (3.27), (3.28), (3.3), (3.31), (3.29) and (3.26). Note that three of the variables above, $\psi_{z^+,t}, \pi_t, \bar{w}_t$, appear in other equations in the benchmark model. So, we really have 13 new variables here. Given that we have 14 equations, there is an extra one relative to these new variables. However, one of the equations in the benchmark model (i.e., wage Phillips curve in the case of the linearized dynamics, and the labor intratemporal Euler equation in the case of the steady state) is lost. So, it appears that we have an equal number of equations and unknowns.

3.3.1. Steady State

Our strategy for computing the steady state begins with the calculations in the steady state discussion in section 1.7.1. In particular, we proceed with the calculations up to equation (1.60). At this point, we are in an 'outer loop' in which φ is fixed. We fix a value for ψ_{z^+}. This, together with $\bar{w}$ (which we have in hand at this point) allows us to solve the steady state variables in the search and matching part of the model. We now describe those calculations.

In steady state each agency's employment is constant, so that $l_{t+1}^{i+1} = l_t^i$, which implies $\chi^i + \rho = 1$, or $\chi^i = \chi^i = 1 - \rho$. We have 14 variables

$$
Q, \tilde{v}, \varsigma, J_{z^+}, w, J_w, V_{z^+}, U_{z^+}, V_w, m, l, v, H, f
$$

in the following 14 equations:

$$
\begin{aligned}
(1) m &= Nl\left(1-\rho\right) \\
(2)\, f &= \frac{m}{1-Nl} \\
(3)\, \bar{w} &= \zeta^h A_L \varsigma^{\sigma_L} \frac{1}{\psi_{z^+}} \\
(4)\, H &= \varsigma Nl \\
(5)\, J_w &= -\varsigma \frac{1-\beta^N}{1-\beta} \\
(6)\, V_w &= \varsigma \frac{1-\left(\beta\rho\right)^N}{1-\beta\rho} \\
(7)\, 1-\rho &= Q^{1-\iota}\tilde{v} \\
(8)\, v &= \frac{Nl}{Q^{\iota}}\tilde{v} \\
(9)\, m &= \sigma_m \left(1-Nl\right)^{\sigma} v^{1-\sigma} \\
(10)\, \kappa\tilde{v}\frac{1}{Q^{1-\iota}} &= \beta\left[\left(\bar{w}-w\bar{w}\right)\varsigma+\kappa\left(\frac{\left(\tilde{v}\right)^2}{2}+\frac{\tilde{v}\rho}{Q}\right)\right] \\
(11)\, J_{z^+} &= \left[\left(\bar{w}-w\bar{w}\right)\varsigma-\frac{\kappa}{2}\left(\tilde{v}\right)^2\right]\frac{1-\beta^N}{1-\beta} \\
(12)\, V_{z^+} &= w\bar{w}\varsigma-\zeta^h A_L\frac{\varsigma^{1+\sigma_L}}{\left(1+\sigma_L\right)\psi_{z^+}}+\beta[\rho V_{z^+}+\left(1-\rho\right)U_{z^+}] \\
(13)\, U_{z^+} &= b^u+\beta[fV_{z^+}+\left(1-f\right)U_{z^+}] \\
(14)\, 0 &= \eta V_w J_{z^+}+\left(1-\eta\right)\left[V_{z^+}-U_{z^+}\right]J_w
\end{aligned}
$$

We solve for a value of σ_m that rationalizes a calibrated value of the unemployment rate, $1-Nl$. The objects, ψ_{z^+} and $\bar{w}$ are treated as known. Solve (1) for m. Solve (2) for f. Solve (3) for ς. Solve (4) for H. Solve (5) for J_w. Solve (6) for V_w. Fix $\tilde{v}>0$..Solve (7) for Q. Solve (8) for v. Solve (9) for σ_m. Solve (10) for w. Solve (11) for J_{z^+}. Solve (12) and (13) for U_{z^+} and V_{z^+}. Adjust $\tilde{v}$ until (14) is satisfied. Equations (12) and (13) are solved as follows. Substitute out for U_{z^+}in (12) from (13) to obtain an expression that can be solved for (3.34). After solving (12) for V_{z^+}, use the result to solve for U_{z^+} in (3.35):

$$V_{z^+}\left[1-\rho\beta\right] = w\bar{w}\varsigma-\zeta^h A_L\frac{\varsigma^{1+\sigma_L}}{\left(1+\sigma_L\right)\psi_{z^+}}+\frac{\beta\left(1-\rho\right)b^u}{1-\beta\left(1-f\right)}+\frac{\beta\left(1-\rho\right)\beta f V_{z^+}}{1-\beta\left(1-f\right)} \tag{3.33}$$

$$V_{z^+}\left[1-\rho\beta-\frac{\beta\left(1-\rho\right)\beta f}{1-\beta\left(1-f\right)}\right] = w\bar{w}\varsigma-\zeta^h A_L\frac{\varsigma^{1+\sigma_L}}{\left(1+\sigma_L\right)\psi_{z^+}}+\frac{\beta\left(1-\rho\right)b^u}{1-\beta\left(1-f\right)} \tag{3.34}$$

$$U_{z^+} = \frac{b^u}{1-\beta\left(1-f\right)}+\frac{\beta f V_{z^+}}{1-\beta\left(1-f\right)} \tag{3.35}$$

The case, $\iota=1$ is the one considered by GT and GST. The formulas above do not work in the limiting case, $\iota\to 1$, so we derive the appropriate modifications here. In this case, steps 1-6 above are unchanged. However, (7) cannot be used to determine Q. Instead, (7) can be used to determine

$$\tilde{v} = 1 - \rho$$

so there is no need for a nonlinear search to determine a value for $\tilde{v}$. Note that the first order condition associated with the Nash bargaining problem, in conjunction with (11)-(13) defines a single non-linear equation in the unknown, w. To see this, substitute out for V_w and $J_w$from (5) and (6), to obtain:

$$0 = \eta V_w \left[(\bar{w} - w\bar{w}) \varsigma - \frac{\kappa}{2} (\tilde{v})^2 \right] \frac{1 - \beta^N}{1 - \beta} + (1 - \eta) [V_{z+} - U_{z+}] J_w.$$

Then, recall from (3.34) and (3.35) that U_{z+} and V_{z+} are functions of w alone. With w determined, solve for Q by rearranging (10):

$$Q = \frac{\kappa \tilde{v} \rho}{\frac{\kappa \tilde{v}}{\beta} - \bar{w} (1 - w) \varsigma - \kappa \frac{(\tilde{v})^2}{2}}.$$

Then, v can be determined from (8) and, finally, σ_m can be computed from (9). This completes the determination of the labor market equations for the case, $\iota = 1$. Note how σ in the matching function plays only a residual role in these calculations. The value of this parameter has no impact on any of the variables in the search and matching part of the model. This is consistent with out earlier discussion in which we conclude that with $\iota = 1$ the firm has hiring costs only in the sense of having costs after a match has been made. It does not have any costs associated specifically with search and matching.

Next, we use equation (1.66) to solve for c :

$$c = \frac{1}{\psi_{z+}} \frac{\zeta^c (\mu_{z+} - \beta b)}{p^c (1 + \tau^c) (\mu_{z+} - b)}.$$

The steady state resource constraint is (1.64), adjusted to include resources used in posting vacancies.

$$\frac{1 - \eta_g - (R^x)^{\eta_x} \left[\omega_x (p^{m,x})^{1-\eta_x} + (1 - \omega_x) \right]^{\frac{\eta_x}{1-\eta_x}} (1 - \omega_x) (p^x)^{-\eta_f} \eta_y}{\lambda^d} H\epsilon \left(\frac{k/H}{\mu_{z+}\mu_\Psi} \right)^\alpha$$
$$= (1 - \omega_c) (p^c)^{\eta_c} c + (1 - \omega_i) (p^i)^{\eta_i} \frac{\bar{k}}{H} \frac{\left[1 - \frac{1-\delta}{\mu_{z+}\mu_\Psi} \right]}{\Upsilon} H + \frac{\kappa}{2} N \tilde{v}^2.$$

Here, $\eta_y = y^*/y$. We adjust ψ_{z+} until the above resource equation is satisfied. As before, we adjust φ until the trade balance, (1.68), is satisfied.

4. Mixing Monthly and Quarterly Data

Suppose the model is specified monthly and at the monthly level, the state evolution equation is

$$\xi_t = F\xi_{t-1} + Bu_t,$$

where t denotes months. Note:

$$\xi_t = F^3\xi_{t-3} + Bu_t + FBu_{t-1} + F^2Bu_{t-2},$$

and note that $Bu_t + FBu_{t-1} + F^2Bu_{t-2}$ is iid across quarters. Consider

$$\begin{pmatrix} \xi_t \\ \xi_{t-1} \\ \xi_{t-2} \end{pmatrix} = \begin{bmatrix} F^3 & 0 & 0 \\ F^2 & 0 & 0 \\ F & 0 & 0 \end{bmatrix} \begin{pmatrix} \xi_{t-3} \\ \xi_{t-4} \\ \xi_{t-5} \end{pmatrix} + \begin{bmatrix} B & FB & F^2B \\ 0 & B & FB \\ 0 & 0 & B \end{bmatrix} \begin{pmatrix} u_t \\ u_{t-1} \\ u_{t-2} \end{pmatrix}.$$

Now define

$$\tilde{\xi}_t = \begin{pmatrix} \xi_t \\ \xi_{t-1} \\ \xi_{t-2} \end{pmatrix}, \tilde{F} = \begin{bmatrix} F^3 & 0 & 0 \\ F^2 & 0 & 0 \\ F & 0 & 0 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B & FB & F^2B \\ 0 & B & FB \\ 0 & 0 & B \end{bmatrix}, \ \tilde{u}_t = \begin{pmatrix} u_t \\ u_{t-1} \\ u_{t-2} \end{pmatrix}.$$

Note that

$$\tilde{\xi}_t = \tilde{F}\tilde{\xi}_{t-3} + \tilde{B}\tilde{u}_t,$$

and note that this is a first order AR process for $\tilde{\xi}_t, \tilde{\xi}_{t+3}, \tilde{\xi}_{t+6}, ...$, and that $\tilde{u}_t$ is iid in quarterly data. This is the quarterly representation of the state evolution equation. In reality, if you observed quarterly observations on $\tilde{\xi}_t$, you would actually be observing the monthly observations on the underlying data, the ξ_t's. This is because the quarterly observations on $\tilde{\xi}_t$ contain quarterly observations on the first month's data, quarterly observations on the second month's data, etc.

Of course, we don't actually observe a quarterly record on $\tilde{\xi}_t$. For some variables, like gdp, we only observe the average of the quarterly observations. But, this fact can be accommodated by suitably choosing the parameters of the observer equation. Consider the following simple example. Suppose

$$\xi_t = \begin{pmatrix} y_t \\ \pi_t \\ h_t \end{pmatrix},$$

where y_t is (log) monthly GDP, π_t is monthly inflation and h_t denotes monthly hours worked.

Now suppose we change our interpretation of the t index, so that it refers to quarters. Then, y_t means GDP in the third month of quarter t, $y_{t-1/3}$ denotes GDP in the 2^{nd} and so on. Our quarterly time series representation for the data is

$$\tilde{\xi}_t = \tilde{F}\tilde{\xi}_{t-1} + \tilde{B}\tilde{u}_t,$$

where $\tilde{u}_t$ is iid for integer values of t, with

$$\tilde{B}E\left(\tilde{u}_t\tilde{u}_t'\right)\tilde{B}' = \tilde{B}\begin{bmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{bmatrix}\tilde{B}'.$$

Suppose we observe:

$$Y_t = \begin{pmatrix} GDP_t \\ \text{monthly inflation}_t \\ \text{monthly inflation}_{t-1/3} \\ \text{monthly inflation}_{t-2/3} \\ \text{hours}_t \\ \text{hours}_{t-1/3} \\ \text{hours}_{t-2/3} \end{pmatrix},$$

where GDP_t denotes quarterly GDP, which we interpret as $y_t + y_{t-1/3} + y_{t-2/3}$, monthly inflation and monthly hours worked. The observer equation has the following form:

$$Y_t = H\tilde{\xi}_t + w_t,$$

where w_t is a vector of measurement error with variance-covariance matrix R ($R = 0$ is possible). The matrix, H, has the following form:

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. Ramsey-Optimal Policy

We find the Ramsey-optimal allocations for our economy using the computer code and strategy used in Levin, Lopez-Salido, (2004) and Levin, Onatski, Williams and Williams (2005). For completeness, we briefly describe this strategy below. Let x_t denote a set of N endogenous variables in a dynamic economic model. Let the private sector equilibrium conditions be represented by the following $N-1$ conditions:

$$\sum_{s^{t+1}|s^t} \frac{\mu\left(s^{t+1}\right)}{\mu\left(s^t\right)} f\left(x\left(s^t\right), x\left(s^{t+1}\right), s_t, s_{t+1}\right) = 0,$$

for all t and all s^t. Here, s^t denotes a history:

$$s^t = \left(s_0, s_1, ..., s_t\right),$$

and s_t denotes the time t realization of uncertainty, which can take on n possible values:

$$\begin{aligned} s_t &\in \{s(1), ..., s(n)\} \\ \mu\left(s^t\right) &= \mathrm{prob}[s^t], \end{aligned}$$

so that $\mu\left(s^{t+1}\right)/\mu\left(s^t\right)$ is the probability of history s^{t+1}, conditional on s^t.

Suppose preferences over $x\left(s^t\right)$ are follows:

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\mu\left(s^t\right)U\left(x\left(s^t\right), s_t\right). \tag{5.1}$$

In our simple monetary model, U is given by (**??**). The Ramsey problem is to maximize preferences by choice of $x\left(s^t\right)$ for each s^t, subject to (**??**). We express the Ramsey problem in Lagrangian form as follows:

$$\max\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\mu\left(s^t\right)\left\{U\left(x\left(s^t\right), s_t\right) + \underbrace{\lambda\left(s^t\right)}_{1\times N-1}\sum_{s^{t+1}|s^t}\frac{\mu\left(s^{t+1}\right)}{\mu\left(s^t\right)}\underbrace{f\left(x\left(s^t\right), x\left(s^{t+1}\right), s_t, s_{t+1}\right)}_{N-1\times 1}\right\},$$

where $\lambda\left(s^t\right)$ is the row vector of multipliers on the equilibrium conditions. Consider a particular history, $s^t = \left(s^{t-1}, s_t\right)$, with $t > 0$. The first order necessary condition for optimality of $x\left(s^t\right)$ is

$$\begin{aligned} \underbrace{U_1\left(x\left(s^t\right), s_t\right)}_{1\times N} + \underbrace{\lambda\left(s^t\right)}_{1\times N-1}\sum_{s^{t+1}|s^t}\frac{\mu\left(s^{t+1}\right)}{\mu\left(s^t\right)}\underbrace{f_1\left(x\left(s^t\right), x\left(s^{t+1}\right), s_t, s_{t+1}\right)}_{N-1\times N} & \\ +\beta^{-1}\underbrace{\lambda\left(s^{t-1}\right)}_{1\times N-1}\underbrace{f_2\left(x\left(s^{t-1}\right), x\left(s^t\right), s_{t-1}, s_t\right)}_{N-1\times N} &= \underbrace{0}_{1\times N} \end{aligned} \tag{5.2}$$

after dividing by $\mu\left(s^t\right)\beta^t$. In less notationally-intensive notation,

$$U_1\left(x_t, s_t\right) + \lambda_t E_t f_1\left(x_t, x_{t+1}, s_t, s_{t+1}\right) + \beta^{-1}\lambda_{t-1}f_2\left(x_{t-1}, x_t, s_{t-1}, s_t\right) = 0.$$

The first order necessary condition for optimality at $t = 0$ is (5.2) with $\lambda_{-1} \equiv 0$. The time-consistency problem occurs when the multipliers associated with the Ramsey problem are non-zero after date 0. Following the Ramsey equilibrium in such a future t requires respecting λ_{t-1}. However, if $\lambda_{t-1} \neq 0$, then utility can be increased at t by restarting the Ramsey problem with $\lambda_{t-1} = 0$. When we study Ramsey allocations, we assume there is a commitment technology that prevents the authorities from acting on this incentive to deviate from the Ramsey plan.

The equations that characterize the Ramsey equilibrium are the $N-1$ equations, (**??**), and the N equations (5.2). The unknowns are the N elements of x and the $N-1$ multipliers,

λ. The equations, (5.2) are computed symbolically using the software prepared for Levin, Lopez-Salido, (2004) and Levin, Onatski, Williams and Williams (2005). The resulting system of equations is then solved by perturbation around steady state using the software package, Dynare.

To apply the perturbation method, we require the nonstochastic steady state value of x. We compute this in two steps. First, fix one of the elements of x, say the inflation rate, π. We then solve for the remaining $N-1$ elements of x by imposing the $N-1$ equations, (**??**). In the next step we compute the $N-1$ vector of multipliers using the steady state version of (5.2):

$$U_1 + \lambda \left[f_1 + \beta^{-1} f_2 \right] = 0,$$

where a function without an explicit argument is understood to mean it is evaluated in steady state.[5] Write

$$\begin{aligned} Y &= U_1' \\ X &= \left[f_1 + \beta^{-1} f_2 \right]' \\ \beta &= \lambda', \end{aligned}$$

so that Y is an $N \times 1$ column vector, X is an $N \times (N-1)$ matrix and β is an $(N-1) \times 1$ column vector. Compute β and u as

$$\begin{aligned} \beta &= \left(X'X \right)^{-1} X'Y \\ u &= Y - X\beta. \end{aligned}$$

Note that this regression will not in general fit perfectly, because there are $N-1$ 'explanatory variables' and N elements of Y to 'explain'. We vary the value of π until $\max |u_i| = 0$. This completes the discussion of the calculation of the steady state and of the algorithm for computing Ramsey allocation

6. Questions that Could be Addressed with Model

- Analysis of the currency crisis
- Evaluating predictions of the model based on Taylor versus Fisher contracts.

[5] This step is potentially very cumbersome, but has been made relatively easy by the software produced for Levin, Lopez-Salido, (2004) and Levin, Onatski, Williams and Williams (2005). This sofware endogenously writes the code necessary to solve for the multipliers.

References

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