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Rough guid to answers for FINAL EXAM

1. (a) The demand curve is:

$$\begin{aligned} \left(\frac{y_t}{x_t(i)}\right)^{\frac{\mu-1}{\mu}} &= p_t(i) \\ x_t(i) &= y_t p_t(i)^{\frac{\mu}{1-\mu}} \end{aligned}$$

(b) Profit maximization, ignoring the presence of the competitive fringe, leads the firm to post a price equal to μ times its marginal cost:

$$p_1(i) = \mu w_1 \frac{c}{\lambda}.$$

The existence of the competitive fringe prevents the firm from setting a price higher than $w_1 c$. If the firm tried to set price higher than this, the competitive fringe sees an opportunity to make profits and would all jump in. So, the firm's price is:

$$\begin{aligned} p_1(i) &= \begin{cases} \frac{\mu}{\lambda} w_1 c & \mu \leq \lambda \\ w_1 c & \mu > \lambda \end{cases} \\ p_1(i) &= \min\left[\mu w_1 \frac{c}{\lambda}, w_1 c\right]. \end{aligned}$$

Notice that $p_1(i) \longrightarrow w_1 c$ as $\mu \longrightarrow +\infty$.
Profits of the firm are given by:

$$\begin{aligned} \pi_1 &= p_1(i) x_1(i) - \frac{w_1 c}{\lambda} x_1(i) = \\ &= y_1 p_1(i)^{\frac{1}{1-\mu}} - \frac{w_1 c}{\lambda} y_1 p_1(i)^{\frac{\mu}{1-\mu}} = \\ &= y_1 p_1(i)^{\frac{1}{1-\mu}} \left[1 - \frac{w_1 c}{\lambda} p_1(i)^{-1}\right] = \\ &= y_1 \left\{\min\left[\frac{\mu}{\lambda} w_1 c, w_1 c\right]\right\}^{\frac{1}{1-\mu}} \left\{1 - \frac{w_1 c}{\lambda} \frac{1}{\min\left[\frac{\mu}{\lambda} w_1 c, w_1 c\right]}\right\} \end{aligned}$$

Note that as $\mu \longrightarrow +\infty$ (which corresponds to the Shleifer model), $\pi_1 \longrightarrow y_1 \left(1 - \frac{1}{\lambda}\right)$.

(c) The PDV of profits is given by:

$$\sum_{t=1}^{+\infty} \frac{1}{\prod_{j=1}^{t-1} R_j} \pi_t = \pi_1$$

since once the idea is implemented it becomes common knowledge, and the probability of getting a new idea is zero.

(d) The firm will face in period 2 a problem analog to the one discussed in (b), and it will make the following profits:

$$\bar{\pi}_2 = y_2 \left\{ \min \left[\frac{\mu}{\lambda} w_2 c, w_2 c \right] \right\}^{\frac{1}{1-\mu}} \left\{ 1 - \frac{w_2 c}{\lambda} \frac{1}{\min \left[\frac{\mu}{\lambda} w_2 c, w_2 c \right]} \right\}.$$

Profits in subsequent periods ($t > 2$) will be zero for the same reason as in (c).

Profits in period 1 are zero.

So, the PDV of profits is:

$$\sum_{t=1}^{+\infty} \frac{1}{\prod_{j=1}^{t-1} R_j} \bar{\pi}_t = \frac{\bar{\pi}_2}{R_1}.$$

(e) The firm will decide to implement the idea at $t = 1$ if:

$$\pi_1 > \frac{\bar{\pi}_2}{R_1}$$

$$y_1 \left\{ \min \left[\frac{\mu}{\lambda} w_1 c, w_1 c \right] \right\}^{\frac{1}{1-\mu}} \left\{ 1 - \frac{w_1 c}{\lambda} \frac{1}{\min \left[\frac{\mu}{\lambda} w_1 c, w_1 c \right]} \right\} >$$

$$> \frac{y_2}{R_1} \left\{ \min \left[\frac{\mu}{\lambda} w_2 c, w_2 c \right] \right\}^{\frac{1}{1-\mu}} \left\{ 1 - \frac{w_2 c}{\lambda} \frac{1}{\min \left[\frac{\mu}{\lambda} w_2 c, w_2 c \right]} \right\}.$$

Note that, even for $\mu > \lambda$, the previous condition differs from the analog condition in Shleifer (1986):

$$y_1 (w_1 c)^{\frac{1}{1-\mu}} \left\{ 1 - \frac{1}{\lambda} \right\} > \frac{y_2}{R_1} (w_2 c)^{\frac{1}{1-\mu}} \left\{ 1 - \frac{1}{\lambda} \right\}.$$

In this setting, the firm needs to forecast also the wage rate in order to decide whether to implement the idea at time 1, or to wait until time 2.

(a) The budget equation of the government can be rewritten,

$$b_{t+1} = b^* + (1+R)(b_t - b^*),$$

so that

$$b_t = b^* + (1+R)^t(b_0 - b^*).$$

Then,

$$\frac{b_t}{(1+R)^t} \to b_0 - b^* = \Delta.$$

When $b_0 > b^*$, then obviously $\Delta > 0$.

(b) Write

$$F(B_t, B_{t+1}) = u(y - \tau + B_t - \frac{1}{1+R}B_{t+1}).$$

Note that F is strictly concave. The Euler equation is

$$F_2(B_t, B_{t+1}) + \beta F_1(B_{t+1}, B_{t+2}) = 0,$$

and the transversality condition is:

$$\lim_{T\to\infty} \beta^T u'(c_T)B_T \to 0.$$

The remainder of the proof proceeds exactly as in Stokey-Lucas, Thm 4.15.

i. From the government's budget constraint, if the household sets $B_t = b_t$ for all t, then

$$c_t = y - \tau + b_t - \frac{b_{t+1}}{1+R} = y - \tau + \tau = y,$$

so that the household's budget equation is satisfied. It is trivial to verify that the Euler equation is satisfied as a consequence of the facts, c_t is a constant, and $\beta = 1/(1+R)$. The transversality is not satisfied because $\beta^T B_T \to b_0 - b^* > 0$.

ii. Consider the strategy whereby the household sets $B_{t+1} = b^*$ for $t = 0, 1, 2, ...$. Note that with this strategy,

$$B_t - \frac{B_{t+1}}{1+R} = \frac{1+R}{R}\tau\left[1 - \frac{1}{1+R}\right] = \tau,\ t = 1, 2, 3, ...$$

Also,

$$B_0 - \frac{B_1}{1+R} = \Delta + b^* - \frac{b^*}{1+R} = \Delta + \tau.$$

Under this debt strategy, the budget constraint implies the following consumption sequence:

$$\begin{aligned} c_0 &= y - \tau + \Delta + \tau = y + \Delta \\ c_t &= y, \ t = 1, 2, 3, ... \end{aligned}$$

This is clearly better than the candidate sequence, since consumption is increased in period 0 without reducing it in any other date.

Here is some intuition for understanding what is going on here. The initial debt, B_0, can be split into two parts, $B_0 = b^* + (B_0 - b^*)$, or

$$B_0 = b^* + \Delta.$$

The first part, b^*, will eventually be paid off[1], whereas the second part is simply rolled over each period. The interest rate is such that the household is willing to purchase the first part. It prefers to sell out the first part and consume the proceeds immediately, to just rolling it over forever.

(c) A sequence of markets equilibrium is a set of quantities, $\{b_{t+1}, B_{t+1}, c_t; t \geq 0\}$ such that $\{b_{t+1}, c_t; t \geq 0\}$ solves the household problem, $\{B_{t+1}; t \geq 0\}$ is consistent with the government's budget equation, and goods and bond market clearing occurs. When $B_0 = b^*$, then the unique sequence of B_t's that solves the government's budget constraint does not solve the household problem.

[1]This is true in the following sense. When $B_0 = b^*$, then repeated substitution with the government's budget constraint yields the result:

$$B_0 = \sum_{t=0}^{\infty} \frac{\tau}{(1+R)^t},$$

which says that the present value of tax receipts equals the current outstanding debt. Note that this does not require literally $B_t \to 0$ as $t \to \infty$.

(a) The budget constraint is:

$$c_t + k_{t+1} - (1-\delta)k_t \leq w_t + \pi_t + r_t k_t.$$

The sequence of market equilibrium is a sequence of prices, quantities and profits such that everybody maximizes and markets clear.

(b) The final goods producer's first order conditions are:

$$\begin{aligned} (1-\alpha)y_t/n_t &= w_t, \\ \alpha\left(\frac{n_t}{x_t(i)}\right)^{1-\alpha} &= p_t(i). \end{aligned} \tag{1}$$

The profits of the intermediate good firms are:

$$p_t(i)x_t(i) - r_t k_t(i),$$

or, after substituting in the demand curve and imposing $n_t = 1$ and the production technology, $x_t(i) = k_t(i)$:

$$\alpha x_t(i)^{\alpha} - r_t x_t(i).$$

Maximizing this, we obtain:

$$\alpha^2 x_t(i)^{\alpha-1} = r_t,$$

so that $x_t(i)$ is constant. The resource constraint on capital implies:

$$k_t = \int_0^{M_t} k_t(i)di = \int_0^{M_t} x_t(i)di = M_t x_t,$$

so that

$$\alpha^2\left(\frac{1}{\tilde{k}_t}\right)^{1-\alpha} = r_t,$$

where $\tilde{k}_t = k_t/M_t$.

(c) The household's intertemporal Euler equation is:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta\left[r_{t+1} + 1 - \delta\right],$$

for $t = 0, 1, 2, \ldots$. so that the intertemporal Euler equation can be written

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta \left[\alpha^2 M_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} + 1 - \delta\right].$$

Let's conjecture that the path involves $c_{t+1}/c_t = (1+\mu)$ for all t. Substitute this into the previous expression:

$$(1+\mu)^{\gamma} = \beta \left[\alpha^2 \left(\tilde{k}_{t+1}\right)^{\alpha-1} + 1 - \delta\right],$$

for $t = 0, 1, 2, \ldots$. This expression can be solved for $\tilde{k}_t = \tilde{k}$ for $t = 1, 2, 3\ldots$. We also need the resource constraint to be satisfied, or,

$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} \frac{\left(\tilde{k}_{t+1}\right)^{-\alpha} - (\delta+\mu)}{\left(\tilde{k}_t\right)^{-\alpha} - (\delta+\mu)} = 1 + \mu,$$

for $t = 0, 1, 2, \ldots$. In a balanced growth path, $k_{t+1}/k_t = 1 + \mu$ for all t. Imposing this on the previous expression, we find, $\tilde{k}_{t+1} = \tilde{k}_t$ for $t = 0, 1, \ldots$. We already know that $\tilde{k}_t = \tilde{k}$ for $t = 1, 2, 3\ldots$, so that this implies $\tilde{k}_0 = \tilde{k}$. So, a necessary condition for a balanced growth path is that the initial conditions be just right.

(d) In the efficient allocations, you replace x_t by k_t/M_t and the technology becomes:

$$c_t + k_{t+1} - (1-\delta)k_t \leq M_t^{1-\alpha} k_t^{\alpha}$$

The efficient allocations optimize the household's preferences subject to this constraint. This is just the exogenous growth model. We know that this has a unique balanced growth path.

$$(1+\mu)^{\gamma} = \beta \left[\alpha \left(\tilde{k}_{t+1}\right)^{\alpha-1} + 1 - \delta\right],$$

Let $\tilde{k}^{eff}$ denote the value in the efficient allocations, and $\tilde{k}^{equi}$ the value in equilibrium. Note:

$$\alpha \left(\tilde{k}^{eff}\right)^{\alpha-1} = \alpha^2 \left(\tilde{k}^{equi}\right)^{\alpha-1},$$

so that $\tilde{k}^{equi} > \tilde{k}^{eff}$ since $\alpha < 1$.

(e) We get monotone convergence when we recall that the efficient allocations correspond to the ones solved by the standard growth model. They display monotone convergence.

(f) The rental rate on capital in the equilibrium is α times the true social marginal product of capital. In this sense the market signal about the rate of return on capital understates its true rate of return. The resulting distortion to incentives results in insufficient capital accumulation.