Homework #2
Economics D11-1
Due Wednesday, October 4
Christiano

1. Consider the problem of choosing a consumption sequence c_t to maximize

$$\sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \gamma \log(c_{t-1})\},\ 0 < \beta < 1,\ \gamma > 0,$$

subject to $c_t + k_{t+1} \leq Ak_t^{\alpha}$, $A > 0$, $0 < \alpha < 1$, and $k_0 > 0$, c_{-1} given. Here, c_t is consumption at time t, and k_t is the capital stock at the beginning of period t. The form of the current utility function $\log(c_t) + \gamma \log(c_{t-1})$ is designed to represent habit persistence in consumption: i.e., the impact of past consumption on current utility.

   (a) Let $v(k_0, c_{-1})$ be the value of $\sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \gamma \log(c_{t-1})\}$ for a consumer who begins time 0 with capital stock k_0 and lagged consumption, c_{-1}, and behaves optimally. Formulate Bellman's functional equation in $v(k, c_{-1})$.

   (b) Prove that the solution of Bellman's equation is of the form $v(k, c_{-1}) = E + F\log(k) + G\log(c_{-1})$ and that the optimal policy is of the form, $k_{t+1} = I + H\log(k_t)$ where E, F, M, G, H, and I are constants. Give explicit formulas for the constants E,F,G,H, in terms of the parameters A, B, α, and γ.

2. Consider the more general version of the preceding problem, to maximize

$$\sum_{t=0}^{\infty} \beta u(c_t, c_{t-1}),\ 0 < \beta < 1,$$

subject to $c_t + k_{t+1} \leq f(k_t)$, $k_0 > 0$, c_{-1} given, where $u(c_t, c_{t-1})$ is twice continuously differentiable, bounded, increasing in both c_t and c_{t-1}, and concave in (c_t, c_{t-1}), and where $f'(0) = +\infty, f' > 0, f'' < 0$.

   (a) Formulate Bellman's functional equation for this problem.

(b) Argue that in general, the optimal consumption plan is to set c_t as a function of both k_t and c_{t-1}. What features of the example in the preceding problem combine to make the optimal consumption plan expressible as a function of k alone?

3. This question is designed to illustrate Blackwell's Theorem, Theorem 3.3 on page 54 of S-L. That theorem represents a set of conditions that are *sufficient* for a mapping, T, to be a contraction, so that $T^j w_0 = w$ as $j \to \infty$ for all w_0 belonging to a specified set. The question draws attention to the fact that the conditions of Blackwell's theorem are not *necessary*.

   Consider the following functional equation:

$$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta \lambda^{(1-\sigma)} v.$$

   Suppose $\sigma > 1$ and $\beta(A+1-\delta)^{1-\sigma} < 1$.

   (a) Show: $T(v) = \infty$ for $v > 0$, $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$.

   (b) Show: the derivative of T at $v = v_0 < 0$ is:

$$\frac{dT(v_0)}{dv} = \beta \lambda(v_0)^{(1-\sigma)},$$

   where

$$\lambda(v_0) = argmax_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta \lambda^{(1-\sigma)} v_0.$$

   (c) Explain why T does not satisfy the conditions of Theorem 3.3 in S-L, page 54. (Hint: does $T : B(X) \to B(X)$, where the 'functions' we consider here are actually points in R? Is discounting satisfied?)

   (d) What happens to $\lambda(v)$ as $v \to -\infty$?

   (e) What does the graph of $T(v)$ versus v for $v \leq 0$ look like? Does it cross a 45^0 line drawn in the negative orthant? Draw this graph by hand, emphasizing its qualitative features (i.e., you need not compute the graph numerically, using numerical values for the parameters of the function.)

   (f) Explain, using the graph you just developed, why $T^j v_0 = v^*$ as $j \to \infty$, for every $v_0 < 0$, where v^* is unique.