Homework #4
Economics D11-1, Fall 2000
Due Wednesday, October 18
Christiano

1. Suppose a planner chooses to maximize, by choice of $c_0, c_1, c_2, \ldots$, the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], u(c_t) = \log(c_t) \tag{1}$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, \;\; k_0 \text{ given},$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

   (a) Let $k_{t+1} = g_t(k_t)$ denote the policy rule that solves this problem, $t = 0, 1, ...$. From the perspective of period 0, the part of the problem from $t = 1$ and on looks exactly like the problem with $\delta = 1$. As a result, you know that the optimized value of $u(c_1) + \beta^2 u(c_2) + \ldots$ has the form, $v(k_1)$, and you know how to compute $v(k_1)$ because it has a simple log-linear form. Use this to show that the optimal choice of k_1 has the form:

$$k_1 = g k_0^\alpha,$$

   where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ. How does the saving rate from period $t = 1$ and on compare with the date 0 saving rate?

   (b) Is there a unique k^* with the property $k_t \rightarrow k^*$ as $t \rightarrow \infty$ for all k_0? Display a formula relating k^* to the parameters of the model.

   (c) Suppose $\beta = 1/1.03, \alpha = .36, \delta = .8$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.

(d) Now suppose that when date 1 happens, the planner decides to reoptimize with respect to $k_2, k_3, \ldots$. The initial condition for this problem is k_1, the decision implemented by the planner last period. From the perspective of $t = 1$, the planner's preferences over $c_t, t \geq 1$ are as follows:

$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \ldots]$$

and the resource constraint is as before. (Note how different the problem for $t \geq 1$ looks from the point of view of period 1 than it does from the point of view of period 0.) What values will the planner choose for k_1, k_2, k_3, k_4, k_5? If the planner chooses to re-optimize in this way every period, to what value will k_t actually tend?

(e) Are the values for k_2, k_3, k_4, k_5 chosen by the planner in date 1 the same as the values planned for these variables as of date 0? Why not? Because the chosen values for these variables differs between time 0 and time 1, this problem is said to be *time inconsistent.* If δ had been set to one, we would not have had this problem. Why not?

(f) Basically, the attitude of the planner is 'I'm very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I'll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I'll consume a lot today and save a lot tomorrow.' Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that 'tomorrow I'll quit smoking, but tonight I'll just have one or two more'. It also has been used to explain the low US saving rate. The notion is that many people say, 'today I'll spend, and tomorrow I'll save', day after day. (See the papers of David Laibson, of Harvard.)

Does the solution concept that we have used make any sense? Would a rational person really make decisions in the time-inconsistent way described in d and e? Here is another idea. The idea is to treat the planner in each period as though they were a different person, not bound by any commitments that may have been made in previous periods. This gives rise to a Nash equilibrium concept in which each date's planner optimizes, taking as given what planners in other periods do. Thus, suppose a planner optimizes today, subject to the constraint that planners at all future dates save according to the rule, $k_{t+1} = dk_t^{\alpha}$. Let $v(k_t; d)$ denote the present value of utility, $u(c_t) + \beta u(c_{t+1}) + \ldots$, that occurs when the saving rate, d, is followed forever.

i. Display an explicit formula for $v(k_t; d)$ (hint: you can find it as the fixed point of a dynamic programming sequence that you can do with paper and pencil, like we did in class).

ii. Let $g(k; d)$ denote the policy rule of a planner with preferences, (1), who expects the saving rate in the future to be d. Show that $g(k; d) = D(d)k^{\alpha}$, and derive an explicit formula for $D(d)$.

iii. A natural equilibrium concept in this setting is that it is a d^*, such that $d^* = D(d^*)$. Display a formula relating d^* to the parameters of the model. Defend this equilibrium concept. How does the utility value of this scenario compare with the one in (a)-(b).

2. Consider the neoclassical growth model with $u(c) = \log(c)$, $\beta = 1/1.03$, $\delta = .10$, $\alpha = 0.36$.

   (a) Compute the value of the steady state stock of capital.

   (b) Compute the first order Taylor series expansion of the policy rule about $k = k^*$. Consider two scenarios, $k_0 = .9 \times k^*$ and $k_0 = .5 \times k^*$. In each case, compute k_1, k_2, k_3, k_4, k_5.

   (c) Compute the second order Taylor series expansion of the policy rule about $k = k^*$. Consider the same two scenarios as in (b). Does going to the second order expansion help much?