Homework #6
Economics D11-1
Due Wednesday, November 1
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1. Suppose the representative household has the following preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t),\ u(c,l) = \frac{[c(1-l)^{\eta}]^{1-\gamma}}{1-\gamma},\ 0 < \beta < 1.$$

The technology for producing consumption goods is:

$$c_t \leq A k_{ct}^{\alpha} l_t^{1-a},\ 0 < \alpha < 1$$

The technology for producing investment goods is:

$$I_t = b k_{it},\ b > 0,$$

while the technology for increasing capital is:

$$k_{t+1} = (1-\delta)k_t + I_t,\ 0 < \delta < 1.$$

The allocation of capital must satisfy:

$$k_t = k_{ct} + k_{it},$$

and k_0 is given. Also,

$$b > \delta,\ \beta(1-\delta+b)^{\alpha(1-\gamma)} < 1.$$

(a) Describe a decentralized, sequence of markets equilibrium for this economy and show that the allocations in that equilibrium coincide with the efficient allocations. Provide an explicit formula for the price of investment goods, $P_{I,t}$, in the equilibrium. Here, $P_{I,t}$ is the price in units of period t consumption goods, of I_t.

(b) Provide a formula for the equilibrium growth rate of capital, $\lambda_k = k_{t+1}/k_t$, in terms of the model paramter values.

(c) Let y_t denote aggregate output for this economy, measured in consumption units. Provide a formula, in terms of model parameters, for the equilibrium growth rate of output, $\lambda_y = y_t/y_{t-1}$. Provide a formula for the growth rate of consumption, $\lambda_c = c_t/c_{t-1}$ and the growth rate of the price of new capital goods, $\lambda_{P_{k'}} = P_{k',t}/P_{k',t-1}$.

(d) Can you find values for the parameters that imply: the share of output paid to labor, $w_t l_t/y_t$, is 2/3; the growth rate of output is 1.5 percent (i.e., $\lambda_y = 1.015$); and $\lambda_{P_{k'}} = 1 - .03$ (i.e., $P_{k'}$ is falling at the rate of 3% per year)?

2. The following three sector exogenous growth model was recently proposed in Kongsamut, Rebelo and Xie (see their paper, 'Beyond Balanced Growth', on the course web site), to explain several key features of long-run (i.e., 1869-1990) growth: (i) K/Y is roughly constant, where K denotes the aggregate stock of capital, and Y denotes aggregate output, (ii) K grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable share in the economy.

   Consider the following technology. Agricultural output, A_t, is produced using the following production function:

$$A_t = B_A K_{At}^{\alpha}(N_{At} z_t)^{1-\alpha},$$

   where $B_A > 0$, $0 < \alpha < 1$, z_t denotes the state of technology, and K_A and N_A denote capital and labor allocated to agriculture. The state of technology evolves according:

$$z_t = exp(g) z_{t-1}, \; g > 0.$$

   The manufacturing sector produces output that can be converted into consumption goods, C_t, or capital goods:

$$C_t + K_{t+1} - (1-\delta)K_t = K_{Mt}^{\alpha}(N_{Mt} z_t)^{1-\alpha},$$

   in obvious notation. Finally, services, S_t, are produced according to:

$$S_t = B_S K_{St}^{\alpha}(N_{St} z_t)^{1-\alpha}.$$

Suppose that at a point in time, households supply K_t units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

$$N_{At} + N_{Bt} + N_{Mt} = 1, \; K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be P_{At}. Let the price of a service be P_{St}. Finally, let r_t and w_t denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

(a) Show that competitive behavior by firms implies: $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$, $P_{At} = 1/B_A$, $P_{St} = 1/B_S$,

$$C_t + K_{t+1} - (1-\delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^{\alpha} z_t^{1-\alpha}. \tag{1}$$

One can treat the latter as a 'reduced form' expression for the economy's resource constraint.

(b) Derive an expression for the rate of return on capital. If K_t grows at the rate, g, will the rate of return on capital be constant?

(c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{\left[\left(A_t - \bar{A}\right)^{\eta} C_t^{\gamma} (S_t + \bar{S})^{\theta}\right]^{1-\sigma}}{1-\sigma}, \; \eta + \gamma + \theta = 1, \; \eta, \gamma, \theta, \sigma > 0.$$

i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.

ii. Derive the household's intertemporal Euler equation associated with capital.

iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \; \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \tag{2}$$

iv. Substitute out for $A_t - \bar{A}$ and $S_t + \bar{S}$ in terms of C_t in the household's intertemporal Euler equation, to get an expression in terms of the growth rate of C_t and the rate of return on capital alone.

v. Show that in general, there is no reason to expect the economy to converge to a 'balanced growth path', i.e., one in which A_t, C, S_t, K_t, Y_t all grow at a constant rate. (Here, $Y_t = K_t^{\alpha} z_t{}^{1-\alpha}$.) Hint: note that if you scale A_t, M_t, K_t, S_t by z_t and work with the scaled versions of (2) and (1) and the household's intertemporal Euler equation, you cannot get rid of z_t.

vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \tag{3}$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace A_t and S_t in (1) by $A_t - \bar{A}$ and $S_t - \bar{S}$, respectively without changing (1). Then, define $a_t = A_t - \bar{A}$ and $s_t = S_t - \bar{S}$ everywhere and impose (2), that a_t and s_t are each proportional to M_t.

vii. Explain why it is that under (3), the following are true:

$$\begin{aligned} \frac{K_t}{z_t} &\rightarrow k^*, \ \frac{A_t - \bar{A}}{z_t} \rightarrow a^*, \ \frac{S_t + \bar{S}}{z_t} \rightarrow s^* \\ \frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} &\rightarrow exp(g), \end{aligned}$$

where k^*, a^*, s^* are finite, positive, constants.

viii. Provide a simple formula for k^*.

ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?

x. What does the model imply for $P_{k'}$, the consumption price of capital, along the growth path.