

Homework #8
Economics D11
Christiano
Due November 27, 2000.

1. We have not worried about how investment is financed. That's because we have always considered decentralizations in which the household buys additions to capital using current income, so there is no need for 'finance'. This way of organizing things makes it impossible for us to think about an asset market variable like the rate of return on equity. In this question we consider a decentralization in which the accumulation of capital is financed by entrepreneurs who put the capital to work with hired labor to produce output. The entrepreneurs issue equity and debt to finance their acquisition of capital. As a result, the environment in this question facilitates thinking about the equilibrium rate of return on equity and on other assets such as a one-period-ahead sure loan (our version of corporate debt).

The environment below has three properties: First, the equilibrium consumption, labor and capital stock quantities in this model are not dependent on the debt-to-equity ratio of the firm. This is a version of the celebrated Modigliani-Miller theorem in finance. Second, the equilibrium rate of return on equity does depend on the debt-to-equity ratio. This is because equity has to absorb all the uncertainty in the firm's revenue stream, and that is riskier as the firm is more leveraged with debt. The premium on the rate of return on equity over debt increases as the debt to equity ratio increases. Third, for this model to account for the empirically observed equity premium requires an implausibly high debt-to-equity ratio (this is consistent with the findings of a celebrated paper, Mehra and Prescott 'The Equity Premium: A Puzzle,' *Journal of Monetary Economics*, 1985).

Consider the following economy with households and firms.

Households

We suppose there are many identical households. The typical household takes prices, wages and rates of return as given. In a sequence-of-markets competitive environment, the household seeks at time t to

maximize expected utility:

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \quad (1)$$

subject to the sequence of budget constraints,

$$c_{t+j} + z_{t+j+1} + b_{t+j+1} = w_{t+j}n + R_{t+j}b_{t+j} + r_{t+j}z_{t+j}, \quad (2)$$

$j \geq 0$, and subject to initial levels of the stock of debt and equity, b_t and z_t . Here, c_t denotes consumption and household supply of labor, n , is assumed to be fixed. Equity and debt acquired in period t , z_{t+1} and b_{t+1} , have rates of return r_{t+1} and R_{t+1} , respectively. The return, r_{t+1} , is a function of date $t+1$ (and possibly earlier) economic shocks, while R_{t+1} is only a function of shocks dated t and earlier. Thus, from the perspective of time t , the time $t+1$ rate of return on debt is constant across date $t+1$ states of nature, while the rate of return on r_{t+1} varies across date $t+1$ states of nature. This feature of the return on government debt leads us to refer to it as ‘conditionally sure’. The household takes r_t, w_t (wages) and R_t as given and beyond its control.

- (a) Write out the first order necessary conditions associated with the household’s choice of z_{t+1} and b_{t+1} .
- (b) Define the date t equity premium, P_t , to be the excess of the date t conditionally expected return on equity, $E_t r_{t+1}$, over the conditional sure return on bonds, R_{t+1} :

$$P_t = \frac{E_t r_{t+1}}{R_{t+1}}.$$

- (c) Show that

$$P_t = 1 - Cov_t(m_{t+1}, r_{t+1}),$$

where $m_{t+1} = \beta u_{c,t+1}/u_{c,t}$, is the intertemporal marginal rate of substitution in consumption and $u_{c,t}$ is the date t marginal utility of consumption. (Hint: use the fact, $Cov_t(x_{t+1}, y_{t+1}) = E_t x_{t+1} y_{t+1} - E_t x_{t+1} E_t y_{t+1}$, and the first order conditions developed in (a).)

- (d) Suppose the conditional covariance in the above expression were positive. Then P_t is less than 1, i.e., the conditionally expected rate of return on equity is less than that on debt. This seems peculiar. Why should people be willing to hold equity when its payoff is uncertain, and lower in expected value than debt? Explain in intuitive terms.

Firms

We suppose there are many firms, all of which are identical. The typical firm takes prices, w_t and rates of return, R_t, r_t , as given. In period $t + 1$ the firm uses capital, K_{t+1} , and labor, N_{t+1} , to produce output, Y_{t+1} , using the following production function:

$$Y_{t+1} = F(K_{t+1}, N_{t+1}, \epsilon_{t+1}), \quad (3)$$

where F is linear homogeneous in its first two arguments and ϵ_{t+1} is a stationary random shock to technology. An entrepreneur who wishes to operate the firm in period $t + 1$ must during period t acquire the purchasing power needed to purchase K_{t+1} from the current firm. The entrepreneur acquires this purchasing power by selling equity shares, Z_{t+1} , and debt, B_{t+1} , to the household. The entrepreneur's period t financing constraint is:

$$Z_{t+1} + B_{t+1} = K_{t+1}. \quad (4)$$

Next period, the entrepreneur brings $Y_{t+1} + (1 - \delta)K_{t+1}$ to the goods market to sell. This includes new production, Y_{t+1} , and the capital stock that remains after depreciation at the end of next period, $(1 - \delta)K_{t+1}$. The entrepreneur's expenses next period include the wage bill, $w_{t+1}N_{t+1}$, and the obligations on debt, $R_{t+1}B_{t+1}$, and equity, $r_{t+1}Z_{t+1}$. Thus, the entrepreneur's total profit at the end of $t+1$ is π_{t+1} , where,

$$\pi_{t+1} = Y_{t+1} + (1 - \delta)K_{t+1} - w_{t+1}N_{t+1} - R_{t+1}B_{t+1} - r_{t+1}Z_{t+1} \quad (5)$$

The entrepreneur's objective is to maximize π_{t+1} . However, at the time Z_{t+1} and B_{t+1} are chosen, ϵ_{t+1} is not known. The entrepreneur weighs the different π_{t+1} 's corresponding to different realizations of ϵ_{t+1} by the product of the probability of that realization and the associated marginal utility of consumption.

- (e) Suppose that at date t there were markets for date $t + 1$ state-contingent goods. Show that the weights that we apply to profits across states of nature correspond to the prices that would obtain in such markets for state-contingent goods.
- (f) The entrepreneur in our model is concerned with $E_t u_{c,t+1} \pi_{t+1}$, where $u_{c,t+1}$ is treated as exogenous. The entrepreneur solves

$$\max_{Z_{t+1}, B_{t+1}} E_t \{ u_{c,t+1} \max_{N_{t+1}} \pi_{t+1} \}. \quad (6)$$

The maximization inside the braces reflects that the firm's employment decision is made after the realization of ϵ_{t+1} . The maximization outside the braces reflects that the financing decision is made before the realization of ϵ_{t+1} . Make sure you understand this point.

- (g) write out the first order necessary conditions associated with the firm's choice of Z_{t+1} , B_{t+1} , and N_{t+1} .

Suppose that there is free entry, so that, in equilibrium, $E_t \{ u_{c,t+1} \pi_{t+1} \} = 0$. Suppose, in addition, that the equilibrium process, r_t , has the property that ex post profits are zero, i.e., $\pi_t = 0$.

- (h) Define a sequence of markets equilibrium and a recursive competitive equilibrium for this economy.
- (i) Show that the debt-to-equity ratio, b_{t+1}/z_{t+1} , is not pinned down in equilibrium. In particular, if there is an equilibrium in which some particular value of $\gamma = b_{t+1}/z_{t+1}$ holds, then there exists an equilibrium with the same allocations of consumption, output and capital, for all other values of γ .
- (j) Show, using the linear homogeneity assumption on F , the firm's first order condition on labor, the entrepreneur's financing constraint, and the zero ex post profit condition, that

$$r_{t+1} = (F_{k,t+1} + 1 - \delta)(1 + \gamma) - \gamma R_{t+1}.$$

Use this to show, using your result in (v), that the equity premium, P_t , increases with γ .

- (k) Let $u(c_t) = \log(c_t)$ and resource constraint, $c_t + k_{t+1} = k_t^\alpha n_t^{(1-\alpha)} \exp(x_t)$, where x_t has a first order autoregressive representation: $x_t = \rho x_{t-1} + \varepsilon_t$, where ε_t iid over time and independent of x_{t-1} . Show that the equity premium, P_t , reduces to:

$$P_t = 1 - Cov_t(\exp(-\varepsilon_{t+1}), \exp(\varepsilon_{t+1}))(1 + \gamma),$$

where γ is the debt/equity ratio, assumed to be fixed.

Suppose ε_t is iid over time with $\varepsilon_t = \sigma$ with probability 1/2 and $\varepsilon_t = -\sigma$ with probability 1/2. It is easily verified that $Var(\varepsilon_t) = (\sigma)^2$, so that σ is the standard deviation of ε_t . Based on his analysis of quarterly U.S. data, Prescott (1986, Federal Reserve Bank of Minneapolis Quarterly Review) has argued that an empirically plausible value for this quantity is .00763. Compute P_t for the case $\gamma = 0$.

Mehra and Prescott argued that the equity premium in the U.S. averages 1.07 percent per annum, or, 1.017 per quarter. What value of γ is necessary to make P_t this large?

2. Following is a deterministic economy composed of one representative, competitive household, and a representative, competitive firm. Preferences of the household are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \quad u(c, n) = \log c + \sigma \log(1 - n),$$

where c_t denotes consumption and n_t denotes hours worked. The household budget equation is

$$c_t + I_t \leq w_t n_t + r_t k_t,$$

where w_t is the wage rate and r_t is the rental rate on capital, k_t . Here, I_t denotes investment, which the household applies to increasing the stock of capital, using the following technology:

$$k_{t+1} = (1 - \delta)k_t + I_t, \quad 0 < \delta < 1.$$

The representative firm has access to the following technology:

$$y = Y^\gamma k^\alpha n^{1-\alpha}, \quad \gamma = 1 - \alpha, \quad \alpha = 1/3,$$

where Y is economy-wide average output, and y, k, n are firm output, capital, and employment, respectively. Note that the firm has constant returns to scale in the variables that it controls directly. Note too, the ‘externality’ in this production function. If all other firms are producing a lot (i.e., Y is big), this raises the productivity of an individual firm. Firms maximize profits given r_t and w_t . In equilibrium, $Y = y$.

- (a) Define a sequence-of-markets equilibrium for this economy.
 (b) Show that the Euler equations for labor and capital are, respectively:

$$w_t = -\frac{u_{n,t}}{u_{c,t}}, \quad u_{c,t} = \beta u_{c,t+1} [r_{t+1} + 1 - \delta],$$

where $u_{x,t}$ is the partial derivative of u with respect to $x = c_t, n_t$. Describe the transversality condition, and sketch how you would prove the sufficiency for household optimization of the transversality conditions and the Euler equations. (Hint: the proof can involve a lot of messy algebra. You need not do this. But, do indicate in a precise way what strategy you would take for establishing the proof.)

- (c) Show that in an equilibrium, the first order conditions for firms and the resource constraint are (hint: impose $y = Y$):

$$r_t = \alpha n_t^2, \quad w_t = \gamma k_t n_t, \quad c_t + k_{t+1} - (1 - \delta)k_t = k_t n_t^2.$$

- (d) Show that by combining the household and firm Euler equations, one obtains:

$$\frac{n_t^2 + 1 - \delta - \frac{\gamma}{\alpha} n_t (1 - n_t)}{n_t (1 - n_t)} = \beta \frac{\alpha n_{t+1}^2 + 1 - \delta}{n_{t+1} (1 - n_{t+1})}, \quad t = 0, 1, 2, \dots \quad (7)$$

Let (7) be represented as $v(n_t, n_{t+1}) = 0$. Note that this implicitly defines a map from n_t to n_{t+1} . Show that this map is composed of two functions, $n_{t+1} = f_i(n_t)$, $i = 1, 2$, where $f_1 > f_2$ for all n_t . Display analytic expressions for these functions. (Hint: remember the formula for the roots of a second order polynomial.)

- (e) The following proposition is true. ‘Suppose a sequence, n_0, n_1, \dots is found which satisfies (7) and also has the property, $a \leq n_t \leq b$ for $a > 0$ and $b < 1$ for all t . Then that sequence corresponds to an equilibrium.’ Sketch a proof of this proposition.

- (f) Explain why there are two stationary (i.e., equilibria with n_t constant) equilibria for this economy.
- (g) Let n^* be the greater of the two stationary equilibria just identified. Show that there is an interval, D , about n^* such that if $n_0 \in D$ and $n_t, t = 1, 2, \dots$, solves (7), then the given sequence, $n_t, t \geq 0$ represents an equilibrium. (Hint: note that $|\partial f_2(n)/\partial n| < 1$ for $n = n^*$.) Note that, since D is an interval, we have identified a continuum of equilibria. Explain why there are likely to be a great many more equilibria than just these two.
- (h) Show that the efficient allocations involve a constant value of n_t, \bar{n} , say. It turns out that $n^* < \bar{n}$. Give an economic interpretation of this.