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D11-1, Fall 2000

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. (20) Consider the following operator on the space of functions:

$$T[w](x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta w(x'),$$

   where $x \in X \subset R^l$, $\Gamma : X \to X$, $F : A \to R$, $A = \{(x, y) : x \in X, y \in \Gamma(x)\}$. Here, X is convex; Γ is non-empty, compact and continuous; F is bounded and continuous. Suppose that (i) F is strictly concave and (ii) Γ is convex. Prove that if w is continuous, bounded and weakly concave, then $T[w]$ is strictly concave. Be sure to make very clear how (i) and (ii) are used to establish the result.

2. (25) Suppose the resource constraint has the form,

$$c_t + g + k_{t+1} \leq f(k_t) + (1 - \delta)k_t, \; 0 < \delta < 1,$$

   where f is increasing, $f(0) = 0$, $f_k \to \infty$ as $k \to 0$ and $f_k \to 0$ as $k \to \infty$, and $g > 0$ is government spending.

   (a) Define the constraint set, $\Gamma(k)$, in the S-L canonical form for this economy. Show that there is a always a lowest number, $k^{lb} > 0$, such that $\Gamma(k)$ is non-empty for all $k \geq k^{lb}$.

   (b) Show that, if g is small enough, there is value for the capital stock, $\underline{k} > k^{lb}$, with the following property. If $k^{lb} < k < \underline{k}$ then the economy is *not viable* in the sense that the only feasible option is for k to fall and eventually drop below k^{lb}.

   (c) Show that if g is large enough, there is no value of k such that the economy is viable.

3. (30) Suppose households are all identical, and have preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

and the resource constraint, expressed in per capita terms, is:

$$c_t + k_{t+1} - (1-\delta)k_t \leq f(k_t, n_t),$$

where $c_t, k_{t+1} \geq 0$ denote consumption and capital, respectively, and $\beta, \delta \in (0,1)$. Also, n_t denotes hours worked, which are constrained by $0 \leq n_t \leq 1$. Utility is strictly increasing, concave and f is linearly homogeneous, strictly increasing. As you answer this question, make up whatever additional assumptions about u, and f you feel you need.

Consider an Arrow-Debreu decentralized market economy. There are three goods at every date: date t output, which has price p_t, date t labor services which have price $p_t w_t$ per period, and date t capital services which rent at $p_t r_t$, per period, t=0,1,... . Households own the initial capital stock, k_0, and make date t investment decisions for t=0,... . That is, they buy date t output and split it between date t consumption and date t investment in new capital. Also, they supply date t labor services. Households must obey their budget equation, which specifies that the value of all goods bought (across all dates) must not exceed the value of all goods sold (again, across all dates), plus profits. Firms rent date t capital services and hire date t labor to operate the production technology and produce date t output which they sell to households, t=0,1,... . Households own the firms, and so receive any profits they generate. Profits are the excess of what firms sell (at all dates) over what they buy (at all dates.) Trading in markets occurs in date 0. The economy is competitive in the sense that all agents take prices as beyond their control.

(a) Formally define the household's and firm's date 0 optimization problems in an Arrow Debreu decentralized market economy. Formally define an Arrow Debreu equilibrium.

(b) Show that: (i) if $p_t, t = 0, 1, ...$ are prices in an Arrow Debreu equilibrium, then $\lambda p_t, t = 0, 1, ...$ are too, for $\lambda > 0$; (ii) all prices,

p_t, r_t, w_t, t=0,1,..., are strictly positive in Arrow-Debreu equilibrium; (iii) firm profits are zero; (iv) the firm's date 0 optimization problem is equivalent to a series of date t, t=0,1,2,... maximization problems; (v) a particular relationship must hold between r_t and w_t.

(c) Define the *efficient* allocations (e.g., quantities $n_t, k_{t+1}, c_t,\ t = 0, 1, \ldots$) as the allocations which maximize the representative household's utility subject to the resource constraint. Show that the allocations in an Arrow-Debreu equilibrium are efficient.

(d) Consider a sequential markets competitive economy. Agents meet in markets at the beginning of every date. In the period t market, households sell labor and capital services, buy output, and split this between consumption and capital accumulation. In the period t market, firms rent labor and capital services, produce output and sell it to households. Any excess of receipts over expenses are passed on to households as dividends in the current period. Households receive these dividends because they own the firms. The economy is competitive in the sense that all agents treat all prices and dividends as beyond their control. Formally define this equilibrium, and show that the allocations in this equilibrium coincide with the allocations in the Arrow-Debreu equilibrium.

(e) Suppose the model economy was replaced by one in which period 1 utility is discounted by $\psi\beta$, period 2 utility is discounted by $\psi\beta^2$, and period j utility, etc. So, this economy is the same as the one above, when $\psi = 1$. Explain why there exists no sequence of markets equilibrium for this economy, while there is no problem with the Arrow-Debreu equilibrium.

4. (25) Suppose the planner maximizes, by choice of c_t, k_{t+1}, and n_t,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} - (1-\delta)k_t &= y_t, \\ y_t &= k_t^{\alpha}[\exp(\mu t) n_t]^{(1-\alpha)}, \\ c_t &\geq 0, k_{t+1} \geq 0, 1 \leq n_t \leq 1 \end{aligned}$$

where $\mu > 0$ and δ, α, β are all positive and less than 1. Suppose u has the constant elasticity of substitution form:

$$u(c, n) = \left(\{ [c^{(1-1/\sigma)} + \gamma(1-n)^{(1-1/\sigma)}]^{1/(1-1/\sigma)} \}^{\psi} - 1 \right) / \psi$$

Show that if c_t, y_t converge to a path in which c_t / y_t and n_t are constant, then it must be that the elasticity of substitution is unity, i.e., $\sigma = 1$.