Christiano
411-1, Fall 2001

FINAL EXAM

The exam has three parts. The points for each part are indicated in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (40) The typical two-period-lived household born in period t solves the following problem:

$$\max_{c_t^t, c_{t+1}^t} \log\left(c_t^t\right) + \beta \log\left(c_{t+1}^t\right),\ 0 < \beta < 1$$

where c_t^t and c_{t+1}^t are the household's period t and period $t+1$ consumption, respectively. In its first period of life, the household inelastically supplies one unit of labor for a wage rate of w_t. It uses this to finance current period consumption and investment. In period $t+1$ it finances its consumption based on the rental earnings of capital and the sale of undepreciated capital. The household does not work in its second period of life. Its first and second period budget constraints are:

$$\begin{aligned} c_t^t + k_{t+1} &\leq w_t, \\ c_t^{t+1} &\leq r_{t+1}k_{t+1} + (1-\delta)k_{t+1}. \end{aligned}$$

The first period of this economy is $t = 0$. The initial stock of capital, k_0, is owned by the initial old households in their second period of life. There is no population growth: each period, the number of new households equals the number of old households.

The firm sector is as it is in Romer's model:

$$r_t = \alpha A,\ w_t = (1-\alpha)Ak_t,\ y_t = Ak_t,$$

where $A > 0$, $0 < \alpha < 1$ are constants, r_t is the rental rate of capital, w_t is the wage rate, and y_t is per capita aggregate output. The resource constraint for the economy is:

$$c_t^t + c_t^{t-1} + k_{t+1} = Ak_t + (1-\delta)k_t,\ t = 0, 1, 2, ..., 0 < \delta < 1.$$

The parameters of the model are A, α, δ, β.

(a) (2) Define an equilibrium for this economy. Display an expression relating the rate of return on capital, r^k, to the parameters of the model.

(b) (6) Derive an expression relating the equilibrium growth rate of household consumption, c_{t+1}^t/c_t^t, to the parameters of the model.

(c) (6) Show that the saving of young households, k_{t+1}, is a fixed fraction of their current period income, w_t. Display an expression relating that fraction to the parameters of the model.

(d) (5) Display an expression relating the growth rate of the aggregate capital stock, λ^*, to the parameters of the model.

(e) (6) Let c_t denote period t aggregate consumption, where $c_t = c_t^t + c_t^{t-1}$. Show that the growth rate of c_t is λ^*. Can λ^* be different from the growth rate of household consumption? Explain.

(f) (15) Suppose we introduce a new asset. It is a claim on an intrinsically useless object: one that does not generate utility and that cannot be used to produce goods of any type. There is a market in which households can buy $a_t \geq 0$ units of the asset when young for $P_t a_t$ and sell it when old for $P_{t+1} a_t$, where P_t is the asset's price. There is a fixed supply, $a > 0$, of the asset which is owned by the period 0 old households. Implicitly, in the previous parts of this question we explored the 'fundamental' equilibrium of the model, in which the asset is valued at its 'fundamental value', with $P_t = 0$. There may also exist 'bubble' equilibria, in which $P_t > 0$.

   i. Modify the household's two budget constraints to reflect the existence in each t of a market for trading a_t.

   ii. Suppose there is an equilibrium in which $P_t > 0$. Derive an expression relating P_{t+1}/P_t to the parameters of the model. Why might a young household want to purchase a_t?

   iii. Does the existence of the new asset affect the fraction of period t income consumed by households born in period t? Does your answer depend on whether P_{t+1}/P_t takes on its equilibrium value described in (ii)? Explain.

   iv. It turns out that if $r^k > \lambda^*$, then the only equilibrium is the one with $P_t = 0$ for all t. When $r^k < \lambda^*$ there exist equilibria with $P_t > 0$. Provide intuition for this result. (You need not provide a rigorous proof.)

2. (20) Consider the following two-period economy. The representative household maximizes

$$u(c_1 + c_2, l)$$

subject to the following two budget constraints:

$$\begin{aligned} c_1 + k &\leq \omega \\ c_2 &\leq (1-\delta)Rk + (1-\tau)\omega l. \end{aligned}$$

Here, ω is the wage rate (which corresponds to the marginal product of labor), R denotes the rental rate of capital (its marginal product), and c_i denotes consumption in period i, $i = 1, 2$. One unit of labor is supplied inelastically in period 1 and l units of labor are supplied in period 2. Finally, k denotes saving in period 0, while δ and τ denote the tax rates, respectively, on capital income and labor income. We suppose that preferences have the following parametric form:

$$u(c_1 + c_2, l) = c_1 + c_2 - \frac{1}{2}l^2.$$

When $(1-\delta)R = 1$, we suppose that the household chooses $c_1 = 0$, $k = \omega$. The government must finance an exogenously given level of consumption, G, in period 2, subject to the following budget constraint:

$$G \leq \delta Rk + \tau\omega l.$$

The government chooses values for δ and τ so that its budget constraint is satisfied and utility of the representative household is as large as possible.

For the problem to be interesting, G must exceed the maximum revenues from capital taxes alone. In addition, it must not be so large as to exceed the maximum possible revenues from capital and labor taxes. We suppose that G satisfies these conditions.

(a) (2) Define the periods 1 and 2 resource constraints for this economy.

(b) (2) Define a Ramsey equilibrium.

(c) (6) Express government revenues as a function of δ, τ, taking into account the private sector allocation rule relating equilibrium allocations to δ, τ. Graph this function with τ on the horizontal axis and revenues on the vertical axis. Note how changes in δ induces parallel shifts in the graph in a vertical direction. What value of δ makes this graph as high as possible?

(d) (6) Derive a government 'utility function' over values of δ and τ. Graph this utility function with τ on the horizontal axis and government utility on the horizontal axis. How does δ shift this function?

(e) (4) Prove rigorously that in a Ramsey equilibrium, $\delta = (R-1)/R$.

3. (40) Consider the following model economy. Utility of the representative household is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \ u(c, l) = \frac{\left[c(1-l)^{\psi}\right]^{1-\gamma} - 1}{1-\gamma}, \ \psi > 0, \ \gamma > 0,$$

where $l_t \geq 0$ is labor supplied to the market and $1 - l_t \geq 0$ is leisure. Households purchase consumption goods, $c_t \geq 0$, and investment goods, i_t. They own the stock of capital, which satisfies the following accumulation technology:

$$k_{t+1} = (1-\delta)k_t + i_t.$$

Households have the following budget constraint:

$$c_t + i_t \leq r_t k_t + w_t l_t + \pi_t,$$

where π_t are lump-sum profits.

Production is carried out by firms at three levels. The most downstream firm is a representative, competitive final good firm which produces y_t. This gives rise to the following goods market clearing condition:

$$c_t + i_t \leq y_t.$$

The representative firm produces y using labor-intensive goods, y^l, and capital-intensive goods, y^k, and the following technology:

$$y = \left(y^l\right)^{\alpha} \left(y^k\right)^{1-\alpha}, \ 0 < \alpha < 1.$$

The price of the labor-intensive good is p^l and the price of the capital-intensive good is p^k. (To save on notation, in describing the firm sector we do not include time subscripts.)

The next level upstream consists of the industries which produce the labor-intensive and capital-intensive goods. These industries are also characterized by perfect competition and each industry's output is produced by a representative firm. They use the following technology:

$$\begin{aligned} y^l &= \left(\int_0^n \left[y^l(i)\right]^\zeta di\right)^{\frac{1}{\zeta}} \\ y^k &= \left(\int_0^m \left[y^k(j)\right]^\zeta dj\right)^{\frac{1}{\zeta}}, \end{aligned}$$

where $0 < \zeta < 1$. The labor intensive good is produced using a range of inputs, $y^l(i)$ for $i \in (0, n)$. The capital-intensive good is produced using a range of inputs, $y^k(j)$, for $j \in (0, m)$. Here, n, $m > 0$. The price of the i^{th} labor and capital-intensive goods is $p^l(i)$ and $p^k(j)$, respectively.

The firms furthest upstream are the ones producing the intermediate goods used in the labor-intensive and capital-intensive industries. These firms are monopolists in the product market, though they are competitive in resource markets. They take the wage rate, w, and rental rate on capital, r, as exogenous. They have the following linear production technologies:

$$y^l(i) = l(i), \ y^k(j) = k(j),$$

where $l(i)$ is the quantity of labor used in the production of the i^{th} intermediate good in the labor-intensive sector. Also, $k(j)$ is the quantity of capital used in the production of the j^{th} intermediate good in the capital-intensive sector.

We assume that n and m each grow exogenously over time at rates γ_n and γ_m, respectively. That is, $n_t = \gamma_n n_{t-1}$, and $m_t = \gamma_m m_{t-1}$ for all t.

(a) (2) Derive the first order necessary conditions for household optimization.

(b) (2) Write out the problem of the final good producer, and derive its first order conditions.

(c) (5) Write out the problem of the representative firms in the capital-intensive and labor-intensive goods industries. Derive their first order conditions. Show how these, together with the relevant production functions, can be used to construct an exact relationship between p^l and $p^l(i)$, $i \in (0, n)$ and between p^k and $p^k(j)$, $j \in (0, m)$.

(d) (3) Write out the problem of the intermediate good producers and derive an expression relating the price they optimally set to their resource costs.

(e) (4) Explain why production is identical among the intermediate good producers within the capital-intensive industry. Similarly for the labor-intensive goods industry. Suppose that the amount of capital supplied by households is k_t and the amount of labor supplied is l_t. Show that market clearing in the resource markets implies the i^{th} intermediate good producer in the labor-intensive industry uses $l(i) = l_t/n_t$ units of labor. Similarly, the j^{th} intermediate good producer in the capital-intensive industry uses $k(j) = k_t/m_t$ units of capital.

(f) (6) Show that aggregate output can be represented as

$$y_t = (z_t l_t)^{\alpha} k_t^{1-\alpha},$$

where z_t represents exogenous technical change and $z_t = \gamma_z z_{t-1}$. Derive an expression relating z_t to n_t and m_t. Also, derive an expression relating γ_z to γ_n and γ_m.

(g) (6) Show that

$$\alpha \frac{y_t}{l_t} = \frac{w_t}{\zeta}, \; (1-\alpha)\frac{y_t}{k_t} = \frac{r_t}{\zeta}.$$

(h) (6) Explain why the equilibrium in this economy is not efficient. Suppose that households received a subsidy on their capital and labor income, which was financed by lump sum taxes on households. That is, instead of receiving $r_t k_t$ and $w_t l_t$ in capital income, households receive $(1+\theta)r_t k_t$ and $(1+\theta)w_t l_t$, where θ is the subsidy rate. Is there a value of θ which will cause the equilibrium of this economy to be efficient? Explain.

(i) (6) Define a balanced growth equilibrium for this economy. What is the growth rate of y_t, k_t and l_t along a balanced growth path equilibrium. Express these growth rates in terms of γ_n and γ_m.