Suggested answers to Fall, 2001 Final for Economics 411-1

1 Question 1.

First, a discussion of the fundamentals equilibrium

Households born in period t solve the following problem:

$$\max_{c_t^t, c_{t+1}^t} \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \beta \frac{\left(c_{t+1}^t\right)^{1-\gamma}}{1-\gamma},$$

subject to

$$\begin{aligned} c_t^t + k_{t+1} &\leq w_t, \\ c_t^{t+1} &\leq r_{t+1} k_{t+1} + (1-\delta) k_{t+1}, \end{aligned}$$

where w_t is the wage rate earned while in the first period of life, supplying one unit of labor inelastically. The households don't work in the second period of life. They finance their consumption in old age with rental income on capital and by selling the undepreciated part of their capital stock.

The first period of this economy is period $t = 0$, when there is an old generation alive who owns k_0, the initial stock of capital.

The first order condition associated with the saving decision while young is:

$$\frac{1}{\beta} \left(\frac{c_{t+1}^t}{c_t^t} \right)^{\gamma} = r_{t+1}^k, \ t = 0, 1, 2, ...$$

where $r_{t+1}^k = r_{t+1} + 1 - \delta$. Suppose, as in the Romer model, that

$$r_t = \alpha A, \ w_t = (1-\alpha) A k_t, \ y_t = A k_t.$$

Then, household consumption growth is:

$$\frac{c_{t+1}^t}{c_t^t} = \lambda = \left[\beta \left(\alpha A + 1 - \delta\right)\right]^{\frac{1}{\gamma}}. \tag{1}$$

To determine consumption of the young, first determine the intertemporal budget constraint:

$$c_t^t + \frac{c_{t+1}^t}{\alpha A + 1 - \delta} \leq (1-\alpha) A k_t, \ t = 0, 1, 2, ... \tag{2}$$

or, imposing the strict equality:

$$c_t^t = \frac{(1-\alpha)A}{1+\beta^{\frac{1}{\gamma}}(\alpha A + 1 - \delta)^{\frac{1}{\gamma}-1}} k_t = \psi k_t,$$

say. So, a characterization of what the individual household does in equilibrium is based on λ, ψ and k_t. From here on, we set $\gamma = 1$, so that

$$\psi = \frac{(1-\alpha)A}{1+\beta} \tag{3}$$

Thus, in its first period of life, the household consumes $1/(1+\beta)$ of its income (i.e., w_t) and saves the rest, so that:

$$k_{t+1} = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta}(1-\alpha)Ak_t$$

Then, the growth rate of the aggregate stock of capital is:

$$\lambda^* \equiv \frac{k_{t+1}}{k_t} = \frac{\beta}{1+\beta}(1-\alpha)A = \beta\psi. \tag{4}$$

Note that the growth rate of the aggregate stock of capital need not be the same as the growth rate of household consumption.

The resource constraint for this economy is:

$$c_t^t + c_t^{t-1} + k_{t+1} = Ak_t + (1-\delta)k_t, \; t = 0, 1, 2, ... \tag{5}$$

Let $c_t = c_t^t + c_t^{t-1}$ denote aggregate consumption, so that

$$c_t = [A + (1-\delta)] k_t - k_{t+1}.$$

From this, one can deduce the growth rate of aggregate consumption:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \frac{[A + (1-\delta)] k_{t+1} - k_{t+2}}{[A + (1-\delta)] k_t - k_{t+1}} \\ &= \frac{[A + (1-\delta)] k_t - k_{t+1}}{[A + (1-\delta)] k_t - k_{t+1}} \lambda^* \\ &= \lambda^*. \end{aligned}$$

The growth rate of aggregate consumption does correspond to the growth rate of the capital stock.

Suppose one period in this model corresponds to 30 years. Then, $\beta = (1/1.03)^{30} = 0.40$. Also, $\delta = 1 - (1 - .1)^{30} = 0.96$. And, $\alpha = 0.36$. Also, assuming consumption growth at the individual level of 3 percent per year, $\lambda = 1.03^{30} = 2.43$. In this case, $\lambda^* = 3.0654$, $r^k = 6.075$.

Now turn to the bubble equilibria.

Let's introduce a new asset, one that is a claim on an intrinsically useless object: one that does not generate utility and cannot be used to produce goods of any type. Individual households can purchase any amount, $a_t \geq 0$, of this asset. The asset is in permanently fixed supply, $a > 0$. It is owned by the initial old households, and let its price be denoted by P_t.

Suppose there is a market in which people can buy a_t units of the asset for $P_t a_t$ when young and sell it when old for $P_{t+1} a_t$. Revising the household's budget constraint to reflect this new market:

$$\begin{aligned} c_t^t + k_{t+1} + P_t a_t &\leq w_t \\ c_{t+1}^t &\leq r_{t+1}^k k_{t+1} + P_{t+1} a_t. \end{aligned}$$

Implicitly, in the previous section we explored the 'fundamental' equilibrium in which $P_t = 0$, i.e., the price of the asset corresponds to its fundamental value. Below we show that if $r^k > \lambda^*$, then the fundamental equilibrium is the only equilibrium. However, when $r^k < \lambda^*$ there exist ('bubble') equilibria with $P_t > 0$.

The intuition for this result is straightforward. Recall from the previous subsection that the growth rate of the aggregate capital stock is determined by the fraction of period t income, $\beta/(1+\beta)$, that households born in period t save. Since all their saving goes towards the accumulation of capital in the fundamental equilibrium we concluded that the growth rate of capital - and of period t income - is $\lambda^* = \beta(1-\alpha)A/(1+\beta)$. Now, in any equilibrium with $P_t > 0$ arbitrage between a_t and k_t requires that $P_{t+1}/P_t = r^k$.[1] Relative to the fundamental equilibrium, in a bubble equilibrium a fraction (this is determined by the level of P_t) of household income is diverted towards a.

[1] Suppose not. If $P_{t+1}/P_t < r^k$, then $a_t = 0 < a$ and market clearing fails. If $P_{t+1}/P_t > r^k$, then demand by young households for k_t is zero in periods 0, 1, 2, ... Market clearing in period 0 when period 0 old households supply $k_0 > 0$ fails.

With the reduction in the fraction of income used to buy capital, the growth rate of the capital stock is reduced.

Suppose that the growth rate in the fundamental equilibrium, λ^*, is less than r^k. With the growth rate of capital in a candidate bubble equilibrium lower than λ^*, this means that eventually the price of a_t will outstrip the ability of young households to purchase it. A bubble cannot be part of an equilibrium in this case. Suppose now that $\lambda^* > r^k$. Then, as long as P_t is not too large, it is possible for there to exist a bubble equilibrium in which young households can always afford to purchase a. It is easy to see that there is a precise level of P_t where the growth rate of capital is r^k and the equilibrium of the economy is constant. We explore these possibilities carefully below.

From the arbitrage relationship mentioned above, we conclude:

$$\frac{P_{t+1}}{P_t} = \alpha A + 1 - \delta.$$

Using this relationship, and substituting out for k_{t+1} in the household's budget constraint, we obtain the same intertemporal budget constraint we had before, (2). This is not surprising, since the inclusion of a_t in the household's budget constraint does not change its intertemporal consumption opportunity set. Then, $c_t^t = \psi k_t$, where ψ is defined in (3). The aggregate resource constraint is still (5). The substitutions done before are less convenient now. Divide both sides of (5) by k_t, impose $c_t^t = \psi k_t$ for all t, and define $\lambda_t^* = k_{t+1}/k_t$, as before:

$$\begin{aligned}
\frac{\lambda\psi}{\lambda_{t-1}^*} + \lambda_t^* &= A + 1 - \delta - \psi \\
&= A + 1 - \delta - \frac{(1-\alpha)A}{1+\beta} \\
&= A\frac{\beta+\alpha}{1+\beta} + 1 - \delta \\
&= \phi,
\end{aligned}$$

for $t = 1, 2, 3, ...,$ where

$$\phi = A\frac{\beta+\alpha}{1+\beta} + 1 - \delta.$$

It is easy to verify that λ^* in (4) and r^k are the fixed points of the above difference equation.[2] Write this as follows

$$\lambda_t^* = g(\lambda_{t-1}^*) = \phi - \frac{\lambda\psi}{\lambda_{t-1}^*}.$$

Since $g' > 0$ and $g'' < 0$, this is a strictly increasing and strictly concave function. Also, $g' \to -\infty$ as $\lambda_{t-1}^* \to 0$ and $g' \to 0$ as $\lambda_{t-1}^* \to \infty$. Thus, if g crosses the 45 degree line then it first crosses from below and then crosses from above. Obviously, when it crosses from below it has slope greater than unity, and when it crosses from above it has slope less than unity. That is, $\lambda^* > r^k$ implies that the slope of g at $\lambda_{t-1}^* = \lambda^*$ is less than unity while $\lambda^* < r^k$ implies that the slope of g at $\lambda_{t-1}^* = \lambda^*$ is greater than unity. These observations are confirmed by direct differentiation:

$$g'(\lambda^*) = \frac{\lambda\psi}{(\lambda^*)^2} = \frac{\lambda\psi}{(\beta\psi)^2} = \frac{\lambda}{\beta^2\psi} = \frac{\lambda}{\beta\lambda^*} = \frac{r^k}{\lambda^*}.$$

In the case of the parameter values discussed above, $g'(\lambda^*) = 1.98$.

[2] Substitute $\lambda_t^* = \lambda_{t-1}^* = \lambda^* = \beta\psi$ into:

$$\frac{\lambda\psi}{\lambda_{t-1}^*} + \lambda_t^* + \psi$$

one obtains

$$\frac{\lambda\psi}{\beta\psi} + (1+\beta)\psi = \alpha A + 1 - \delta + (1-\alpha)A,$$

so that λ^* is indeed a fixed point of the difference equation. We can also try and find the other fixed point, $\bar{\lambda}^*$, by noting

$$(x - \lambda^*)(x - \bar{\lambda}^*) = x^2 - \phi x + \lambda\psi,$$

so that

$$\begin{aligned} \lambda^* + \bar{\lambda}^* &= \phi \\ \lambda^*\bar{\lambda}^* &= \lambda\psi. \end{aligned}$$

The latter result says that

$$\bar{\lambda}^* = \frac{\lambda\psi}{\lambda^*} = \frac{\lambda}{\beta} = r^k.$$

So, the fixed points are r^k and λ^*.

The difference equation determines λ_t^* for $t = 1, 2, ...$ given λ_0^*. But, λ_0^* is still to be determined. For this, use the period 0 resource constraint:

$$\psi k_0 + c_0^{-1} + k_1 = (A + 1 - \delta) k_0, \tag{6}$$

where ψk_0 is c_0^0, the first period consumption of the people born in period 0. The budget constraint in period 0 for the period 0 old is:

$$c_0^{-1} = [\alpha A + 1 - \delta] k_0 + P_0 a,$$

so the resource constraint becomes:

$$\psi k_0 + [\alpha A + 1 - \delta] k_0 + P_0 a + k_1 = (A + 1 - \delta) k_0.$$

Divide this by k_0 :

$$\psi + [\alpha A + 1 - \delta] + \frac{P_0 a}{k_0} + \lambda_0^* = (A + 1 - \delta),$$

or

$$\begin{aligned} \lambda_0^* &= (1-\alpha)A - \frac{(1-\alpha)A}{1+\beta} - \frac{P_0 a}{k_0} \\ &= (1-\alpha)A\frac{\beta}{1+\beta} - \frac{P_0 a}{k_0} \\ &= \lambda^* - \frac{P_0 a}{k_0} \end{aligned}$$

From this, it is evident that we cannot have a bubble if $r^k > \lambda^*$. If this were the case, then g' would be greater than unity at $\lambda_{t-1}^* = \lambda^*$, and so if $P_0 > 0$, so that $\lambda_0^* < \lambda^*$, then $\lambda_t^* \to -\infty$ which cannot be an equilibrium. If, on the other hand, $r^k < \lambda^*$, then P_0 positive but not too large (we don't want $\lambda_0^* < r^k$), there may be bubble equilibria. These would produce a temporary period of low λ_t^*, which would permanently reduce the size of the capital stock.

Consider the parameter values, $\beta = 0.80$, $A = 31.05$, $\delta = 0.96$, $\alpha = 0.20$, $\lambda = 5$. In this case, $r^k = 6.25$, $\lambda^* = 11.04$, $\lambda = 5$. With $P_0 = 0$, then the unique equilibrium is one in which the aggregate economy grows at the gross rate 11.04, while individual consumption grows at the gross rate, 5. But, there are also equilibria in which $P_0 > 0$ and $P_{t+1}/P_t = r^k$. We now describe two examples.

Example 1:

Suppose, as in the previous example, that $r^k < \lambda^*$. There is one stationary equilibrium in which the growth rate of the economy is r^k while that is the growth rate of P_t too. We find this by setting P_0 so that

$$\lambda_0^* = \lambda^* - \frac{P_0 a}{k_0} = r^k,$$

so that

$$P_0 = \frac{\left(\lambda^* - r^k\right) k_0}{a}.$$

To verify that $k_{t+1} = r^k k_t$, $t = 0, 1, 2, 3, \ldots$ corresponds to an equilibrium when P_0 takes on this value, we need to verify that the other equilibrium objects can be found such that these, together with the given k_t's satisfy all the equilibrium conditions.

The budget constraint of the old in period 0 implies:

$$c_0^{-1} = [\alpha A + 1 - \delta] k_0 + P_0 a = \left[\alpha A + 1 - \delta + \lambda^* - r^k\right] k_0.$$

Their consumption is high in a $P_0 > 0$ equilibrium, because a value is assigned to their intrinsically worthless asset, a. Expenditures of the period 0 young on consumption goods and capital is:

$$\begin{aligned} c_0^0 + k_1 &= (1-\alpha) A k_0 - P_0 a \\ &= \left[(1-\alpha) A - (\lambda^* - r^k)\right] k_0. \end{aligned}$$

Note, their consumption of goods and capital is less than what it is when $P_0 = 0$, reflecting the diversion to a. That the period 0 resource constraint is satisfied is easily verified. Similarly, the resource constraint for each t is satisfied as long as each person's budget constraint is satisfied, consistent with Walras' law.

Let's see if household optimization is satisfied, i.e., $c_t^t = \psi k_t$ and $c_{t+1}^t / c_t^t = \lambda$ for $t = 0$. Imposing the given growth rate of capital in young households' budget constraint:

$$c_0^0 + r^k k_0 = \left[(1-\alpha) A - (\lambda^* - r^k)\right] k_0,$$

or

$$
\begin{aligned}
\frac{c_0^0}{k_0} &= (1-\alpha)A - \lambda^* \\
&= (1-\alpha)A - \beta\frac{(1-\alpha)A}{1+\beta} \\
&= \frac{(1-\alpha)A}{1+\beta} = \psi.
\end{aligned}
$$

An implication of this result is that the entire amount of the fall in consumption by the young in period 0 comes from a diversion of purchases on k_1 to a. Their consumption, being ψk_0, is the same, whether P_0 is positive or zero. The consumption in old age of the generation born in period 0 is:

$$
\begin{aligned}
c_1^0 &= [\alpha A + 1 - \delta]\, k_1 + P_1 a = r^k \left\{[\alpha A + 1 - \delta]\, k_0 + P_0 a\right\} \\
&= r^k k_0 \left\{[\alpha A + 1 - \delta] + (\lambda^* - r^k)\right\} \\
&= r^k k_0 \lambda^* \\
&= k_0 \lambda \psi \\
&= \lambda c_0^0.
\end{aligned}
$$

This establishes efficiency for the generation born in period 0. Since the quantity that they consume over their lifetime is invariant to P_0, their utility is unaffected by the presence of a bubble. It is easy to show that the efficiency conditions also hold for later generations, though their utility levels are affected by the presence of a bubble.

Example 2

It is easy to see that for any $P_0 > 0$ but smaller than the one just described, the economy's low growth rate is just temporary. Eventually it will return to 11.04. There isn't a problem with the household's budget constraint here because $P_t a$ is growing at the rate r^k whereas aggregate variables are growing at the gross rate, $\lambda^* > r^k$. Thus, there exists a continuum of nonstationary equilibria.

2 Question 2

Utility is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t),\ u(c,l) = \frac{\left[c(1-l)^{\psi}\right]^{1-\gamma} - 1}{1-\gamma},\ \psi > 0,\ \gamma > 0,$$

where $l_t \geq 0$ is labor supplied to the market and $1 - l_t \geq 0$ is leisure. Households purchase consumption goods, $c_t \geq 0$, and investment goods, i_t. They own the stock of capital, which satisfies the following accumulation technology:

$$k_{t+1} = (1-\delta)k_t + i_t.$$

Households have the following budget constraint:

$$c_t + i_t \leq r_t k_t + w_t l_t + \pi_t,$$

where π_t are lump-sum profits. The household's first order condition for capital is:

$$u_{c,t} = \beta u_{c,t+1} \left[r_{t+1} + 1 - \delta\right],\ t = 0, 1, 2, 3... \tag{7}$$

The first order condition for labor is:

$$\frac{-u_{l,t}}{u_{c,t}} = \frac{\psi c_t}{1 - l_t} = w_t. \tag{8}$$

Production in this economy is carried out by firms at three levels. The most downstream firm is a representative, competitive final good firm which produces y_t. This gives rise to the following goods market clearing condition:

$$c_t + i_t \leq y_t.$$

The representative firm producing y_t uses labor services, y^l, and capital services, y^k, and the following technology:

$$y = \left(y^l\right)^{\alpha} \left(y^k\right)^{1-\alpha},\ 0 < \zeta < 1,\ 0 < \alpha < 1,$$

where y^l is a labor-intensive good and y^k is a capital-intensive good. The price of the labor-intensive good is p^l and the price of the capital-intensive good is p^k. The objective of the final good producer is to choose y, y^l, y^k to maximize profits:

$$y - p^l y^l - p^k y^k$$

subject to the technology for producing y. The first order conditions corresponding to this problem are:

$$\alpha \frac{y}{y^l} = p^l, \ (1-\alpha)\frac{y}{y^k} = p^k. \tag{9}$$

The next level upstream is composed of the industries producing the labor-intensive and capital-intensive goods. These industries are also characterized by perfect competition and each industry's output is produced by a representative firm. They use the following technology:

$$\begin{aligned} y^l &= \left(\int_0^n \left[y^l(i)\right]^\zeta di\right)^{\frac{1}{\zeta}} \\ y^k &= \left(\int_0^m \left[y^k(j)\right]^\zeta dj\right)^{\frac{1}{\zeta}}. \end{aligned}$$

The labor intensive good is produced using a range of inputs, $y^l(i)$ for $i \in (0, n)$. The capital-intensive good is produced using a range of inputs, $y^k(j)$, for $j \in (0, m)$. Here, n, $m > 0$. The price of the i^{th} labor and capital-intensive goods is $p^l(i)$ and $p^k(j)$, respectively. The first order conditions for these goods are:

$$\left(\frac{y^l}{y^l(i)}\right)^{1-\zeta} = \frac{p^l(i)}{p^l}, \ \left(\frac{y^k}{y^k(j)}\right)^{1-\zeta} = \frac{p^k(j)}{p^k}$$

We obtain a relationship among the various prices by substituting these expressions back into the production function:

$$\begin{aligned} y^l &= \left(\int_0^n \left[y^l \left(\frac{p^l}{p^l(i)}\right)^{\frac{1}{1-\zeta}}\right]^\zeta di\right)^{\frac{1}{\zeta}} = y^l \left(p^l\right)^{\frac{1}{1-\zeta}} \left(\int_0^n \left(p^l(i)\right)^{\frac{-\zeta}{1-\zeta}} di\right)^{\frac{1}{\zeta}} \\ y^k &= \left(\int_0^m \left[y^k \left(\frac{p^k}{p^k(i)}\right)^{\frac{1}{1-\zeta}}\right]^\zeta dj\right)^{\frac{1}{\zeta}} = y^k \left(p^k\right)^{\frac{1}{1-\zeta}} \left(\int_0^m \left(p^k(i)\right)^{\frac{-\zeta}{1-\zeta}} di\right)^{\frac{1}{\zeta}}. \end{aligned}$$

Cancelling output on both sides, we obtain

$$\begin{aligned}\left(p^l\right)^{\frac{-1}{1-\zeta}} &= \left(\int_0^n \left(p^l(i)\right)^{\frac{-\zeta}{1-\zeta}} di\right)^{\frac{1}{\zeta}} \\ \left(p^k\right)^{\frac{-1}{1-\zeta}} &= \left(\int_0^m \left(p^k(i)\right)^{\frac{-\zeta}{1-\zeta}} di\right)^{\frac{1}{\zeta}}.\end{aligned}$$

or,

$$\begin{aligned}p^l &= \left(\int_0^n \left(p^l(i)\right)^{\frac{-\zeta}{1-\zeta}} di\right)^{\frac{1-\zeta}{-\zeta}} \\ p^k &= \left(\int_0^m \left(p^k(i)\right)^{\frac{-\zeta}{1-\zeta}} di\right)^{\frac{1-\zeta}{-\zeta}}.\end{aligned}$$

Note that if all the $p^l(i)$'s and $p^k(j)$'s are the same then they must equal p^l and p^k, respectively.

The firms furthest upstream are the ones producing the intermediate goods used in the labor-intensive and capital-intensive industries. These firms are monopolists in the product market, though they are competitive in resource markets. They have the following linear production technologies:

$$y^l(i) = l(i),\ y^k(j) = k(j),$$

where $l(i)$ is the quantity of labor used in the production of the i^{th} intermediate good in the labor-intensive sector. The price of this factor, w, is taken as given. Also, $k(j)$ is the quantity of capital used in the production of the j^{th} intermediate good in the capital-intensive sector. The rental rate of capital, r, is also taken as given.

Intermediate good firms maximize profits subject to satisfying the demand curve. Profits in the intermediate good sector supplying the labor-intensive industry are:

$$\pi^l(i) = \max p^l(i)y^l(i) - wl(i),$$

Substituting in the demand curve into this problem:

$$\max y^l \left(p^l\right)^{\frac{1}{1-\zeta}} \left(p^l(i)\right)^{\frac{-1}{1-\zeta}+1} - wy^l \left(p^l\right)^{\frac{1}{1-\zeta}} \left(p^l(i)\right)^{\frac{-1}{1-\zeta}}.$$

The first order condition is:

$$y^l \left(p^l\right)^{\frac{1}{1-\zeta}} \left[\left(\frac{-1}{1-\zeta} + 1 \right) \left(p^l(i)\right)^{\frac{-1}{1-\zeta}} + \frac{w}{1-\zeta} \left(p^l(i)\right)^{\frac{-1}{1-\zeta}-1} \right] = 0$$

or, after rearranging:

$$p^l(i) = \frac{w}{\zeta}. \tag{10}$$

Similarly for the intermediate good suppliers in the capital-intensive sector:

$$p^k(i) = \frac{r}{\zeta}. \tag{11}$$

Since all the intermediate good firms in a given industry set prices in the same way, they supply the same amount of output. Given that l and k are the total amount of labor and capital, respectively, supplied by households, market clearing for resources implies:

$$k = \int_0^m k(j)dj, \ l = \int_0^n l(i)di.$$

Since $k(j)$ is the same for all j, clearing in the resource market implies $k(j) = k/m$ and $l(i) = l/n$. Then the quantity produced of the capital-intensive good is

$$\begin{aligned} y^l &= \left(\int_0^n \left[\frac{l}{n} \right]^\zeta di \right)^{\frac{1}{\zeta}} = \left(n \left[\frac{l}{n} \right]^\zeta \right)^{\frac{1}{\zeta}} = n^{\frac{1}{\zeta}} l \qquad (12) \\ y^k &= m^{\frac{1}{\zeta}} k. \end{aligned}$$

Final output then is

$$\begin{aligned} y &= \left(n^{\frac{1}{\zeta}} l \right)^\alpha \left(m^{\frac{1}{\zeta}} k \right)^{1-\alpha} \\ &= (zl)^\alpha k^{1-\alpha}, \end{aligned}$$

where

$$z = n^{\frac{1}{\zeta}} \left(m^{\frac{1}{\zeta}} \right)^{\frac{1-\alpha}{\alpha}}.$$

To obtain expressions for the real wage and for the rental rate of capital, substitute out for y^k and y^l from (12) into (9) to obtain:

$$\alpha \frac{y}{l} = p^l, \ (1-\alpha) \frac{y}{k} = p^k.$$

Then, making use of (11) and (10),

$$\alpha \frac{y}{l} = \frac{w}{\zeta},\ (1-\alpha)\frac{y}{k} = \frac{r}{\zeta}. \tag{13}$$

This completes the discussion of the three levels of firms in this economy: the furthest downstream are the final good producers, the first level up are the capital- and labor-intensive firms and the next level up is the intermediate good producers.

Combining the household's first order conditions with (13), we obtain:

$$u_{c,t} = \beta u_{c,t+1}\left[\zeta(1-\alpha)\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right],\ t = 0, 1, 2, 3...$$

$$\frac{\psi c_t}{1-l_t} = \zeta\alpha\frac{y_t}{l_t}.$$

In the equilibrium, households do not receive the marginal product of capital and the marginal product of labor. This is why this equilibrium is not efficient. A subsidy on rental income from capital and on wage income from labor would correct this. If instead of receiving $r_t k_t$ and $w_t l_t$ in capital income, households received $(1+\theta)r_t k_t$ and $(1+\theta)w_t l_t$, where $(1+\theta)\zeta = 1$, then the equilibrium would be efficient. That is because the first order conditions coincide with those of the planning problem whose solution yields the efficient allocations. Note that this subsidy needs to be financed. For this scheme to work, it must be financed with a lump-sum tax.

We assume that n and m each grow over time at rates γ_n and γ_m, respectively. We've shown that our model economy is equivalent to a standard real business cycle model with exogenous technical change. The only exception is that in the RBC model, $\zeta = 1$. This does not change the conclusion we reached in the analysis of that model, however, that the growth rate of output, consumption, investment and capital corresponds to the growth rate of z. The growth rate of z is γ_z

$$\gamma_z = \frac{1}{\zeta}\gamma_n + \frac{1-\alpha}{\alpha}\frac{1}{\zeta}\gamma_m.$$

Note that the smaller is ζ, the larger is the growth rate of the economy. The reason for this is that with a smaller ζ, the gains from variety are greater.