

### Optimal Monetary Policy in the Presence of Price Frictions

In the analysis of the first two lectures, we described an environment in which volatility in the aggregate price level is desirable as part of an efficient government financing scheme. This observation has received some close scrutiny in the literature, because it is at variance with the general consensus that price stability is desirable.

A key feature of the economic environment in which this result obtains is that there are no costs associated with price volatility. For example, if there were frictions in price-setting, shocks to the aggregate price level would induce welfare-reducing misallocations in resources. Incorporating frictions like this into the economic environment can have a substantial impact on the properties of Ramsey equilibrium. This was recently shown in papers written independently by Henry Siu and by Stefanie Schmitt-Grohe and Martin Uribe. Each shows that when price frictions are introduced into the analysis, it is no longer optimal to finance business cycle shocks to the government budget constraint with shocks to the aggregate price level. Instead, the Ramsey equilibrium resembles the policy suggested in the analysis of Barro: when a bad shock occurs, raise the labor tax rate by the smallest possible amount so that if you keep it constant at that level forever, you can finance the present value of the shock.<sup>1</sup> Of course, this policy implies that in the short run, you must raise the level of government debt. It also implies that the labor tax rate has time series properties that resemble a random walk.<sup>2</sup>

The purpose of these notes is to investigate, at as simple a level as possible, the role of price frictions in monetary policy. The environment corresponds to the ‘sticky price model’ in Christiano, Eichenbaum and Evans (1997, *European Economic Review*). A detailed explanation of the model is provided there.

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<sup>1</sup>As emphasized recently by Correia, Nicolini and Teles (2001), this conclusion depends sensitively on the absence of other fiscal tools for absorbing shocks to the government budget constraint. Most of the references in this paper can be found on the web site for this course.

<sup>2</sup>Interestingly, initial results by Henry Siu suggest that when there is a really big shock, such as a war shock, then you *do* want to use movements in the aggregate price level to finance shocks.

The model has households, firms, a financial intermediary, and a government. The government is composed of a fiscal authority and a monetary authority. The former has no outstanding debt, and always finances purchases of consumption goods with lump-sum taxes. The monetary authority injects and withdraws cash into the economy by lump sum transfers to (from, if negative) households. The firm sector has the now-standard Dixit-Stiglitz structure introduced by Blanchard and Kiyotaki. There is a representative, competitive final good firm that uses a continuum of intermediate inputs to produce a homogeneous output good. Each intermediate input is produced by a monopolist. A fraction of monopolists, the *sticky price firms*, set their current period price before the current period realization of the monetary transfer and of the exogenous shocks. The others, the *flexible price firms*, set their price afterward. The presence of sticky price firms has the implication that a shock to the aggregate price level induces a distortion in relative prices, which in turn leads to a suboptimal allocation of resources across intermediate good firms.

We ask what is the optimal monetary policy in our environment. In the first section below, we consider the case in which the government has access to a tax-subsidy scheme that it can use to eliminate distortions stemming from the presence of monopoly power in the economy. Our results suggest that in this environment the optimal policy is to set the nominal rate of interest to zero, and eliminate shocks to the aggregate price level.<sup>3</sup>

We then go on to consider the case where the tax-subsidy scheme used in the first section is not available. We then discuss the possibility, emphasized by Dupor ( ), that the optimal monetary policy may be random. The idea is that a random monetary policy can partially duplicate the effects of the tax-subsidy scheme in removing the ill effects of monopoly power. Dupor's argument is quite simple, and is briefly summarized here.

Consider a version of our model in which there are no exogenous shocks. There is one equilibrium in which the rate of interest is zero and the gross money

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<sup>3</sup>This result may at first appear to contradict the finding of Khan, King and Wolman (2001), who show that with sticky prices, a zero nominal rate of interest is not desirable. The difference between results has to do with the nature of price stickiness assumed. Khan, King and Wolman specify an environment in which a deterministic deflation results in distortions due to the nature of price setting. They specify that all firms set prices for two periods, in a staggered way. Then, when there is a deterministic deflation, relative prices are distorted, and the distortion increases with the magnitude of the deflation. In our setup, a deterministic deflation produces no distortions in relative prices.

growth rate is equal to the discount rate. In this equilibrium, the monetary distortion associated with a positive interest rate has been eliminated.<sup>4</sup> However, the monopoly distortion is still present, and this implies that economic activity is suboptimally low. Can monetary policy somehow reduce the monopoly distortion? Dupor provides an example where the answer is ‘yes’, and we explore this possibility in these notes. The idea is this. Suppose there is an unexpected jump in the money growth rate. In that state of the world, output and employment expand and we may expect that utility in that state goes up. However, for the jump in the money supply to be a surprise, there must be other states of the world in which the money supply falls. And, in those states utility goes down. Dupor presents an example in which the rise in utility outweighs - when expected utility is computed - the fall in utility. We will find out if this is true in our model environment too. As Dupor notes, the conventional wisdom that price-setting frictions rationalize stabilizing the price level is undercut if randomization turns out to be optimal in a wide class of reasonable models.

## 1. A Cash in Advance Model

Following is a discussion of the agents in the model.

### 1.1. Households

Household preferences are:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t, l_t).$$

The household begins the period holding money,  $M_t$ . After seeing the realization of the current period shocks and the current period action of the monetary authority, it divides  $M_t$  between deposits,  $D_t$ , with a financial intermediary and cash set aside for consumption expenditures,  $M_t - D_t$ . It faces the following non-negativity constraints on  $D_t$ ,  $0 \leq D_t \leq M_t$ . The household then purchases a final good and makes its labor supply decision. The household is ‘small’ in that it takes the price level, wage rate and interest rate as exogenous. It faces the following cash

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<sup>4</sup>Actually, we consider an equilibrium in which the gross nominal rate of interest is  $1 + \varepsilon$ , where  $\varepsilon > 0$  is a small number. This has the consequence that the cash in advance constraint is satisfied as an equality.

constraint in the goods market:

$$P_t c_t \leq M_t - D_t + W_t l_t - T_t, \quad (1.1)$$

where  $T_t$  denotes lump-sum taxes paid by the household to the fiscal authority. These are taxes used to finance government purchases of goods and services.

The household's cash evolution equation is:

$$M_{t+1} = (1 + R_t)(X_t + D_t) + M_t - D_t + W_t l_t - P_t c_t - T_t + \text{Profits}_t, \quad (1.2)$$

where  $X_t$  denotes a monetary transfer from the central bank. Also, 'Profits $_t$ ' denotes lump sum profits received from intermediate good firms, net of any tax payments needed to finance subsidies to intermediate good firms. These subsidies are discussed further below.

The necessary and sufficient conditions for household optimality include the following Euler equations:

$$\frac{u_{c,t}}{P_t} = \beta(1 + R_t) E_t \frac{u_{c,t+1}}{P_{t+1}} \quad (1.3)$$

$$\frac{-u_{l,t}}{u_{c,t}} = \frac{W_t}{P_t} \quad (1.4)$$

The expectation operator in (1.3) is conditioned on the realization of date  $t$  uncertainty. This reflects our assumption that the household deposit decision is made after the realization of date  $t$  uncertainty, including any uncertainty about the realization of monetary policy. Thus, this model does not have the 'limited participation' sticky-deposit friction. In addition, a transversality condition is included among the necessary and sufficient conditions for optimization, plus the following 'complementary slackness' condition:

$$\begin{aligned} P_t c_t &\leq M_t - D_t + W_t l_t - T_t, \\ P_t c_t &= M_t - D_t + W_t l_t - T_t, \text{ if } R_t > 0. \end{aligned} \quad (1.5)$$

## 1.2. Firms

We adopt a variant of the Dixit-Stiglitz model of production, in which perfectly competitive firms produce a homogeneous final good,  $Y_t$ , using a continuum of differentiated intermediate goods,  $y_t^i$ ,  $i \in (-\infty, \infty)$ :

$$Y_t = \left[ \int_0^1 (y_t^i)^\kappa di \right]^{\frac{1}{\kappa}}, \quad 0 < \kappa \leq 1.$$

Final good firms' optimality condition is:

$$\left(\frac{P_t}{P_t^i}\right)^{\frac{1}{1-\kappa}} = \frac{y_t^i}{Y_t},$$

where  $P_t^i$  is the price of the  $i^{\text{th}}$  intermediate input good.

Each intermediate good is produced by a firm that is a monopolist in the output market and is competitive in the market for the single productive factor, labor. There is no entry or exit into the production of intermediate goods. A typical intermediate good,  $y_t^i$ , is produced using the linear technology,

$$y_t^i = z_t l_t^i,$$

where  $z_t$  is an exogenous shock to technology, and  $l_t^i$  denotes labor services. Intermediate good firms pay  $W_t$  currency units for one unit of labor. There are two types of intermediate good firms. A fraction,  $\nu$ , is called sticky price firms because they set their prices before the realization of all current period uncertainty. The other  $1 - \nu$  firms are called flexible price firms. They set their prices after all current period uncertainty is realized.

Each intermediate good firm must finance its wage bill at a net nominal rate of interest,  $R_t$ . As a result of this, and of the nature of the production technology, the marginal cost, in currency units, of one unit of intermediate goods is  $W_t(1 + R_t)/z_t$ . The  $i^{\text{th}}$  flexible price intermediate good firm solves the following problem:

$$\max_{y_t^i, P_t^i} \tau P_t^i y_t^i - \frac{W_t(1 + R_t)}{z_t} y_t^i,$$

subject to the demand curve for their product, given by the final good firm's first order condition. Here,  $\tau$  denotes a tax subsidy to firms. We denote real marginal

cost by  $\kappa_t$ , where<sup>5</sup>

$$\kappa_t = \frac{W_t(1+R_t)}{z_t P_t}. \quad (1.6)$$

If  $P_t$  were the price of the intermediate good, then the reciprocal of  $\kappa_t$  would be the markup of an intermediate good producer. However, final and intermediate good prices will in general differ outside of a deterministic steady state. Flexible price intermediate good firms set their price,  $P_t^f$ , as a fixed markup,  $1/\kappa$ , over marginal cost adjusted for the subsidy,  $P_t \kappa_t / \tau$ :

$$P_t^f = \frac{\kappa_t}{\kappa \tau} P_t. \quad (1.7)$$

The variable,  $0 < \kappa \leq 1$ , is a parameter of the model.

Sticky price firms set their price,  $P_t^s$ , before the realization of period  $t$  uncertainty, so they must weight cash flow in different states. We assume they do so using the relevant Arrow-Debreu prices. This leads them to set their price,  $P_t^s$ , as a markup over a weighted expectation of marginal cost:<sup>6</sup>

$$P_t^s = E_{t-1} \omega_t \frac{\kappa_t}{\kappa \tau} P_t \quad (1.8)$$

where

$$\omega_t = \frac{\left[ E_t \frac{u_{c,t+1}}{P_{t+1}} \right] P_t^{\frac{1}{1-\kappa}} Y_t}{E_{t-1} \left[ \frac{u_{c,t+1}}{P_{t+1}} P_t^{\frac{1}{1-\kappa}} Y_t \right]}$$

or,

$$\omega_t = \frac{\frac{u_{c,t}}{1+R_t} P_t^{\frac{\kappa}{1-\kappa}} Y_t}{E_{t-1} \left[ \frac{u_{c,t}}{1+R_t} P_t^{\frac{\kappa}{1-\kappa}} Y_t \right]}, \quad (1.9)$$

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<sup>5</sup>To see that this is indeed ‘marginal cost’, note that marginal cost is the extra expense of producing one extra unit of output. Call this  $\Delta C_t / \Delta Y_t$ , ‘the change in cost per unit change in output’. Now,  $W_t(1+R_t)/P_t$  is  $\Delta C_t / \Delta l_t$ , ‘the change in cost per change in units of labor hired’. Also, the marginal product of labor,  $\Delta Y_t / \Delta l_t$  is  $z_t$ , ‘the change in output per change in units of labor hired’. Then,

$$\frac{\Delta C_t}{\Delta Y_t} = \frac{\frac{\Delta C_t}{\Delta l_t}}{\frac{\Delta Y_t}{\Delta l_t}} = \frac{W_t(1+R_t)}{z_t},$$

which corresponds to  $\kappa_t$ .

<sup>6</sup>See Christiano, Eichenbaum and Evans (1997) for a derivation.

after substituting from (1.3). Here,  $E_{t-1}$  is the expectation, conditional on period  $t-1$  uncertainty. Also,  $u_{c,t+1}$  is the marginal utility of consumption at date  $t+1$ . Intermediate good firms treat  $u_{c,t+1}$ ,  $P_t$  and  $Y_t$  as exogenous. Note that if there is no uncertainty, then  $\omega_t = 1$  and (1.7) and (1.8) coincide.

The price of  $Y_t$ , denoted by  $P_t$ , is related to  $P_t^f$  and  $P_t^s$  as follows:

$$P_t = \left[ \nu (P_t^f)^{\frac{\kappa}{\kappa-1}} + (1-\nu) (P_t^s)^{\frac{\kappa}{\kappa-1}} \right]^{\frac{\kappa-1}{\kappa}}, \quad (1.10)$$

where  $0 < \nu < 1$ . Substituting (1.7) into this expression and rearranging, we obtain:

$$P_t = \left\{ \frac{1-\nu}{1-\nu \left(\frac{\kappa_t}{\tau\kappa}\right)^{\frac{\kappa}{\kappa-1}}} \right\}^{\frac{\kappa-1}{\kappa}} P_t^s \equiv h\left(\frac{\kappa_t}{\tau\kappa}\right) P_t^s, \quad (1.11)$$

say. The function,  $h(x)$ , is well defined for  $x > \nu^{(1-\kappa)/\kappa}$ , so that any equilibrium must have the property,

$$\frac{\kappa_t}{\tau\kappa} > \nu^{(1-\kappa)/\kappa}.$$

Since  $h$  is strictly increasing over its domain and  $h(1) = 1$ , it follows that  $h = 1$  if, and only if,  $\kappa_t = \tau\kappa$ . When  $\nu = 0$  then, of course,  $h = 1$  always. To understand (1.11), recall that  $P$  is an average of  $P^f$  and  $P^s$ . Expression, (1.11), says that the larger is  $P^f$  as a proportion of  $P$  (recall, (1.7)), the smaller is  $P^s$  as a proportion of  $P$ . It is useful to note that there is a simple relationship between  $\kappa_t/(\tau\kappa)$  and  $P_t^f/P_t^s$ . We obtain this by combining (1.10) and (1.11):

$$\frac{P_t^f}{P_t^s} = \left[ \frac{h\left(\frac{\kappa_t}{\tau\kappa}\right)^{\frac{\kappa}{\kappa-1}} - (1-\nu)}{\nu} \right]^{\frac{\kappa-1}{\kappa}},$$

which is increasing in  $\kappa_t/(\tau\kappa)$ . The intuition is simple. A higher  $\kappa_t$  signifies a higher markup of marginal cost over the aggregate price index. Given that flexible price firms always choose a fixed markup, a higher  $\kappa_t$  can only occur if  $P_t^f$  is rising relative to  $P_t^s$ .

Final good output is related to the output,  $y_t^s$ , of sticky price intermediate firms and the output,  $y_t^f$ , of flexible price firms, by the following aggregate production

function:<sup>7</sup>

$$Y_t = \left[ \nu (y_t^f)^\kappa + (1 - \nu) (y_t^s)^\kappa \right]^{\frac{1}{\kappa}} = z_t l_t H\left(\frac{\kappa_t}{\tau \kappa}\right), \quad (1.12)$$

where

$$H\left(\frac{\kappa_t}{\tau \kappa}\right) = \frac{\left(\frac{\kappa_t}{\tau \kappa}\right)^{\frac{1}{1-\kappa}}}{\nu + (1 - \nu)^{1-\frac{1}{\kappa}} \left[ \left(\frac{\kappa_t}{\tau \kappa}\right)^{\frac{\kappa}{1-\kappa}} - \nu \right]^{\frac{1}{\kappa}}}.$$

The second equality in (1.12) makes use of (1.7) and (1.11). It is easy to verify that  $H(1) = 1$  and  $H'(x) = 0$  for  $x = 1$ .<sup>8</sup>

Equation (1.12) is a key expression in the model. Recall that there is a monotone relationship between  $P_t^f/P_t^s$  and  $\kappa_t/(\tau \kappa)$ . That is, deviations between  $\kappa_t$  and

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<sup>7</sup>We obtain this expression as follows:

$$\begin{aligned} Y_t &= \left[ \nu (y_t^f)^\kappa + (1 - \nu) (y_t^s)^\kappa \right]^{\frac{1}{\kappa}} \\ Y_t &= z_t l_t^f \left[ \nu + (1 - \nu) \left( \frac{l_t^s}{l_t^f} \right)^\kappa \right]^{\frac{1}{\kappa}} \\ Y_t &= z_t l_t^f \left[ \nu + (1 - \nu) \left( \left[ \frac{\frac{\kappa_t}{\kappa} P_t}{P_t^s} \right]^{\frac{1}{1-\kappa}} \right)^\kappa \right]^{\frac{1}{\kappa}} \\ Y_t &= z_t l_t^f \left[ \nu + (1 - \nu) \left( \frac{\kappa_t}{\kappa} h\left(\frac{\kappa_t}{\kappa}\right) \right)^{\frac{\kappa}{1-\kappa}} \right]^{\frac{1}{\kappa}} \\ Y_t &= z_t l_t \left( \frac{l_t^f}{l_t} \right) \left[ \nu + (1 - \nu) \left( \frac{\kappa_t}{\kappa} h\left(\frac{\kappa_t}{\kappa}\right) \right)^{\frac{\kappa}{1-\kappa}} \right]^{\frac{1}{\kappa}} \\ Y_t &= z_t l_t \frac{\left( \nu + (1 - \nu) \left[ \frac{\kappa_t}{\kappa} h\left(\frac{\kappa_t}{\kappa}\right) \right]^{\frac{\kappa}{1-\kappa}} \right)^{\frac{1}{\kappa}}}{\nu + (1 - \nu) \left[ \frac{\kappa_t}{\kappa} h\left(\frac{\kappa_t}{\kappa}\right) \right]^{\frac{1}{1-\kappa}}} = z_t l_t H\left(\frac{\kappa_t}{\kappa}\right). \end{aligned}$$

<sup>8</sup>To do this, note that

$$H(x) = \left[ \nu x^{\frac{1}{\kappa-1}} + (1 - \nu) h(x)^{\frac{1}{1-\kappa}} \right]^{-1},$$

so that, trivially,  $H(1) = 1$ . Also,

$$H'(x) = -H(x)^2 \left[ \frac{\nu}{\kappa - 1} x^{\frac{1}{\kappa-1}-1} + \frac{1 - \nu}{1 - \kappa} h(x)^{\frac{1}{1-\kappa}-1} h'(x) \right].$$

$\tau\kappa$  correspond to distortions between the prices of flexible and sticky price firms (recall the discussion after (1.11)). This in turn induces an asymmetric allocation of labor across these two types of firms, which is inefficient when aggregate output is the aggregate of intermediate goods given in (1.12). This inefficiency is minimized when  $H = 1$  and  $\kappa_t = \tau\kappa$ . The function,  $H$ , falls as  $\kappa_t$  rises above  $\tau\kappa$ , or falls below  $\tau\kappa$ .<sup>9</sup> Other things the same, it is desirable to somehow ‘set’  $\kappa_t = \tau\kappa$ . The variable,  $H$ , is referred to as ‘Okun’s gap’ by King and Wolman (1999). Alternatively, we can think of  $z_t l_t$  as ‘potential output’. It is what output would be, if resources were allocated efficiently across all the sectors. In this model, ‘efficiently’ means symmetrically to all sectors. So,  $H$  is a measure of the utilization of resources.

Unfortunately, the degree of variation in  $H$  is relatively small. This can perhaps be guessed from the observation,  $H'(1) = 0$ . The graph of  $H$  is depicted in Figures 1 - 3, setting  $\tau = 1$  (no subsidy) and  $\kappa = 1/1.4$ ,  $\nu = 0.8$ . Figures 1 and 3 display the variation in  $H$  as a function of  $P^f/P^s$ . Figure 2 shows how  $H$  varies as  $\kappa_t$  varies from 0.65 to nearly unity. Over this range,  $H$  does vary considerably down to about 0.77 for  $\kappa_t = 1$ . This range for  $\kappa_t/\kappa$  corresponds to a range in  $P_t^f/P_t^s$  from nearly zero to nearly 2.5. Figure 3 focuses on a subset of the graph in Figure 1, the part pertaining to  $P_t^f/P_t^s$  in the range, 0.9 – 1.1. In this range,  $H$  fluctuates between 1 and 0.997, or 0.3 percent.

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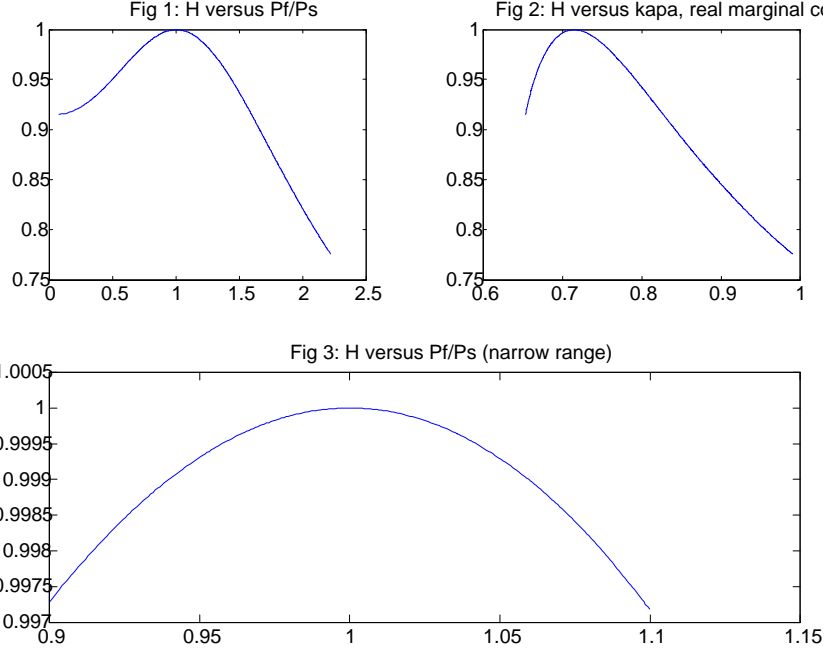
Since  $H(1) = h(1) = 1$ , and  $h'(1) = \nu/(1 - \nu)$ ,

$$H'(1) = - \left[ \frac{\nu}{\kappa - 1} + \frac{\nu}{1 - \kappa} \right] = 0.$$

<sup>9</sup>This has been verified numerically with several examples. To establish this rigorously, one needs to study the function,

$$f(y) = \frac{y}{\nu + (1 - \nu)^{1 - \frac{1}{\kappa}} [y^\kappa - \nu]^{\frac{1}{\kappa}}},$$

where  $y = (\kappa_t/\kappa)^{1/(1-\kappa)}$ . The function,  $f$ , is defined for  $y^\kappa > \nu$ . One needs to verify that over this domain,  $f$  achieves its maximum value of  $f = 1$  at  $y = 1$ .



We proceed now to obtain an expression for  $\omega_t$  that is in terms of allocations and  $\kappa_t$ . Thus, divide the numerator and denominator of (1.9) by  $(P_t^s)^{\kappa/(1-\kappa)}$ , and take into account that this is in the date  $t - 1$  information set, to obtain:

$$\omega_t = \frac{\frac{u_{c,t}}{(1+R_t)} h\left(\frac{\kappa_t}{\tau\kappa}\right)^{\frac{\kappa}{1-\kappa}} Y_t}{E_{t-1} \left[ \frac{u_{c,t}}{(1+R_t)} h\left(\frac{\kappa_t}{\tau\kappa}\right)^{\frac{\kappa}{1-\kappa}} Y_t \right]}.$$

Combining (1.4) and (1.6) to substitute out for  $1 + R_t$  in the above expression, and making use of (1.12):

$$\omega_t = \frac{\frac{u_{l,t} l_t}{\kappa_t} h\left(\frac{\kappa_t}{\tau\kappa}\right)^{\frac{\kappa}{1-\kappa}} H\left(\frac{\kappa_t}{\tau\kappa}\right)}{E_{t-1} \left[ \frac{u_{l,t} l_t}{\tau\kappa_t} h\left(\frac{\kappa_t}{\tau\kappa}\right)^{\frac{\kappa}{1-\kappa}} H\left(\frac{\kappa_t}{\tau\kappa}\right) \right]} \quad (1.13)$$

Divide both sides of (1.8) by  $P_t^s$  and impose (1.11), to obtain:

$$E_{t-1} \omega_t \frac{\kappa_t}{\tau\kappa} h\left(\frac{\kappa_t}{\tau\kappa}\right) = 1. \quad (1.14)$$

This represents a single equation, restricting the variation across states of the world in  $\kappa_t$ ,  $c_t$  and  $l_t$ . It summarizes the optimality condition of sticky price intermediate good firms. Two additional observations are worth making about (1.14). First, in the case in which there is no uncertainty, so that  $\omega_t \equiv 1$ , then, (1.14) reduces to:

$$1 = h\left(\frac{\kappa_t}{\tau\kappa}\right)\frac{\kappa_t}{\tau\kappa},$$

which requires  $\kappa_t = \tau\kappa$ . Second, if we set  $\nu = 1$ , so that there are no sticky price firms, then (1.8) and (1.10) imply  $P_t = (\kappa_t / (\tau\kappa))P_t$ , or,  $\kappa_t = \tau\kappa$ . It is not obvious (to me!) what happens to (1.14) as  $\nu \rightarrow 1$ . Still, it seems safe to presume that (1.14) somehow reduces to the condition  $\kappa_t = \tau\kappa$ .

### 1.3. Financial Intermediary

The total nominal demand for funds in financial markets is  $W_t l_t$ . The amount of funds that the financial intermediary has available for lending is  $D_t + X_t$ . When  $R_t > 0$ , they lend out all they have. So, the market clearing condition in financial markets is:

$$\begin{aligned} W_t l_t &\leq D_t + X_t, \\ W_t l_t &= D_t + X_t, \text{ if } R_t > 0 \end{aligned} \tag{1.15}$$

### 1.4. Fiscal and Monetary Authorities

The fiscal authority sets lump sum taxes,  $T_t$ , to balance its budget in each period, so that

$$T_t = P_t g. \tag{1.16}$$

There is no outstanding government debt, and no debt is ever issued.

The monetary authority transfers an amount of cash to households (actually, taxes them if  $X_t < 0$ ) in the amount  $X_t$ . Monetary policy is a stochastic process for  $X_t$ .

### 1.5. Equilibrium

Conditional on a stochastic process for  $X_t$  and  $z_t$ , an *equilibrium* is a set of 7 stochastic processes,  $P_t$ ,  $\kappa_t$ ,  $R_t$ ,  $Y_t$ ,  $l_t$ ,  $D_t$ , and  $W_t$  that satisfy the following nec-

essary and sufficient conditions for equilibrium. There is the resource constraint:

$$c_t + g \leq Y_t. \quad (1.17)$$

There are the household Euler equations, (1.3), (1.4), and its complementary slackness condition, (1.5); and the equations pertaining to the firms, (1.6), (1.14), (1.12). The household transversality condition is also a requirement. This condition is non-binding given that we limit ourselves to considering stationary equilibria, which we define below.

We find it is convenient to begin the analysis by restricting the stochastic process for  $z_t$  to be *iid*, with:

$$\begin{aligned} \text{prob}(z_t = z^h) &= \mu \\ \text{prob}(z_t = z^l) &= 1 - \mu. \end{aligned}$$

In addition, it is convenient to define the nominal variables relative to the beginning of period money stock. Thus, the objects to be determined in equilibrium are  $p_t$ ,  $\kappa_t$ ,  $R_t$ ,  $Y_t$ ,  $l_t$ ,  $d_t$ , and  $w_t$ , where a lower case price means it's been divided by the beginning of period stock of money,  $M_t$ . We define a *stationary equilibrium* as two sets of 7 numbers each,  $p(z^i)$ ,  $\kappa(z^i)$ ,  $R(z^i)$ ,  $Y(z^i)$ ,  $l(z^i)$ ,  $d(z^i)$ , and  $w(z^i)$ ,  $i = h, l$ , which satisfy the equilibrium conditions. Thus, there is a total of 14 unknowns to be determined in a stationary equilibrium. Note that there are also 14 equations.

## 2. Equilibrium Computation

We adopt the following specification of household utility:

$$u(c_t, l_t) = \frac{\left[ c_t - \frac{\psi_0}{1+\psi} l_t^{1+\psi} \right]^{1-\sigma}}{1-\sigma}.$$

We can substantially reduce the dimension of the problem from the way it was characterized in the previous section. In particular, the equilibrium conditions have a recursive structure, which allows us to solve for the four variables,  $l_t$ ,  $c_t$ ,  $p_t$  and  $\kappa_t$  first, and then solve for the others afterward. These three variables, in a stationary equilibrium, represent 6 unknowns. We now rearrange the equilibrium conditions to obtain six equations that can be used to solve for these 6 unknowns.

Combining (1.12) and (1.17), we obtain

$$c_t + g = z_t l_t H\left(\frac{\kappa_t}{\tau \kappa}\right). \quad (2.1)$$

This represents two equations, one for  $z^h$  and one for  $z^l$ . Combine (1.6) and (1.4), to obtain

$$1 + R_t = \frac{z_t \kappa_t}{\psi_0 l_t^\psi}. \quad (2.2)$$

A restriction that must be satisfied in any equilibrium, is the requirement that  $R_t \geq 0$ , i.e.,

$$\frac{z_t \kappa_t}{\psi_0 l_t^\psi} \geq 1. \quad (2.3)$$

Substitute the expression for  $1 + R_t$  into (1.3), to obtain:

$$\frac{u_{c,t}}{\beta E_t u_{c,t+1} / \pi_{t+1}} = \frac{z_t \kappa_t}{\psi_0 l_t^\psi}.$$

Note:

$$\pi_{t+1} = \frac{P_{t+1}}{P_t} = \frac{M_{t+1} p_{t+1}}{M_t p_t} = (1 + x_t) \frac{p_{t+1}}{p_t},$$

so that

$$\frac{1 + x_t}{p_t} \frac{u_{c,t}}{\beta E_t u_{c,t+1} / p_{t+1}} = \frac{z_t \kappa_t}{\psi_0 l_t^\psi}. \quad (2.4)$$

Here, we have taken into account that  $x_t$  and  $p_t$  are contained in the information set implicit in  $E_t$  (recall the discussion above, after (1.4)). Equation (2.4) represents two equations, one for  $z^h$  and one for  $z^l$ . Here,  $E_t u_{c,t+1} / p_{t+1}$  is a constant because of the iid assumption, and corresponds to:

$$E_t \frac{u_{c,t+1}}{p_{t+1}} = \mu \frac{u_c^h}{p^h} + (1 - \mu) \frac{u_c^l}{p^l},$$

where

$$u_c^i = \left[ c^i - \frac{\psi_0}{1 + \psi} (l^i)^{1+\psi} \right]^{-\sigma}, \quad i = l, h.$$

For convenience, we reproduce (1.14) here:

$$E_{t-1} \omega_t \left( \frac{\kappa_t}{\tau \kappa} \right) h \left( \frac{\kappa_t}{\tau \kappa} \right) = 1. \quad (2.5)$$

This represents just one equation, because of our *iid* assumption. Writing this out explicitly:

$$\mu\omega^h\frac{\kappa^h}{\tau\kappa}h\left(\frac{\kappa^h}{\tau\kappa}\right) + (1-\mu)\omega^l\frac{\kappa^l}{\tau\kappa}h\left(\frac{\kappa^l}{\tau\kappa}\right) = 1,$$

where  $\omega^i$  is defined in (1.13),  $i = f, s$ . Finally, combine (1.15), (1.5) and (1.16) to obtain:

$$p_t(c_t + g) \leq 1 + x_t, \text{ with } p_t(c_t + g) = 1 + x_t \text{ if } R_t > 0. \quad (2.6)$$

Notice that (2.1), (2.4), (2.5) and (2.6) represents 7 equations in our 8 unknowns,  $l^i = l(z^i)$ ,  $c^i = c(z^i)$ ,  $p^i = p^i(z^i)$  and  $\kappa^i = \kappa(z^i)$ ,  $i = l, h$ . We need one additional equation. This is given by the requirement that  $P^s$  not be sensitive to the realization of the current period state of nature. In particular, from (1.11),

$$p^s = \frac{h\left(\frac{\kappa^h}{\tau\kappa}\right)}{p^h} = \frac{h\left(\frac{\kappa^l}{\tau\kappa}\right)}{p^l}. \quad (2.7)$$

We compute an equilibrium as follows. First, impose (2.6) as a strict equality and solve (2.1), (2.4), (2.5), (2.6) and (2.7) ignoring restriction, (2.3). If the solution satisfies (2.3), then we have an equilibrium. If (2.3) is violated for some state, then for this state, replace (2.6) with (2.3) evaluated at equality.

In practice, the number of equations and unknowns to be solved by nonlinear methods can be reduced to four equations in the four unknowns,  $l^l, l^h, \kappa^l, \kappa^h$ . Conditional on these variables,  $c^i$  can be computed using (2.1), and  $p^i$  can be computed using (2.6),  $i = l, h$ . When the equations being solved replace (2.6) with (2.3) evaluated at equality, then the nonlinear solution method has to involve a different system of variables. In this case, the variables should be  $l^l, l^h, p^l, p^h$ . Given  $l^i, \kappa^i$  can be computed using (2.3) and  $c^i$  can be computed using (2.1),  $i = l, h$ . Equations (2.4) and (2.5) can be used to solve for  $l^l, l^h, p^l, p^h$ .

Nonlinear equation solving routines require a good initial guess. The steady state values of the variables, conditional on money growth,  $x$ , being  $\mu x^h + (1-\mu)x^l$  and technology,  $z$ , being  $\mu z^h + (1-\mu)z^l$ , can be obtained like this. Using (2.4) and the steady state value,  $\kappa_t/(\tau\kappa) = 1$ , solve the following equation for  $l$ :

$$\frac{1+x}{\beta} = \frac{z\tau\kappa}{\psi_0 l^\psi}.$$

Note that as long as  $1+x \geq \beta$ , then (2.3) is satisfied. From (2.1),

$$c = zl - g.$$

Finally, (2.6),

$$p = \frac{1+x}{c+g}.$$

Given a solution to these equations, it is easy to solve for the other variables in an equilibrium. The rate of interest can be obtained from (2.3). Flexible and sticky prices, respectively, can be obtained from (1.7) and (1.8). Finally, inflation is a function of the current and previous realization of technology. To see this:

$$\pi_t = \frac{P_t}{P_{t-1}} = \frac{M_t p_t}{M_{t-1} p_{t-1}} = \frac{(1+x_{t-1})p_t}{p_{t-1}}.$$

### 3. Ramsey Problem

We define the *Ramsey Problem* as follows. Choose  $x^h$  and  $x^l$  so that expected utility is maximized, subject to (2.1), (2.4), (2.5), (2.6) and (2.7). Following is a candidate solution to this problem. We develop this solution under the assumption that the date  $t$  conditional distribution of  $z_{t+1}$  is a function of  $z_t$  alone, and not a function of  $z_{t-s}$ ,  $s > 0$ .

First, define the set  $D$  as the set of state-contingent sequences,  $c_t, l_t$ ,  $t = 0, 1, 2, \dots$  which are competitive equilibria corresponding to some state contingent sequence,  $x_t$ , and some setting for  $\tau$ . The Ramsey Allocation problem is:

$$\max_{c_t, l_t \in D} \sum_{t=0}^{\infty} \beta u(c_t, l_t).$$

The solution to the Ramsey problem is found once  $x_t$  and  $\tau$  are found that support the solution to the above problem as an equilibrium.<sup>10</sup>

We identify a candidate solution to the Ramsey problem as follows. Consider the less constrained set:

$$\tilde{D} = \{c_t, l_t : c_t, l_t \geq 0, c_t + g \leq z_t l_t\}.$$

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<sup>10</sup>This approach to solving the Ramsey problem is standard in monetary economics. It is the approach taken in the 1991 JMCB paper of Chari-Christiano-Kehoe. In addition, it is pursued in the papers by King and Wolman (1999), and discussed further in Kahn, King and Wolman (2000), as well as Siu (2001) and Schmitt-Grohe and Uribe (2001) (these references appear on the web page).

Points in  $D$  evidently belong to  $\tilde{D}$ , but not the other way around. Next, consider the solution to the less constrained problem:

$$\max_{c_t, l_t \in \tilde{D}} \sum_{t=0}^{\infty} \beta u(c_t, l_t).$$

The solution to this problem corresponds to the  $c_t$  and  $l_t$  that solve the following two equations:

$$\frac{-u_{l,t}}{u_{c,t}} = z_t, \quad c_t + g \leq z_t l_t. \quad (3.1)$$

This solution solves the more constrained Ramsey problem if it turns out that the constraints in  $D$ , but not in  $\tilde{D}$ , are non-binding. This is the case if we can find  $\tau, x_t, R_t, D_t, p_t$  which, together with the  $c_t, l_t$  defined by (3.1) constitute an equilibrium. For this, we need to verify  $R_t \geq 0$ ,  $0 \leq D_t \leq M_t$ , (2.1), (2.2), (2.4), (2.5), (2.6), and (2.7). Here is a candidate. Let:

$$\tau = 1/\kappa, \quad R_t = 0, \quad \kappa_t = 1, \quad p_t = p,$$

where  $p$  is a constant to be determined. Evidently, (2.1) and (2.2) are satisfied.

Next, we set  $x_t$  so that (2.4) is satisfied, taking into account that  $z_t \kappa_t / (\psi_0 l_t^\psi) = 1$  and  $p_t = p$ :

$$1 + x_t = \frac{\beta E_t u_{c,t+1}}{u_{c,t}}$$

Under the assumption about  $z_t$ , the conditional expectation is a (perhaps trivial) function of  $z_t$ . So is,  $u_{c,t}$ . This expression can therefore be solved for an  $x_t$  process that is a function of  $z_t$ .

Since  $\kappa_t = \tau \kappa$ ,  $h = 1$  and  $p^s = p^h = p^l = p$ , and so (2.7) is satisfied. Finally, set

$$p \leq \min \frac{1 + x_t}{c_t + g}, \quad (3.2)$$

which assures that (2.6) is satisfied. We solve for  $d_t \equiv D_t/M_t$  using the money market clearing condition that is relevant to the case,  $R_t = 0$ :

$$w_t l_t \leq d_t + x_t,$$

where

$$w_t \equiv \frac{W_t}{M_t} = p \frac{-u_{l,t}}{u_{c,t}}.$$

It remains only to verify that  $p$  can be chosen to satisfy (3.2) and  $0 \leq d_t \leq 1$ . That (3.2) can be satisfied is obvious. That the restriction on  $d_t$  is satisfied is less obvious, and is left as exercise 3 below. Assuming  $d_t$  can be found that satisfies the indicated restriction, then it has been established that the solution to the Ramsey problem is to set  $R_t = 0$  and eliminate all shocks to the aggregate price level.

## 4. Optimal Monetary Policy In the Presence of Monopoly Distortions

We now repeat the analysis of section 3.1, under the assumption that  $\tau \equiv 1$ . That is, we assume that fiscal policy cannot be used to ‘fix’ the monetary distortion. We ask whether that distortion could somehow be ‘fixed’ by a random monetary policy. To focus the analysis, we shut down all sources of uncertainty apart from the monetary policy shock. Since we will obviously consider uncertainty in monetary policy, the equilibrium conditions relevant to the case of uncertainty will be used. We will start with a simple benchmark, an equilibrium in which the money growth rate is a constant and supports an interest rate of zero. We will then consider a class of deviations from that benchmark, in the direction of randomizing monetary policy, to find a policy which increases utility above the benchmark.

To simplify the analysis, we work with a different utility function:

$$u(c_t, l_t) = \log(c_t) + \psi \log(1 - l_t), \psi > 0,$$

with the restriction,  $0 \leq l_t \leq 1$ . Note,

$$\frac{-u_{l,t}}{u_{c,t}} = \frac{\psi c_t}{1 - l_t}$$

In addition, we set  $g = T_t = 0$ , and  $z_t \equiv 1$ .

### 4.1. The Equilibrium Conditions

We now summarize the equilibrium conditions for this economy. The resource constraint is (2.1), with  $\tau = 1$ :

$$c_t \leq l_t H\left(\frac{\kappa_t}{\kappa}\right). \tag{4.1}$$

The analog of equation (2.2) is:

$$1 + R_t = \frac{\kappa_t(1 - l_t)}{\psi l_t H(\frac{\kappa_t}{\kappa})} \geq 1, \quad (4.2)$$

since the household's labor Euler equation is  $W_t/P_t = \psi c_t/(1 - l_t)$ . The analog of (2.4) is:

$$\frac{1 + x_t}{c_t p_t} \frac{u_{c,t} c_t}{\beta E_t u_{c,t+1}/p_{t+1}} = \frac{\kappa_t(1 - l_t)}{\psi l_t H(\frac{\kappa_t}{\kappa})}. \quad (4.3)$$

Here, we have substituted out for  $c_t$  using (4.1). Equation (2.5) with  $\tau \equiv 1$  is:

$$E_{t-1} \omega_t \frac{\kappa_t}{\kappa} h(\frac{\kappa_t}{\kappa}) = 1, \quad (4.4)$$

where  $\omega_t$  is given by (1.13).

We will focus on equilibria in which  $R_t = \varepsilon$ , where  $\varepsilon > 0$ , but arbitrarily small. As a result, the expression that summarizes the cash in advance constraint and the money market clearing condition must be satisfied as a strict equality:

$$p_t c_t = 1 + x_t \quad (4.5)$$

Substituting (4.5) into (4.3), and making use of the fact,  $u_{c,t} c_t = 1$ ,

$$\frac{1}{\beta E_t u_{c,t+1}/p_{t+1}} = \frac{\kappa_t(1 - l_t)}{\psi l_t H(\frac{\kappa_t}{\kappa})}. \quad (4.6)$$

We shall restrict our attention to equilibria in which  $x_t$  can take on two values,  $x^h$  with probability  $\mu$  and  $x^l$  with probability  $1 - \mu$ . In addition, we consider equilibria in which the variables are functions of the realized values of  $x_t$ . Thus,  $p^i$  is the (scaled) price level when  $x_t = x^i$ ,  $i = h, l$ . The sticky price restriction, (2.7), requires:

$$\frac{h(\frac{\kappa^h}{\kappa})}{h(\frac{\kappa^l}{\kappa})} = \frac{p^h}{p^l}, \quad (4.7)$$

From (4.6), we have  $E_t u_{c,t+1}/p_{t+1} = 1$ , or, after taking into account the form of the utility function, and multiplying through by  $p^l$ :

$$\beta \left[ \frac{\mu}{\frac{p^h}{p^l} l^h H(\frac{\kappa^h}{\kappa})} + \frac{1 - \mu}{l^l H(\frac{\kappa^l}{\kappa})} \right] = p^l.$$

This can be solved for  $p^l$ , once  $p^h/p^l$  is taken from (4.7). With  $p^h, p^l, c^h, c^l$  in hand, we can compute:

$$x^i = p^i c^i - 1, \quad i = h, l.$$

To verify that we have an equilibrium, it needs to be confirmed that  $d^i$  lies inside the unit interval,  $i = h, l$ .

With the previous results we can define a mapping from  $\kappa^h$  to expected utility. When  $\kappa^h = \kappa$ , there is no randomization and expected utility boils down to the level of utility in the benchmark equilibrium. Randomization improves things relative to this benchmark equilibrium, if expected utility increases with an increase in  $\kappa^h$  above  $\kappa$ . The results in Dupor suggest that this may indeed be the case.

## 4.2. A Benchmark Equilibrium

We consider the following benchmark equilibrium:

$$1 + x^h = 1 + x^l = \beta.$$

The equilibrium conditions are satisfied by setting  $\kappa_t = \kappa$ ,  $H(\kappa_t/\kappa) = 1$ ,  $\beta E_t u_{c,t+1}/p_{t+1} = 1$

$$c_t = l_t = \frac{\kappa}{\psi + \kappa}, \quad p = \frac{\beta}{c}, \quad w = \kappa p.$$

Finally, deposits (scaled by  $M_t$ ) can be solved using the relation,  $w_t l_t = d_t + x_t$ . Since  $w_t l_t - x_t = \kappa\beta - (\beta - 1)$ ,

$$d = 1 - (1 - \kappa)\beta.$$

Note that since  $\kappa < 1$ , it follows that  $d \leq 1$ . In addition, since  $1 - \kappa$  and  $\beta$  are each less than unity, it follows that  $d \geq 0$ . This establishes that the posited benchmark is indeed an equilibrium.

## 4.3. Equilibrium with Random Monetary Policy

The benchmark equilibrium is inefficient relative to the one that could be achieved with a tax-subsidy scheme that removes the monopoly distortion. In such an equilibrium, employment would be higher, with  $l = 1/(\psi + 1)$ . We now investigate the possibility that there exists a random, *iid*, monetary policy that does a better job at producing an equilibrium that approximates this more efficient outcome.

To construct a random, *iid*, monetary policy, we shall search over values of  $\kappa^h > \kappa$ . The motivation for this may be seen by solving (4.2) for  $l_t$  :

$$\frac{\psi l_t}{1 - l_t} = \frac{\kappa_t}{H(\frac{\kappa_t}{\kappa})},$$

which is  $\kappa$  in the benchmark equilibrium. Under the efficient outcome,  $\psi l_t / (1 - l_t)$  is unity. One way to try and ‘get’ there would be to move  $\kappa_t$  up towards unity. This is why we consider  $\kappa^h > \kappa$ . With any particular value of  $\kappa^h$ , the above expression can be solved for  $l^h$  :

$$\frac{\psi l^h}{1 - l^h} = \frac{\kappa^h}{H(\frac{\kappa^h}{\kappa})}.$$

We can see here, how raising  $\kappa^h$  may or may not improve things. On the one hand, holding  $H$  fixed, increasing  $\kappa^h$  raises employment in the direction of the efficient level. On the other hand, this higher level of employment is less effective at producing output, since  $H$  falls.

Next, solve (4.2) for the low money growth state with  $R_t = 0$  and (4.4) for  $l^l$  and  $\kappa^l$ . As long as  $\kappa^h$  is not too much greater than  $\kappa$ , there should exist a solution by continuity. This is because when  $\kappa^h = \kappa$ , then  $\kappa^l = \kappa$  and  $l^l = l$  solve this system of equations.

With  $\kappa^i$ ,  $i = h, l$  in hand, (4.7) can be used to compute  $p^h/p^l$ . and e level of employment becomes less efficient at productioight improve things. Now, with  $\kappa^h > \kappa$ , we must have  $\kappa^l < \kappa$ , according to With a random, *iid* monetary policy, the object on the left side of (4.6) is invariant to the realization of monetary policy. This implies that the object on the right is too. That object, by (4.2), is the nominal rate of interest. So, if we

## 5. Homework Exercise

1. Compute two equilibria in the model. In each case, report the values of  $R^h$ ,  $R^l$ ,  $c^h$ ,  $c^l$ ,  $l^h$ ,  $l^l$ ,  $H^h$ ,  $H^l$ ,  $\kappa^l$ ,  $\kappa^h$ . Also, report  $Eu = \mu u^h + (1 - \mu)u^l$ , the expected value of utility.

Let  $z^h = 1 + .01$ ,  $z^l = 1 - .01$  and set  $\mu = 0.5$ . Let the mean money growth rate be  $0.10 = \mu x^h + (1 - \mu)x^l$ , i.e., 10 percent. Set  $\sigma = 1$  (log case),  $\psi = 1$ ,  $\kappa = 0.97$  (that is, the steady state markup is 3%),  $\beta = 1/1.06$  (i.e., the

discount rate is 6%),  $\nu = 0.8$ . Set  $\psi_0$  so that the steady state value of  $l$  is unity (by steady state, I mean the value taken on by the variables when  $z$  and  $x$  are held constant at their mean values of unity and 0.1, respectively).

In the first equilibrium, hold  $x_t$  constant at its expected value, independent of the realization of  $z_t$ ,  $x^h = x^l = 0.10$ . In the second equilibrium, let  $x^h = 0.10 + 0.23$ ,  $x^l = 0.10 - 0.23$ . Note that expected utility is higher in the second equilibrium. Provide the intuition for this.

To do these calculations, you should use the equation solver, `fzero`, in MATLAB. Do not hesitate to contact me in case you run into trouble with this.

2. Compute the candidate Ramsey equilibrium discussed in section 3, using the parameter values from question 1. Is it possible to choose  $p$  so that the restriction on  $d_t$  is satisfied? Display  $l_t$ ,  $c_t$ ,  $\pi_{t+1} = P_{t+1}/P_t$ ,  $d_t$ ,  $w_t$  in the various states of the world. Also, what is the average inflation rate and money growth rate,  $E\pi_t$ ,  $Ex_t$ ?
3. Following the argument in section 4.3, see if you can find a random monetary policy that increases utility above its level in the benchmark equilibrium discussed in section 4.2. Be sure and verify that you consider only *bona fide* equilibria, that is, verify that the condition on  $d_t$  is satisfied.

## References

- [1] Christiano, Lawrence J, Martin Eichenbaum, and Charles Evans, 1997, 'Sticky Price and Limited Participation Models of Money: A Comparison,' European Economic Review.