Homework #2
Economics 411-1
Due Wednesday, October 10
Christiano

1. Consider the problem of choosing a consumption sequence c_t to maximize
$$\sum_{t=0}^{\infty}\beta^t\{\log(c_t)+\gamma\log(c_{t-1})\},\ 0<\beta<1,\ \gamma>0,$$
subject to $c_t + k_{t+1} \leq Ak_t^{\alpha}$, $A>0$, $0<\alpha<1$, and $k_0>0$, c_{-1} given. Here, c_t is consumption at time t, and k_t is the capital stock at the beginning of period t. The form of the current utility function $\log(c_t)+\gamma\log(c_{t-1})$ is designed to capture the notion that consumption purchases generate a stream of utility that extends for more than one period.

   (a) Let $v\,(k_0,c_{-1})$ be the optimized value of $\sum_{t=0}^{\infty}\beta^t\{\log(c_t)+\gamma\log(c_{t-1})\}$, given the initial conditions, k_0 and c_{-1}. Formulate the FE representation of the problem.

   (b) Use the results in S-L to argue that the solution to the functional equation described in the previous question is of the form $v(k,c_{-1})=E+F\log(k)+G\log(c_{-1})$ and that the optimal policy is of the form, $k_{t+1}=I+H\log(k_t)$ where $E,F,M,G,H,$ and I are constants. Give explicit formulas for the constants E,F,G,H, in terms of the parameters $A,B,\alpha,$ and γ. Do the results in S-L apply literally to this problem, or do you have to cut corners here and there in order to apply them? Where do you have to cut those corners?

2. This question asks you to redo Theorem 4.15 in a model that incorporates hours worked as a choice variable. Consider the following utility function:
$$\sum_{t=0}^{\infty}\beta^t u(c_t,n_t), \tag{1}$$

where $c_t \geq 0$ and $0 \leq n_t \leq 1$ denote date t consumption and employment, respectively. The resource constraint is:

$$c_t + k_{t+1} \leq f(k_t, n_t), \tag{2}$$

with $k_t \geq 0$. Here, u is strictly concave, differentiable, strictly increasing in c and strictly decreasing in n. Also, f is strictly increasing in each argument, is linearly homogeneous of degree one, differentiable and concave. Suppose $c_t^*, k_t^* > 0$, $0 < n_t^* < 1$ satisfy (2) at each t and $k_0^* = \bar{k}_0$, the given initial stock of capital. Also, these numbers satisfy the 'Euler equations':

$$\begin{aligned} u_c(c_t^*, n_t^*) &= \beta u_c(c_{t+1}^*, n_{t+1}^*) f_k(k_{t+1}^*, n_{t+1}^*), \\ u_c(c_t^*, n_t^*) f_n(k_t^*, n_t^*) + u_n(c_t^*, n_t^*) &= 0, \end{aligned}$$

for $t = 0, 1, 2, ...$, and the 'transversality condition':

$$\lim_{T \to \infty} u_c(c_T^*, n_T^*) f_k(k_T^*, n_T^*) k_T^* = 0.$$

Here, u_c and u_n denote the derivatives of u with respect to its first and second argument, and similarly for f. Show that the given sequences $\{c_t^*, k_t^*, n_t^*; t \geq 0\}$ produce the highest value of (1) within the set of sequences which satisfy (2) and the inequality constraints on consumption, labor and capital. (Hint: imitate the proof to Theorem 4.15 in the text.)

3. Consider an economy with a large number of identical households, each having preferences, $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Suppose the resource constraint is $c_t + i_t \leq f(k_t)$, where $k_{t+1} = i_t + (1-\delta)k_t$, f is strictly increasing and concave, $f'(k) \to \infty$ as $k \to 0$, $f'(k) \to 0$ as $k \to \infty$, $0 < \delta < 1$. Assume investment, i_t, is irreversible, i.e., it must be that $i_t \geq 0$. In addition, suppose $c_t, k_t \geq 0$ and that $k_0 > 0$ is given. Consider the functional equation associated with this problem:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} u\left(f(k) + (1-\delta)k - k'\right) + \beta v(k') \\ \Gamma(k) &= \{k' : (1-\delta)k \leq k' \leq f(k)\}. \end{aligned}$$

(a) State a set of assumptions on β and u that guarantee there is a unique, differentiable, concave v that solves the above functional equation. For each property of v, explain which assumptions are used to get it.

(b) Show that monotonicity of $\Gamma(k)$, Assumption 4.6 in S-L, fails so that one of the conditions of Theorem 4.7 which guarantee strictly increasing v, is not satisfied.

(c) Show that the feasible set for this economy satisfies the following 'quasi-monotonicity property': if $\tilde{k} \geq k$, then $\Gamma(k) + (1-\delta)(\tilde{k} - k) \subseteq \Gamma(\tilde{k})$. Here, the sum of a set, say X, and a number, say a, is a new set, $X + a$, where $X + a \equiv \{x + a : x \in X\}$.

(d) Show: v is an increasing function in k. (Hint: (i) following the basic strategy of the proof of Theorem 4.7, it's enough to establish that the assumptions of Theorem 4.7 with the monotonicity assumption on Γ replaced by quasi-monotonicity guarantee Tw is increasing if w is; (ii) make use of the fact that if $k' \in \Gamma(k)$, then $\tilde{k}' = k' + (1-\delta)(\tilde{k} - k) \in \Gamma(\tilde{k})$, $\tilde{k}' > k'$, and $f(\tilde{k}) + (1-\delta)\tilde{k} - \tilde{k}' > f(k) + (1-\delta)k - k'$.) Can you provide intuition for the fact that v is increasing even though Γ fails to satisfy monotonicity?