

Homework #5  
 Economics 411-1  
 Due Wednesday, October 31  
 Christiano

1. Consider an economy with capital of different vintages. At time  $t$ , the amount of capital of vintage  $\tau$ ,  $k_{t,\tau}$ ,  $\tau = 1, 2, 3, \dots$ , is

$$k_{t,\tau} = \gamma^{t-\tau}(1 - \delta)^{\tau-1}i_{t-\tau},$$

where  $\gamma > 1$ ,  $0 < \delta < 1$ ,  $i_{t-\tau}$  is the amount of investment, in time  $t - \tau$  consumption units, applied in period  $t - \tau$ . Capital which has vintage  $\tau$  in period  $t$  has vintage  $\tau + 1$  in period  $t + 1$ . Investment expenditures at time  $t$ ,  $i_t$ , must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in  $k_{t+1,1} = \gamma^t i_t$  units of new-vintage period  $t + 1$  installed capital goods. Consider a given amount of investment,  $i$ . Note that this investment applied in period  $t + 1$  produces more new-vintage installed capital (i.e.,  $\gamma^{t+1}i$ ) than the same level of investment applied in period  $t$  (i.e.,  $\gamma^t i$ ). This reflects the assumption,  $\gamma > 1$  which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good,  $y_{t,\tau}$  according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^\alpha n_{t,\tau}^{1-\alpha}, \quad 0 < \alpha < 1, \quad \tau = 1, 2, 3, \dots$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate,  $r_t$ , since capital is measured in common efficiency units. Similarly, the wage rate is  $w_t$ .

- (a) Show that a firm's profit maximizing choice of  $n_{t,\tau}$  gives rise to the following relationships:

$$y_t = k_t^\alpha n_t^{1-\alpha}, \quad (1 - \alpha) \left( \frac{k_t}{n_t} \right)^\alpha = w_t, \quad \alpha \left( \frac{k_t}{n_t} \right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \quad k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \quad n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(Hint: refer to your class notes on the indeterminacy of firm size under constant returns to scale.)

(b) Show that ‘aggregate capital’,  $k_t$ , evolves according to:

$$k_{t+1} = (1 - \delta)k_t + \gamma^t i_t.$$

2. The following three sector exogenous growth model was recently proposed in Kongsamut, Rebelo and Xie (‘Beyond Balanced Growth’, National Bureau of Economic Research Working Paper number 6159, available on the web at <http://www.nber.org/>), to explain several key features of long-run (i.e., 1869-1990) growth: (i)  $K/Y$  is roughly constant, where  $K$  denotes the aggregate stock of capital, and  $Y$  denotes aggregate output, (ii)  $K$  grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable share in the economy.

Consider the following technology. Agricultural output,  $A_t$ , is produced using the following production function:

$$A_t = B_A K_{A_t}^\alpha (N_{A_t} z_t)^{1-\alpha},$$

where  $B_A > 0$ ,  $0 < \alpha < 1$ ,  $z_t$  denotes the state of technology, and  $K_A$  and  $N_A$  denote capital and labor allocated to agriculture. The state of technology evolves according:

$$z_t = \exp(g) z_{t-1}, \quad g > 0.$$

The manufacturing sector produces output that can be converted into consumption goods,  $C_t$ , or capital goods:

$$C_t + K_{t+1} - (1 - \delta)K_t = K_{M_t}^\alpha (N_{M_t} z_t)^{1-\alpha},$$

in obvious notation. Finally, services,  $S_t$ , are produced according to:

$$S_t = B_S K_{S_t}^\alpha (N_{S_t} z_t)^{1-\alpha}.$$

Suppose that at a point in time, households supply  $K_t$  units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

$$N_{At} + N_{Bt} + N_{Mt} = 1, \quad K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be  $P_{At}$ . Let the price of a service be  $P_{St}$ . Finally, let  $r_t$  and  $w_t$  denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

- (a) Show that competitive behavior by firms implies:  $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$ ,  $P_{At} = 1/B_A$ ,  $P_{St} = 1/B_S$ ,

$$C_t + K_{t+1} - (1 - \delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^\alpha z_t^{1-\alpha}. \quad (1)$$

One can treat the latter as a ‘reduced form’ expression for the economy’s resource constraint.

- (b) Derive an expression for the rate of return on capital. If  $K_t$  grows at the rate,  $g$ , will the rate of return on capital be constant?
- (c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{[(A_t - \bar{A})^\eta C_t^\gamma (S_t + \bar{S})^\theta]^{1-\sigma}}{1 - \sigma}, \quad \eta + \gamma + \theta = 1, \quad \eta, \gamma, \theta, \sigma > 0.$$

- i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.
- ii. Derive the household’s intertemporal Euler equation associated with capital.
- iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \quad \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \quad (2)$$

- iv. Substitute out for  $A_t - \bar{A}$  and  $S_t + \bar{S}$  in terms of  $C_t$  in the household's intertemporal Euler equation, to get an expression in terms of the growth rate of  $C_t$  and the rate of return on capital alone.
- v. Show that in general, there is no reason to expect the economy to converge to a 'balanced growth path', i.e., one in which  $A_t, C, S_t, K_t, Y_t$  all grow at a constant rate. (Here,  $Y_t = K_t^\alpha z_t^{1-\alpha}$ .) Hint: note that if you scale  $A_t, M_t, K_t, S_t$  by  $z_t$  and work with the scaled versions of (2) and (1) and the household's intertemporal Euler equation, you cannot get rid of  $z_t$ .
- vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \quad (3)$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace  $A_t$  and  $S_t$  in (1) by  $A_t - \bar{A}$  and  $S_t - \bar{S}$ , respectively without changing (1). Then, define  $a_t = A_t - \bar{A}$  and  $s_t = S_t - \bar{S}$  everywhere and impose (2), that  $a_t$  and  $s_t$  are each proportional to  $M_t$ .

- vii. Explain why it is that under (3), the following are true:

$$\frac{K_t}{z_t} \rightarrow k^*, \quad \frac{A_t - \bar{A}}{z_t} \rightarrow a^*, \quad \frac{S_t + \bar{S}}{z_t} \rightarrow s^*$$

$$\frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} \rightarrow \exp(g),$$

where  $k^*, a^*, s^*$  are finite, positive, constants.

- viii. Provide a simple formula for  $k^*$ .
- ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?
- x. What does the model imply for  $P_{k^t}$ , the consumption price of capital, along the growth path.