

Homework #7

Economics 411-1, Due Monday, November 19

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1. Consider the model of Matsuyama, in the handout available on the website. Matsuyama denotes a situation in which $k < \theta F / (1 - \alpha)$ as a ‘Solow regime’ and a situation in which $k > \theta F / (1 - \alpha)$ as a ‘Romer regime’. Let

$$G = \beta \left[\alpha \left(\frac{\theta F}{1 - \alpha} \right)^{\alpha - 1} + 1 - \delta \right].$$

- (a) Suppose $G < 1$. Show that there is a steady state value of k in the Solow regime, call it k^* . That is, for any $M_{-1} > 0$ if the initial stock of capital is $K_0 = k^* M_{-1}$, then there is a no growth equilibrium with

$$K_{t+1} = K_0, \text{ for } t = 0, 1, 2, \dots$$

Note that in this steady state equilibrium, there is never any innovation. This regime is more likely the larger is F , which makes sense because this represents the fixed cost of innovation.

Let $M_{-1} = 1$, $\beta = 1/1.03$, $\alpha = 0.36$, $F = 100$, so that $G = 0.8833$, after rounding. Compute k^* . If K_0 is in a sufficiently small neighborhood of $k^* M_{-1}$, show that there exists an equilibrium in which $\lim_{t \rightarrow \infty} K_t = k^* M_{-1}$. (Hint: consider the household’s intertemporal Euler equation after substituting out for the rental rate of capital using equation (12) in the handout. Use the following facts: (i) for K_t sufficiently close to $k^* M_{-1}$, the Taylor series expansion of this equation about $K_t = K_{t+1} = K_{t+2} = k^* M_{-1}$ is an arbitrarily good approximation to this equation, and write this as

$$V_0 \tilde{K}_t + V_1 \tilde{K}_{t+1} + V_2 \tilde{K}_{t+2} = 0, \quad t = 0, 1, 2, \dots$$

where $\tilde{K}_t \equiv K_t - k^* M_{-1}$; (ii) the set of solutions to a linear difference equation like this is given by $\tilde{K}_t = (\tilde{K}_0 - a)\lambda_1^t + a\lambda_2^t$, for $t = 0, 1, 2, \dots$, for arbitrary a , where the λ_i ’s solve:

$$V_0 + V_1 \lambda_i + V_2 \lambda_i^2 = 0, \quad i = 1, 2;$$

- (b) Suppose $G > 1$. Show that there is a steady state value of k in the Romer regime, call it k^{**} . That is, given $M_{-1} > 0$ and $K_0 > 0$, there is an equilibrium in which

$$\frac{K_t}{M_{t-1}} = k^{**}, \quad \frac{c_{t+1}}{c_t} = \frac{K_{t+1}}{K_t} = \frac{M_t}{M_{t-1}} = G, \quad t = 0, 1, 2, \dots$$

Provide a formula for computing k^{**} and verify $k^{**} > \theta F / (1 - \alpha)$.

- (c) Think about the possibility of equilibria that fluctuate between the Romer and Solow regimes: in a Solow regime the relatively low amount of physical capital results in a high rental rate on capital. This discourages innovation but encourages capital accumulation (just like in the neoclassical growth model when you are below steady state). When capital becomes relatively abundant (so that $k > \theta F / (1 - \alpha)$) then innovators have an incentive to enter: the Romer regime begins and M starts to grow. If M grows fast enough relative to K (this will depend upon parameter values) then k is driven back down towards the Solow regime, and the process starts all over again. Along such a growth path there will be alternating periods of fast growth during which there is no innovation and slow growth, during which there is a lot of innovation. Interestingly, the same conditions that encourage high growth in capital and output, i.e., a high rental rate of capital, discourage innovation. This model generates all sorts of empirical hypotheses that would be interesting to test (patent applications come in bursts, and at times of low growth?).
2. The purpose of this question is to give you a ‘hands on’ acquaintance with linearization as a strategy for solving a model. In addition, you will use this tool to shed light on one possible factor, lack of substitutability between capital and labor, that may help account for comovement of labor inputs over the business cycle.

Consider a model in which the preferences of the representative agent are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \quad u(c, n) = \frac{[c(1-n)^\psi]^{1-\sigma}}{1-\sigma}, \quad 0 < \beta < 1,$$

where $\psi, \sigma > 0$, and satisfy the other restrictions needed for strict concavity (see the handout on bringing hours worked into the neoclassical growth model.) Suppose the resource constraint is given by

$$c_t + k_{t+1} - (1 - \delta)k_t \leq \left[\alpha k_t^{\frac{\nu-1}{\nu}} + (1 - \alpha)n_t^{\frac{\nu-1}{\nu}} \right]^{\frac{\nu}{\nu-1}} \exp(z_t) \equiv f(k_t, n_t, z_t),$$

where $\delta, 1 - \delta, \nu > 0$ and z_t has the following statistical representation:

$$z_t = \rho z_{t-1} + \varepsilon_t, \quad |\rho| < 1,$$

where ε_t is a zero mean random variable that is independently distributed over time. Note that the mean of z_t is zero. Here, $k_0 > 0$ is given and the restrictions are $n_t, k_{t+1}, 1 - n_t, c_t \geq 0$, for $t = 0, 1, 2, \dots$.

- (a) The solution to the planning problem associated with the above model is a set of policy rules, $n = h(k, z)$, and $k' = g(k, z)$. Explain why these also characterize the aggregate employment and capital decisions in the underlying Arrow-Debreu or Sequence of Markets equilibrium.
- (b) We can interpret the above model as a ‘reduced form’ for a particular two sector model with the following specification of technology:

$$c_t \leq f(k_{ct}, n_{ct}, z_t), \quad i_t \leq f(k_{it}, n_{it}, z_t),$$

and the following additional restrictions:

$$n_{it}, n_{ct}, k_{ct}, k_{it} \geq 0, \quad k_{ct} + k_{it} = k_t, \quad n_{it} + n_{ct} = n_t.$$

Show that the policy rule for n_c is as follows:

$$n_c(k, z) \equiv \left(\frac{1 - \alpha}{\psi} \right)^{\nu} (1 - h(k, z))^{\nu} \left(\frac{c(k, z)}{\exp(z)} \right)^{1-\nu}, \quad (1)$$

where $c(k, z)$ is the policy rule for consumption:

$$c(k, z) = f(k, h(k, z), z) + (1 - \delta)k - g(k, z).$$

(Hint: think about the labor first order condition in the two-sector version of the economy.) Explain why it is that when $\nu = 1$, this model is incapable of accounting for business cycle *comovement*, i.e., the fact that labor across different sectors rises and falls together.

(c) The share of income paid to labor, s_t , is defined as

$$s_t = \frac{W_t n_t}{y_t},$$

where $y_t = f(k_t, n_t, z_t)$ and W_t is the wage rate in the underlying market economy. Show that, in nonstochastic steady state,

$$1 - s = \alpha^\nu \left[\frac{1}{\beta} + \delta - 1 \right]^{1-\nu}.$$

From this relationship we see a potentially important difference between the $\nu = 1$ and the $\nu \neq 1$ cases. When $\nu = 1$, then $s = 1 - \alpha$. Both of these are independent of the time unit in the model. However, when $\nu \neq 1$, the formula involves a mixture of variables denominated in units of time (β and δ) and variables that do not have a time dimension (s , α , ν). For example, if in an annual model $\beta = 1/1.03$ makes sense, then $\beta = 1/1.03^{.25}$ is the corresponding quarterly discount factor. If $\delta = .08$ makes sense at an annual frequency, then $\delta = 1 - (1 - .08)^{.25} \cong .08/4$ is the corresponding quarterly rate. These observations have an important implication. To see this, rewrite the above expression:

$$\alpha = \left\{ (1 - s) \left[\frac{1}{\beta} + \delta - 1 \right]^{\nu-1} \right\}^{\frac{1}{\nu}}.$$

Suppose our strategy for estimating α is to set it to the value that is compatible with the sample average of labor's share, s , the Department of Commerce's estimate of δ and values for β and ν based on other information. If for some reason, say to get the right amount of comovement of labor across sectors, we wanted to make ν small, but we didn't want to violate the constraint, $\alpha < 1$, then we might prefer to go to an annual, rather than a quarterly specification for the model. This would make $1/\beta + \delta - 1$ large and, hence, increase the likelihood that $(1 - s)/[1/\beta + \delta - 1] < 1$, so that $\alpha \leq 1$ as $\nu \rightarrow 0$. Thus, when we allow $\nu \neq 1$, the choice of time unit for the model can matter when trying to 'fit' various first moments of the data. For example, suppose $s = 0.64$, $\delta = .10$,

$\beta = 1/1.05$. Then, $\alpha = 0.7$, after rounding. But, if we use the quarterly analogs of the parameters with a time dimension, so that $\delta = .026$ and $\beta = 1/1.0123$, then the above formula implies $\alpha = 1.95$, after rounding.

- (d) The fact that when $\nu = 1$, there cannot be comovement in labor inputs across sectors is a very big problem, since comovement is perhaps *the* most fundamental feature of business cycles. We shall investigate whether comovement is possible with $\nu \neq 1$. We want n_c and n_i to both go up in response to a positive shock to z which raises output. Give an intuitive explanation, using (1), why raising ν might increase the likelihood of getting comovement.
- (e) The only way to find out if there is comovement, is to do some computations. We shall compute

$$\frac{1}{n_c(k, z)} \frac{dn_c(k, z)}{dz}, \quad \frac{1}{n_i(k, z)} \frac{dn_i(k, z)}{dz},$$

evaluating all variables in non-stochastic (replacing z by $Ez = 0$) steady state. The objective is to get these to have the same sign and the second to be about twice as large as the first. The computation will rely fundamentally on the linearization solution strategy we discussed in class. You should use the following parameter values: $\psi = 3$, $\sigma = 2$, $\rho = .9$, $\beta = 1/1.05$, $\delta = .10$, $\nu = 2.5$, $\alpha = .58$.

- (f) Compute the steady state values of n , c , y , n_c , n_i , k , s , k/y .
- (g) Compute u_c , u_{cc} , u_{cn} , u_n , u_{nn} , $F = y + (1 - \delta)k$, F_k , F_{kk} , F_n , F_{nn} , F_z , F_{nz} , F_{kz} . These are the derivatives of the various functions, evaluated in nonstochastic steady state. Here, F is $f + (1 - \delta)k$, where f is the function defined above. Also it is convenient to define the function $w(c, n) = -u_n(c, n)/u_c(c, n)$. Compute w , w_c , w_n . Again, these are evaluated in the nonstochastic steady state. All these numbers are things that go into constructing the linearized Euler equations. It is convenient to compute these separately.
- (h) Write the Euler equations as:

$$E [R(k_t, k_{t+1}, k_{t+2}, n_t, n_{t+1}, z_t, z_{t+1})|z_t] = 0,$$

$$h(k_t, k_{t+1}, n_t, z_t) = 0.$$

Obtain the coefficients in the first order Taylor series expansion of R and h about the non-stochastic, steady state values of the arguments. Use the linearized h to solve for n_t in terms of k_t , k_{t+1} , z_t . Use this expression to substitute out for n_t , n_{t+1} in the linearized version of R . The resulting expression can be written:

$$0 = E[g_k(k_t - k) + g_{k'}(k_{t+1} - k) + g_{k''}(k_{t+2} - k) + g_z z_t + g_{z'} z_{t+1} | z_t].$$

Report the values for g_i , $i = k, k', k'', z, z'$. Here, k is the non-stochastic steady state value of the capital stock.

- (i) Posit the following policy rule for capital:

$$k_{t+1} = k + \alpha_0(k_t - k) + \alpha_1 z_t.$$

Find the values of $|\alpha_0| < 1$ and α_1 which solve the linearized Euler equation. Using the expression for n_t in terms of k_t , k_{t+1} , z_t and the policy rule for capital to find β_0 β_1 in:

$$n_t = n + \beta_0(k_t - k) + \beta_1 z_t.$$

- (j) Compute the derivatives discussed above by linearizing the mapping to the employment decision and the investment decision, and making use of the values you obtained for α_1 and β_1 .
- (k) So, do you get comovement?
- (l) Suppose $\varepsilon_1 = 1$, and $\varepsilon_t = 0$ for $t \neq 1$. Set $s_0 = 1$ and compute s_1 , s_2 , s_3 , s_4 . Set k_1 and z_0 equal to their steady state values. Compute c_t , n_t , I_t , for $t = 1, 2, 3, 4$, and calculate the percent difference between these and their steady state values (i.e., what they would have been if ε_t had been held constant at zero.) Note how much stronger the percentage reponse of investment is, compared with the response in consumption.