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411-1, Fall 2001

MIDTERM EXAM

There are five questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. (20) Household preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t - c^*)^{1-\gamma} - 1}{1-\gamma}, \ \gamma > 0$$

where $c^* > 0$ and $c_t \geq c^*$ is consumption. The resource constraint is:

$$c_t + k_{t+1} - (1-\delta)k_t \leq k_t^{\alpha}, \ \alpha, \delta \in (0,1).$$

In addition, k_0 is given and $k_{t+1} \geq 0$. Define a steady state stock of capital as a value of the initial capital stock, k_0, such that the efficient allocations are, $k_{t+1} = k_0$ for $t = 0, 1, 2, ...$ Provide a rigorous proof that there are two such values of k_0. You may appeal to a theorem in your argument. If you do so, be sure to state the assumptions and proposition in the theorem precisely.

2. (15) Consider an economy with the following technology:

$$c_t + k_{t+1} \leq Ak_t + k_t^{\alpha}, \ 0 < \alpha < 1, \ A > 0,$$

with $k_0 > 0$ given. Also, we require $k_t, c_t \geq 0$ for all t. Prove rigorously that when $A < 1$, there is a $\bar{k} < \infty$ such that feasibility implies $k_t \leq \bar{k}$ for all t.

3. (15) Suppose we have a value function which satisfies the following functional equation:

$$v(k) = \max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \ k \in K, \tag{1}$$

where K is a convex subset of R^l. Present a set of assumptions and prove rigorously that they are sufficient to guarantee that v is strictly increasing in k. State clearly the role of each assumption you make in your proof.

4. (25) Consider the following two-sector model of optimal growth. A planner seeks to maximize the utility of the representative agent given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where c_t is consumption of good 1 at t. Sector 1 produces consumption goods using capital, k_{1t}, and labor, n_{1t}, according to the production function, $c_t \leq f_1(k_{1t}, n_{1t})$. Sector 2 produces the capital good according to the production function $k_{t+1} \leq f_2(k_{2t}, n_{2t})$. The functions, u, f_1 and f_2 are strictly increasing and concave, and $f_1(0,0) = f_2(0,0) = 0$. The constraint on labor is $n_{1t} + n_{2t} = 1$, where 1 denotes the total amount of labor supplied. The other constraints include $n_{it}, k_{it} \geq 0$, $i = 1, 2$, and $k_{t+1} \geq 0$. The sum of the amounts of capital used in each sector cannot exceed the initial capital in the economy, that is, $k_{1t} + k_{2t} \leq k_t$, and $k_0 > 0$, given.

   Let $v(k_0)$ be the function which characterizes how the maximized value of the representative agent's discounted utility varies with k_0.

   (a) Write the sequence problem in the following form:

   $$v^*(k_0) = \max_{k_{t+1} \in \Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

   for $k_0 \in K$. Develop expressions for K, F and Γ. Give a mathematically precise definition of the domain of F. State a restriction on u, f_1, f_2 that is sufficient to guarantee that F is a bounded function.

   (b) Argue informally, by appealing to the structure of the sequence problem, that the v which solves (1) with the K, F and Γ from (a) has the property, $v(k) = v^*(k)$ for $k \in K$.

   (c) Prove that there is a unique v which satisfies (1) and display an iterative algorithm for finding it. In your proof, you may use Blackwell's theorem. If so, state the theorem precisely and prove rigorously that the conditions of the theorem are satisfied.

5. (25) Consider a household which solves the following problem:

   $$v(k, r, w) = \max_{c, l \in B(k, r, w)} u(c, l),$$

   where $u : \Re_+^2 \rightarrow \Re$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption,

c, and leisure, l. The constraints the household must obey in selecting c, l are summarized by B:

$$B(k, r, w) = \{c, l : 0 \leq c \leq rk + w(1 - l),\ 0 \leq l \leq 1\}.$$

Here, $r > 0$ is the market rental rate on capital and $w > 0$ is the market wage rate, neither of which the household can control. Also, $k > 0$ is the household's stock of capital. Prove that v is concave and that the derivative of v with respect to k exists for 'interior k'. An interior k means $k > 0$ and that the optimal choice of l satisfies $0 < l < 1$. Also, display a formula for the derivative of v. If you make use of a theorem to help prove your result, be sure to state it clearly.