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FINAL EXAM

The exam has three parts. The points for each part are indicated in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (Theory of TFP.) For this question, you may suppose that the aggregate supply of labor, l, and of capital, k, are given. Consider a Dixit-Stiglitz firm sector, in which final goods are produced by a representative, competitive firm using the following technology:

$$y = \exp\left[\int_0^1 \log y(i)di\right]$$

(a) Let $p(i)$ denote the price of $y(i)$. Express the problem of the final good producers and derive their first order conditions. What is the elasticity of demand for $y(i)$, with respect to $p(i)$?

There is free entry into the production of the i^{th} intermediate good, although we only consider equilibria in which output is produced by a single intermediate good producer. To produce any positive amount of the i^{th} good, the firm must pay a fixed cost, ϕ, denominated in final goods. After paying the fixed cost, the i^{th} intermediate good firm produces using this production function:

$$y(i) = \theta k(i)^{\alpha} l(i)^{1-\alpha},\ 0 < \alpha < 1,$$

where $\theta > 0$ represents the state of technology and $l(i)$ and $k(i)$ represent the amounts of labor and capital rented by the firm. The firm is competitive in factor markets and takes the wage rate, w, and the rental rate on capital, r^k, as given.

(b) Consider the cost function for the intermediate good producer who produces a given level of output, $y(i)$ (hint: this is derived from the relevant cost minimization problem.) Derive the marginal cost of producing $y(i)$, as a function of w, r^k and - potentially - $y(i)$. What is that producer's total cost? Display a set of efficiency conditions for the choice of labor and capital.

(c) Show that if the intermediate good producer were a monopolist without threat of competition, there does not exist a solution to its profit maximization problem.

(d) Define aggregate output as y, after fixed costs have been netted out - $Y = y - \phi$. Show that this is related to aggregate factor inputs by the following relation:

$$Y = MC \times \theta k^{\alpha} l^{1-\alpha}.$$

Prove that $MC \times \theta$ is total factor productivity (TFP), where this is defined as:

$$\text{TFP}{=}\frac{Y}{k^{a_k} l^{a_l}},$$

where a_k is the share of total income paid to labor, and a_l is the share of total income paid to capital.

2. Answer...(Theory of TFP) We begin by solving the equilibrium of the firm sector, for given aggregate factor supplies, k and l. The equilibrium involves total output and factor prices. In addition, we obtain an aggregate output relation, which incorporates a theory of TFP. The structure of output resembles the standard Dixit-Stiglitz form. There is a slight twist, in that the final goods production function is specified so that demand for intermediate goods is unit elastic. If intermediate good suppliers were monopolists there would be no equilibrium in this case. However, we assume that - although there is only one producer for each intermediate input - there is free entry into intermediate good production. This has the effect of making output in this sector determinate, and driving profits to zero. In our 'theory of tfp', tfp is marginal cost.

   The production function for final output has the following form:

$$y = \exp\left[\int_0^1 \log y(i)di\right]$$

   The maximization problem of final good firms is:

$$\max_{y(i)} y - \int_0^1 p(i)y(i)di,$$

which leads to the following unit-elastic demand curve:

$$\frac{y}{y(i)} = p(i).$$

The i^{th} intermediate good firm must pay a fixed cost, ϕ, denominated in final goods, in order to produce any positive amount. After paying this fixed cost, it produces intermediate goods using this production function:

$$y(i) = \theta k^{\alpha} l^{1-\alpha},\ 0 < \alpha < 1,$$

where $\theta > 0$ represents the state of technology and $l(i)$ and $k(i)$ represent the amounts of labor and capital rented by the firm. It is useful to consider the intermediate good firm's cost minimization problem, conditional on producing a given level of output, $y(i)$:

$$\min_{n(i),k(i)} wn(i) + r^k k(i) + \lambda \left[y(i) - \theta k\left(i\right)^{\alpha} l\left(i\right)^{1-\alpha} \right].$$

The first order necessary conditions for this problem are

$$\begin{aligned} w &= \lambda \left(1-\alpha\right)\theta \left(\frac{k(i)}{l(i)}\right)^{\alpha}, \\ r^k &= \lambda \alpha \theta \left(\frac{k(i)}{l(i)}\right)^{\alpha-1}, \\ y(i) &= \theta k\left(i\right)^{\alpha} l\left(i\right)^{1-\alpha} \end{aligned}$$

The ratio of the first two equations implies:

$$\frac{w}{r^k} = \frac{1-\alpha}{\alpha}\frac{k(i)}{l(i)}.$$

This equation, together with the technology, can be solved for $k(i)$ and $l(i)$:

$$l(i) = \frac{y(i)}{\theta}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{r^k}{w}\right)^{\alpha} \tag{1}$$

$$k(i) = \frac{y(i)}{\theta}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha-1}\left(\frac{r^k}{w}\right)^{\alpha-1}. \tag{2}$$

These are the demand by the i^{th} intermediate good firm for $l(i)$ and $k(i)$, if it produces $y(i)$ and faces factor prices, r^k and w. These expressions can be substituted into the firm's factor costs, $wn(i)+r^k k(i)$, to obtain an expression for the firm's cost function (abstracting from the fixed cost):

$$Cost = MC \times y(i),$$

where

$$MC = \left(\frac{1}{\alpha}\right)^{\alpha}\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\frac{\left(r^k\right)^{\alpha} w^{1-\alpha}}{\theta}. \tag{3}$$

(It is a standard result, easily verified, that $\lambda = MC$.) Given the cost function, it is possible to express the profit maximization problem of the intermediate good firm in terms of a choice of $p(i)$ and $y(i)$:

$$\max_{p(i),y(i)} p(i)y(i) - MC \times y(i) - \phi,$$

subject to the demand curve coming from final good producers and the given functional form for MC. Substituting in the demand curve:

$$\max_{y(i)} y - MC \times y(i) - \phi,$$

where y is treated as exogenous by the firm. This problem obviously has no solution, since the firm can always increase profits by reducing $y(i)$, subject to $y(i) > 0$ (if $y(i) = 0$, then profits are zero).

If there were no entry into the industry, then the profit maximizing firm would set $y(i)$ arbitrarily close to zero. However, the firm cannot do this, because there is free entry. Free entry implies that $y(i)$ cannot fall below the level where profits (including the fixed costs) are positive. Free exit also implies that profits cannot be negative. Thus:

$$y - MC \times y(i) = \phi. \tag{4}$$

Since all intermediate good firms face the same fixed costs, it follows that $y(i) = \tilde{y}$ for all i. The final good production function then implies $\tilde{y} = y$. So, we have:

$$y - MC \times y = \phi,$$

or,

$$y - \phi = MC \times y.$$

From this it is clear that for this example to be interesting, it must be that $y = k^\alpha l^{1-\alpha} > \phi$. That is, gross output, $k^\alpha l^{1-\alpha}$, must be at least as big as the fixed cost. Otherwise, there would be no output in this economy!

If we let Y denote aggregate output (after netting out fixed costs), then there exists a relationship in which aggregate output and aggregate factor inputs are related:

$$Y = MC \times \theta k^\alpha l^{1-\alpha}. \tag{5}$$

To summarize, given l and k, factor prices are determined by (1) and (2). Then, marginal cost is determined by (3) and output is determined by (5).

It is useful to determine what is TFP for this economy. The zero profit condition (i.e., (4)) implies that payments to capital and labor equal Y. Then, the share of income going to capital and labor are:

$$\begin{aligned}
\frac{r^k k}{Y} &= \frac{r^k k}{MC \times y} = \frac{\frac{y(i)}{\theta}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha-1}\left(\frac{r^k}{w}\right)^{\alpha-1} r^k}{\left(\frac{1}{\alpha}\right)^{\alpha}\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\frac{\left(r^k\right)^{\alpha} w^{1-\alpha}}{\theta} y} \\
&= \frac{\left(\frac{1-\alpha}{\alpha}\right)^{\alpha-1}}{\left(\frac{1}{\alpha}\right)^{\alpha}\left(\frac{1}{1-\alpha}\right)^{1-\alpha}} = \frac{\left(\frac{1}{\alpha}\right)^{\alpha-1}}{\left(\frac{1}{\alpha}\right)^{\alpha}} = \alpha \\
\frac{wl}{Y} &= 1 - \alpha.
\end{aligned}$$

It follows that MC is TFP in this economy.