Homework #3
Economics 411-1, fall 2002
Due Wednesday, October 16
Christiano

1. (Boldrin-Montrucchio 1986). Consider the policy rule, $g : [0,1] \to [0,1]$:
$$g(x) = 4x(1-x).$$
Find an economy, (F, Γ, β, X), for which the above function is the policy rule, where the economy satisfies *all* of our assumptions (i.e., A4.3-A4.9). Some hints: Recall, $\Gamma, \beta \in (0,1)$ and F satisfy
$$g(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y),$$
where
$$v(x) = F(x, g(x)) + \beta v(g(x)).$$
Recall that defining a function or correspondence requires not just the functional forms, but also domains and ranges. Recall too that a function is strictly concave iff the second derivative of the function w.r.t. each argument is negative everywhere, and its Hessian is positive everywhere. Also, note that
$$g(x) = \arg\max_{y \in \Gamma(x)} \Psi(x,y),$$
where
$$\Psi(x,y) = -\frac{1}{2}y^2 + yg(x) - \frac{1}{2}Lx^2 + ax$$
Finally, note that $v(x) = \Psi(x, g(x))$, and use this to back out F. You can think of your task as having to identify values of a and L that ensure A4.3-A4.9 are satisfied. How many such values are there?

2. (This is a continuation of question 1.) Let
$$\begin{aligned} a &= 150 \\ L &= 48 \\ \beta &= 0.01072 \\ d &< 0.0062 \end{aligned}$$
Consider the following two-sector economy: in the final good sector the production function is $c \le f(k_c, l_c)$, while the capital good is produced according to the Leontieff production function $k' \le \min\{k_k/d, l_k\}$, where d is a positive real constant. Utility from consumption is linear $u(c) = c$ and the aggregate constraints are $l_c + l_k \le 1$, $k_c + k_k \le k$. Let $f(k_c, l_c) = F(k_c + d(1-l_c), 1-l_c)$ from the previous question, and argue that the g function of the previous question is the policy rule for this economy. Find the economic intuition for the fact that for low values of k, g is increasing in k, whereas it is decreasing in k for larger values of k (hint: recall our discussion of this point in class).

3. Plot the sequence of efficient x's for the above economy, for $x_0 = 1$, 0.5, 0.75 and 0.70. Do you observe monotone convergence to a unique steady state?

4. According to Theorem 4.15, the Euler and transversality conditions (equations (2) and (3) on page 98 of S-L) are sufficient for an interior sequence of x_t's to constitute an optimum of the SP problem. Necessity of the Euler equation is fairly obvious (a standard variational argument establishes this). Here is a sketch of an argument that establishes necessity of the transversality condition. This argument requires, in addition to the usual assumptions, $x_t = 0 \in X$, and $0 \in \Gamma(x_t)$ for $x_t \in X$. I sketch this argument below. Convert this sketch into a rigorous proof. In your proof you may use, without proof, any results from the book that you wish. However, you must be absolutely clear always about which assumptions you are using.

   Let x_t^*, $t = 0, 1, 2, ...$, denote a sequence of x_t's that solve the SP. Then,

   $$v(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v(x_{t+1}^*),\ t = 0, 1, 2, ...,$$

   where, of course, $x_0^* = x_0$, the initial condition. Let x_t be any other sequence that satisfies $(x_t, x_{t+1}) \in A$, the domain of F, for all t (does this imply that x_t is in the domain of v?). Then,

   $$v(x_t) - v(x_t^*) \geq F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*).$$

   The proof follows in a straightforward (not *trivial*) way by replacing x_t with 0, for all t, by multiplying both sides of this expression by β^t, and driving $t \to \infty$.