

Homework #4
Economics 411-1
Due Wednesday, October 23
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1. Consider the neoclassical growth model studied in class, with $\beta = 1/1.03$, $\alpha = 1/3$, $\delta = 0.10$, $\gamma = 1$, where preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

and the aggregate resource constraint is given by:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq k_t^\alpha.$$

What is the steady state value of k ? How long does it take to close 95 percent of a gap between an initial value of the capital stock, k_0 , and the steady state value?

Consider the Solow model. This assumes that people save and invest a fixed fraction, s , of gross output, k^α :

$$k_{t+1} - (1 - \delta)k_t = sk_t^\alpha$$

What value of s is required in order for the steady states of the neoclassical and Solow models to coincide? How much time does it take for 95 percent of the gap between k_0 and steady state capital to be closed in the Solow model?

2. Suppose a planner chooses to maximize, by choice of c_0, c_1, c_2, \dots , the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \dots], \quad u(c_t) = \log(c_t) \quad (1)$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, \quad 0 < \alpha < 1, \quad c_t, k_{t+1} \geq 0, \quad k_0 \text{ given,}$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

- (a) Let $k_{t+1} = g_t(k_t)$ denote the policy rule that solves this problem, $t = 0, 1, \dots$. From the perspective of period 0, the part of the problem from $t = 1$ and on looks exactly like the problem with $\delta = 1$. As a result, you know that the optimized value of $u(c_1) + \beta^2 u(c_2) + \dots$ has the form, $v(k_1)$, and you know how to compute $v(k_1)$ because it has a simple log-linear form. Use this to show that the optimal choice of k_1 has the form:

$$k_1 = g k_0^\alpha,$$

where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ . How does the saving rate from period $t = 1$ and on compare with the date 0 saving rate?

- (b) Is there a unique k^* with the property $k_t \rightarrow k^*$ as $t \rightarrow \infty$ for all k_0 ? Display a formula relating k^* to the parameters of the model.
- (c) Suppose $\beta = 1/1.03$, $\alpha = 1/3$, $\delta = 0.85$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.
- (d) Now suppose that when date 1 happens, the planner decides to reoptimize with respect to k_2, k_3, \dots . The initial condition for this problem is k_1 , the decision implemented by the planner last period. From the perspective of $t = 1$, the planner's preferences over c_t , $t \geq 1$ are as follows:

$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \dots]$$

and the resource constraint is as before. (Note how different the problem for $t \geq 1$ looks from the point of view of period 1 than it does from the point of view of period 0.) What values will the planner choose for k_1, k_2, k_3, k_4, k_5 ? If the planner chooses to reoptimize in this way every period, to what value will k_t actually tend?

- (e) Why are the values for k_2, k_3, k_4, k_5 chosen by the planner in date 1 different from the values planned for these variables as of date 0? Because of this difference, the problem is said to be *time inconsistent*. If δ had been set to one, we would not have had this problem. Why not?

- (f) Basically, the attitude of the planner is ‘I’m very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I’ll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I’ll consume a lot today and save a lot tomorrow.’ Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that ‘tomorrow I’ll quit smoking, but tonight I’ll just have one or two more’. It also has been used to explain the low US saving rate. The notion is that many people say, ‘today I’ll spend, and tomorrow I’ll save’, day after day. (See the papers of David Laibson, of Harvard.)

Does the solution that we have used in (d) make any sense? Would a rational person really make decisions in the time-inconsistent way described there? Maybe. But, here is another approach. The idea is to treat the planner in each period as though they were a different person, not bound by any commitments that may have been made in previous periods. This gives rise to a Nash equilibrium concept in which each date’s planner optimizes, taking as given what planners in other periods do. Thus, suppose a planner optimizes today, subject to the constraint that planners at all future dates save according to the rule, $k_{t+1} = dk_t^\alpha$. Let $v(k_t; d)$ denote the present value of utility, $u(c_t) + \beta u(c_{t+1}) + \dots$, that occurs when the saving rate, d , is followed forever.

- g. Display an explicit formula for $v(k_t; d)$ (hint: you can find it as the fixed point of a dynamic programming sequence that you can do with paper and pencil, like we did in class)
- h. Let $g(k; d)$ denote the policy rule of a planner with preferences, (1), who expects the saving rate in the future to be d . Show that $g(k; d) = D(d)k^\alpha$, and derive an explicit formula for $D(d)$.
- i. A natural equilibrium concept in this setting is that it is a d^* , such that $d^* = D(d^*)$. Display a formula relating d^* to the parameters of the model. Defend this equilibrium concept.

- j. Call the decisions, k_0, k_1, \dots emerging from the solution in (i) the *Nash solution*. Call the decisions in (c) the *commitment solution*. Call the decisions in (d) the naive, *time inconsistent solution*. Compare the value of utility in each of these solutions, assuming $k_0 = 1$ in each case.
- k. Which of these allocations corresponds to the allocations in a sequence of markets equilibrium, and which corresponds to the allocations in an Arrow-Debreu equilibrium? Explain.