Homework #5
Economics 411-1
Due Wednesday, October 30
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1. This question will derive an AK model from what initially may look like a non-AK environment. In some respects, the model resembles Romer's. However, it is different in several ways. It does not rely on 'increased variety'. In addition, there is no imperfect competition in the model. The model's answer to the growth puzzle - what prevents the rate of return on capital from falling along a growth path? - is that diminishing returns from capital are avoided by opening up production at new locations as more capital becomes available.

   Our analysis will proceed as for the Romer model. Thus, we will suppose that there is a given aggregate supply of capital, K. There is no labor, initially. Final goods are produced by a representative final goods producer using the following production function:

$$y = \int_0^M y(i) di^{\alpha},$$

   where M indicates the range of intermediate inputs being produced. Intermediate good output, i, can be thought of as output produced at a particular location. The number of locations is the number of points on the interval, $(0, M)$.Note that the 'variety' effects in the Romer model are absent here: its only the total amount of intermediate goods that matter. Denominate all prices in terms of the final output good, so that the price of this good is unity. Let the price of the i^{th} intermediate good be $p(i)$.

   a. Write out the final good producer's first order condition respect to the choice of $p(i)$. What is the elasticity of demand for the i^{th} intermediate input? Explain.

   Production at location i occurs using the following technology:

$$y(i) = \begin{cases} k_i^{\rho} - \psi & k_i^{\rho} \geq \psi \\ 0 & \text{otherwise} \end{cases}, \quad 0 < \rho < 1,$$

where k_i is the quantity of capital used in the i^{th} location. This technology is similar to that in the Romer model, in that production involves a fixed cost. A difference is that the fixed cost is not in the form of capital itself. Instead, the fixed cost is in the form of intermediate good output. For a technology to generate intermediate goods for sale to final good producers, the technology, k_i^ρ, must first be operated to produce ψ. That is, an amount of capital equal to $k_i^* = \psi^{1/\rho}$ must be rented from a capital supply market. It is the additional output produced from $k_i > k_i^*$ that can be sold to the final good producers.

Production at the i^{th} location is controlled by and intermediate good producer. The i^{th} intermediate good producer's profits are given by:

$$\text{profits} = \begin{cases} \bar{p}(k_i^\rho - \psi) - rk_i & k_i^\rho \geq \psi \\ -rk_i & k_i^\rho < \psi \end{cases}.$$

The producer will operate only if profits are non-negative. If the best the producer can do with $k_i > 0$ is receive negative profits, they would do better to set $k_i = 0$. Although only one producer can produce at a given location, anyone is free to open up a location and produce. The production locations where production occurs are designated $i \in (0, M)$.

b. Derive the first order condition for producers at production locations where $k_i > 0$. Explain why a zero profit condition must be satisfied, and display that condition.

c. There are 5 variables to be determined: y, r, M the amount of capital used in active productive sites, and the price of intermediate good output. Display 5 equations for use in solving for these variables. Show that aggregate output can be written in the form, $y = AK$, where A is a function of model parameters.

d. Suppose the production technology for producing final goods is replaced by

$$y = n^{1-\alpha}\left[\int_0^M y(i)di\right]^\alpha,\ 0 < \alpha < 1.$$

Suppose that the aggregate level of employment, n, is given. Suppose that the wage rate is w. This version of the model adds one unknown, w, and one equation, the final good firm's efficiency condition for labor. Do we still have an AK model? Why not?

2. The following three sector exogenous growth model was recently proposed in Kongsamut, Rebelo and Xie ('Beyond Balanced Growth', National Bureau of Economic Research Working Paper number 6159, available on the web at http://www.nber.org/), to explain several key features of long-run (i.e., 1869-1990) growth: (i) K/Y is roughly constant, where K denotes the aggregate stock of capital (actually, this is what people *used* to think - since Gordon's work it is now thought that K/Y rises when K is correctly adjusted for quality change), and Y denotes aggregate output, (ii) K grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable share in the economy.

   Consider the following technology. Agricultural output, A_t, is produced using the following production function:

$$A_t = B_A K_{At}^{\alpha} (N_{At} z_t)^{1-\alpha},$$

   where $B_A > 0$, $0 < \alpha < 1$, z_t denotes the state of technology, and K_A and N_A denote capital and labor allocated to agriculture. The state of technology evolves according:

$$z_t = exp(g) z_{t-1}, \;\; g > 0.$$

   The manufacturing sector produces output that can be converted into consumption goods, C_t, or capital goods:

$$C_t + K_{t+1} - (1-\delta)K_t = K_{Mt}^{\alpha}(N_{Mt} z_t)^{1-\alpha},$$

   in obvious notation. Finally, services, S_t, are produced according to:

$$S_t = B_S K_{St}^{\alpha} (N_{St} z_t)^{1-\alpha}.$$

   Suppose that at a point in time, households supply K_t units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

$$N_{At} + N_{Bt} + N_{Mt} = 1, \;\; K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be P_{At}. Let the price of a service be P_{St}. Finally, let r_t and w_t denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

(a) Show that competitive behavior by firms implies: $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$, $P_{At} = 1/B_A$, $P_{St} = 1/B_S$,

$$C_t + K_{t+1} - (1-\delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^{\alpha} z_t^{1-\alpha}. \quad (1)$$

One can treat the latter as a 'reduced form' expression for the economy's resource constraint.

(b) Derive an expression for the rate of return on capital. If K_t grows at the rate, g, will the rate of return on capital be constant?

(c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{\left[\left(A_t - \bar{A}\right)^{\eta} C_t^{\gamma} (S_t + \bar{S})^{\theta}\right]^{1-\sigma}}{1-\sigma}, \ \eta + \gamma + \theta = 1, \ \eta, \gamma, \theta, \sigma > 0.$$

i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.

ii. Derive the household's intertemporal Euler equation associated with capital.

iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \ \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \quad (2)$$

iv. Substitute out for $A_t - \bar{A}$ and $S_t + \bar{S}$ in terms of C_t in the household's intertemporal Euler equation, to get an expression in terms of the growth rate of C_t and the rate of return on capital alone.

v. Show that in general, there is no reason to expect the economy to converge to a 'balanced growth path', i.e., one in which A_t, C, S_t, K_t, Y_t all grow at a constant rate. (Here, $Y_t = K_t^\alpha {z_t}^{1-\alpha}$.) Hint: note that if you scale A_t, M_t, K_t, S_t by z_t and work with the scaled versions of (2) and (1) and the household's intertemporal Euler equation, you cannot get rid of z_t.

vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \tag{3}$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace A_t and S_t in (1) by $A_t - \bar{A}$ and $S_t - \bar{S}$, respectively without changing (1). Then, define $a_t = A_t - \bar{A}$ and $s_t = S_t - \bar{S}$ everywhere and impose (2), that a_t and s_t are each proportional to M_t.

vii. Explain why it is that under (3), the following are true:

$$\frac{K_t}{z_t} \to k^*, \ \frac{A_t - \bar{A}}{z_t} \to a^*, \ \frac{S_t + \bar{S}}{z_t} \to s^*$$
$$\frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} \to exp(g),$$

where k^*, a^*, s^* are finite, positive, constants.

viii. Provide a simple formula for k^*.

ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?

x. What does the model imply for $P_{k'}$, the consumption price of capital, along the growth path.