Homework #6
Economics 411-1
Due Wednesday, November 6 (you may delay turning this until Friday).
Christiano

1. Consider the model economy associated with Romer's model of growth through specialization. That is, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The technology for producing final goods is:

$$y_t = \int_0^{M_t} x_t(i)^\alpha di, \; M_t > 0, \; 0 < \alpha < 1,$$

where M_t is a scalar such that for $i > M_t$, $x_t(i) = 0$. To produce $x_t(i)$ units of the i^{th} intermediate good requires

$$\frac{1}{2}(1 + x_t(i)^2)$$

units of capital if $x_t(i) > 0$ and zero units of capital if $x_t(i) = 0$. The following constraint must be satisfied:

$$\int_0^{M_t} \frac{1}{2}(1 + x_t(i)^2) di = k_t,$$

where k_t is the beginning-of-period t aggregate stock of capital. The initial capital stock, $k_0 > 0$, is given. The resource constraint is:

$$c_t + I_t \leq y_t,$$

and the aggregate capital accumulation technology is given by:

$$k_{t+1} = (1 - \delta)k_t + I_t.$$

The efficient allocations for this economy solve the planning problem, maximize utility with respect to $\{M_t, k_{t+1}, y_t, c_t, x_t(i), i \in (0, M_t)\}_{t=0}^{\infty}$, subject to the various constraints. You may assume that efficiency is consistent with $x_t(i) = \bar{x}_t$ for $i \in (0, M_t)$.

(a) Show that the planning problem for the Romer economy coincides with the planning problem for the Ak model. In particular, show that the problem can be written,

$$max_{\{k_{t+1}\in B(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

where

$$F(k,k') = max_{c_t,\bar{x}_t,M_t} \frac{c_t^{1-\gamma}}{1-\gamma} = \frac{\left[(A+1-\delta)\,k - k'\right]^{1-\gamma}}{1-\gamma},$$

and $A = (2-\alpha)\left(\frac{\alpha}{2-\alpha}\right)^{\frac{\alpha}{2}}$. In addition to verifying the form of F, show what B is.

(b) Identify a set of parameter values under which positive growth is efficient, although the growth rate in the market decentralization analyzed in class is zero.

(c) The problem with monopoly power is that it results in an inefficiently low level of activity. In the Romer model we have just seen that this manifests itself in the form of inefficiently low growth. The pace at which new varieties of specialized inputs (e.g., specialized manufactured goods, specialized labor) are introduced is too slow in the market economy. Some sort of intervention in the market economy is desirable. One possibility is to subsidize the activities of monopolists. Accordingly, let $p(i)x(i)$ be the revenues of the i^{th} monopolist in the absence of taxes or subsidies. A subsidy rate, τ_t, raises the revenues of the i^{th} monopolists to $p(i)x(i)(1+\tau_t)$. The total cost, G_t, to the government of this subsidy scheme is

$$G_t = \int_0^{M_t} p(i)x(i)\tau_t di.$$

Suppose G_t is financed by a lump sum tax applied to households. That is, the household budget constraint is modified as follows:

$$c_t + k_{t+1} - (1-\delta)k_t = r_t k_t + w_t n_t - T_t,$$

where T_t represents taxes paid by the representative household to the government. Suppose the government balances its budget period by period:

$$T_t = G_t.$$

Find the subsidy rate, τ_t, that causes the allocations in the market economy to coincide with the efficient allocations.

These results have to be interpreted with caution. You have identified an ideal form of government intervention, which makes the private market economy efficient. However, the intervention we investigated abstracts from any social inefficiencies induced by having to raise the revenues needed to finance the subsidy to monopolists. We abstracted from this by assuming that the tax on households is administered in lump-sum form. In practice, such taxes are not available. So, the problem of 'fixing' the inefficiency in the Romer model is actually more complicated than this question makes it out to be.

2. Consider an economy with capital of different vintages. At time t, the amount of capital of vintage τ, $k_{t,\tau}$, $\tau = 1, 2, 3, ...$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1-\delta)^{\tau-1} i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t-\tau$ consumption units, applied in period $t-\tau$. Capital which has vintage τ in period t has vintage $\tau+1$ in period $t+1$. Investment expenditures at time t, i_t, must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t+1$ installed capital goods. Consider a given amount of investment, i. Note that this investment applied in period $t+1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1} i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^{\alpha} n_{t,\tau}^{1-\alpha}, \; 0 < \alpha < 1, \; \tau = 1, 2, 3, \dots .$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t, since capital is measured in common efficiency units. Similarly, the wage rate is w_t.

(a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^{\alpha} n_t^{1-\alpha}, \; (1-\alpha)\left(\frac{k_t}{n_t}\right)^{\alpha} = w_t, \; \alpha\left(\frac{k_t}{n_t}\right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \; k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \; n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(Hint: refer to your class notes on the indeterminacy of firm size under constant returns to scale.)

(b) Show that 'aggregate capital', k_t, evolves according to:

$$k_{t+1} = (1-\delta)k_t + \gamma^t i_t.$$