

Homework #7

Economics 411-1, Due Thursday, November 14

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1. Consider the model of Matsuyama, in the handout available on the website. Matsuyama denotes a situation in which $k < \theta F / (1 - \alpha)$ as a ‘Solow regime’ and a situation in which $k > \theta F / (1 - \alpha)$ as a ‘Romer regime’. Let

$$G = \beta \left[\alpha \left(\frac{\theta F}{1 - \alpha} \right)^{\alpha - 1} + 1 - \delta \right].$$

- (a) Suppose $G < 1$. Show that there is a steady state value of k in the Solow regime, call it k^* . That is, for any $M_{-1} > 0$ if the initial stock of capital is $K_0 = k^* M_{-1}$, then there is a no growth equilibrium with

$$K_{t+1} = K_0, \text{ for } t = 0, 1, 2, \dots$$

Note that in this steady state equilibrium, there is never any innovation. This regime is more likely the larger is F , which makes sense because this represents the fixed cost of innovation.

Let $M_{-1} = 1$, $\beta = 1/1.03$, $\alpha = 0.36$, $F = 100$, so that $G = 0.8833$, after rounding. Compute k^* . If K_0 is in a sufficiently small neighborhood of $k^* M_{-1}$, show that there exists an equilibrium in which $\lim_{t \rightarrow \infty} K_t = k^* M_{-1}$. (Hint: consider the household’s intertemporal Euler equation after substituting out for the rental rate of capital using equation (12) in the handout. Use the following facts: (i) for K_t sufficiently close to $k^* M_{-1}$, the Taylor series expansion of this equation about $K_t = K_{t+1} = K_{t+2} = k^* M_{-1}$ is an arbitrarily good approximation to this equation, and write this as

$$V_0 \tilde{K}_t + V_1 \tilde{K}_{t+1} + V_2 \tilde{K}_{t+2} = 0, \quad t = 0, 1, 2, \dots$$

where $\tilde{K}_t \equiv K_t - k^* M_{-1}$; (ii) the set of solutions to a linear difference equation like this is given by $\tilde{K}_t = (\tilde{K}_0 - a) \lambda_1^t + a \lambda_2^t$, for $t = 0, 1, 2, \dots$, for arbitrary a , where the λ_i ’s solve:

$$V_0 + V_1 \lambda_i + V_2 \lambda_i^2 = 0, \quad i = 1, 2;$$

- (b) Suppose $G > 1$. Show that there is a steady state value of k in the Romer regime, call it k^{**} . That is, given $M_{-1} > 0$ and $K_0 > 0$, there is an equilibrium in which

$$\frac{K_t}{M_{t-1}} = k^{**}, \quad \frac{c_{t+1}}{c_t} = \frac{K_{t+1}}{K_t} = \frac{M_t}{M_{t-1}} = G, \quad t = 0, 1, 2, \dots$$

Provide a formula for computing k^{**} and verify $k^{**} > \theta F/(1 - \alpha)$.

- (c) Think about the possibility of equilibria that fluctuate between the Romer and Solow regimes: in a Solow regime the relatively low amount of physical capital results in a high rental rate on capital. This discourages innovation but encourages capital accumulation (just like in the neoclassical growth model when you are below steady state). When capital becomes relatively abundant (so that $k > \theta F/(1 - \alpha)$) then innovators have an incentive to enter: the Romer regime begins and M starts to grow. If M grows fast enough relative to K (this will depend upon parameter values) then k is driven back down towards the Solow regime, and the process starts all over again. Along such a growth path there will be alternating periods of fast growth during which there is no innovation and slow growth, during which there is a lot of innovation. Interestingly, the same conditions that encourage high growth in capital and output, i.e., a high rental rate of capital, discourage innovation. This model generates all sorts of empirical hypotheses that would be interesting to test (patent applications come in bursts, and at times of low growth?).
2. (Dynamic Inefficiency in OG Models). Consider the overlapping generations model in which the utility of the generation born at t is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate, w_t . They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \quad c_{t+1}^t \leq r_{t+1}k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are r_t and w_t , respectively. Their production technology is $y = k^\alpha n^{1-\alpha}$, $0 < \alpha < 1$.

- (a) Define a sequence of markets equilibrium. Provide expressions for w_t and r_t in terms of k_t .
- (b) Consider a steady state equilibrium in which the aggregate stock of capital is constant, k , the consumption of each period's young is constant, c^y , and the consumption of each period's old is a constant, c^o . Time starts up in period 0, with the initial old generation owning the (steady state) capital stock, which they sell to the period 0 young. Show that the equilibrium rate of return on capital is

$$r_{k,t} = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}, \text{ for all } t.$$

Interpret this expression. Why is the interest rate infinite if $\beta = 0$? Why is it zero if $\alpha = 0$?

- (c) Show that, for parameter values where $r_{k,t} < 1$, the competitive equilibrium is inefficient. That is, prove the following: it is possible to deviate from the equilibrium consumption allocations by reallocating consumption between each period's old and the same period's young in a way that is compatible with the resource constraint and which makes everyone (i.e., the first generation, the second generation, the third, etc.) better off. How might the result be affected if there were a last date in the economy?
- (d) Identify parameter values for which the growth rate of household consumption deviates from the (zero) growth rate of aggregate consumption that occurs in the steady state equilibrium.