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411-1, Fall 2002

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. (35) Suppose that the total amount of labor and capital supplied by households is given and fixed at n and K, respectively. Taking these as given, we can compute an equilibrium for the firm sector of the economy.

   A final, homogeneous output good is produced using the following production function:
$$y = \int_J^1 y_j dj,\ 0 \quad J < 1.$$
   Note that the intermediate input goods are perfect substitutes. Intermediate good j $\quad (0,1)$ is produced by entrepreneur j. We will find that only entrepreneurs, j $\quad (J,1)$ will produce a non-zero quantity of their intermediate good, where J is a variable that is determined in equilibrium. The final good producers, who are competitive, take the number of intermediate goods that are in production as a given, and they treat J as a variable that is beyond their control. Let p_j be the price of the j^{th} intermediate good.

   a. State the problem of the representative final good producer and write down its first order condition. Show that the demand curve for the j^{th} intermediate good is perfectly elastic, and explain what this means.

   Entrepreneur j has ability level $A(j)$, where $A(j)$ is a strictly increasing function of j. That is, the most able entrepreneurs are the ones with values of j near unity, and the least able ones are the ones with values of j near zero. Entrepreneurs maximize profits and any entrepreneur may enter production if it wishes to do so. An entrepreneur that wishes not to produce makes zero profits. The technology available to an entrepreneur is as follows. If the j^{th} entrepreneur hires k_j units of capital and n_j units of labor, then it can produce the following output:

$$y(j) = A(j)^{1 \quad \gamma_k \quad \gamma_n} k_j^{\gamma_k} n_j^{\gamma_n}.$$

The rental rate on capital is r, and the wage rate is w. The entrepeneur not only has to pay for its labor inputs, but in order to produce at all, the entrepreneur must also pay a fixed cost. In particular, it must hire a fixed amount of labor (in addition to the n_j appearing in the above equation), ψ. The cost of this labor is ψw, while the cost of its other labor inputs is wn_j. The cost of the capital inputs is rk_j. The entrepreneur is competitive in the labor and capital markets. It is the only seller of the j^{th} intermediate input, whose price is p_j.

b. Write down the profits of the j^{th} intermediate good producer, and derive its first order conditions with respect to labor and capital. Use these, and the fact:

$$\frac{x_i}{z_i} = \frac{x_j}{z_j}, \text{ all } i, j \qquad \frac{x_i}{z_i} = \frac{\int x_j dj}{\int z_j dj},$$

to show that

$$\frac{n_i}{k_i} = \frac{n}{k}.$$

where

$$\begin{aligned} n &= \int_J^1 n_l dl \\ K &= \int_J^1 k_l dl \end{aligned}$$

c. Show that the intermediate good firm's production can be written like this:

$$y_j = \left(\frac{A(j)}{k_j}\right)^{1 \quad \gamma_k \quad \gamma_n} \left(\frac{n_j}{k_j}\right)^{\gamma_n} k_j.$$

d. Show that $A(j)/k_j$ is the same across all active entrepreneurs

$$\frac{A(j)}{k_j} = \frac{\bar{A}(J)}{k}, \; j \quad (J, 1),$$

where

$$\bar{A}(J) = \int_J^1 A(j)dj.$$

(Hint: exploit the result you obtained in (c), as well as the capital Euler condition.)

d. Show that the profits of the j^{th} firm can be written

$$[1 - \gamma_k - \gamma_n]\left(\frac{A(j)}{k_j}\right)^{1-\gamma_k-\gamma_n}\left(\frac{n_j}{k_j}\right)^{\gamma_n} k_j - w\psi,$$

and explain how this relation can be used to show that profits are increasing in j. Explain the fact that there is exactly one value of q such that profits are zero for $j = q$, positive for $j > q$ and zero for $j < q$. Explain why $J = q$.

e. Show that aggregate output can be written as follows:

$$y = \bar{A}(J)^{1-\gamma_k-\gamma_n} k^{\gamma_k} n^{\gamma_n}.$$

(Note that $\bar{A}(J)$ 'looks like' disembodied technical change in the exogenous growth model. In fact it is an endogenous variable.)

2. (30) The typical household can engage in two types of activities: producing current output and studying at home. Although time spent on studying at home sacrifices current production, it augments future output by increasing the household's future stock of human capital, k_{t+1}. The household has one unit of time available to split between home study and current production. Any given amount of human capital accumulation, k_{t+1}/k_t, leaves an amount of time, h_t, left over for producing current output, where $h_t = \phi(k_{t+1}/k_t)$. Here, ϕ is strictly decreasing, strictly concave, and continuously differentiable, with

$$\phi(1-\delta) = 1 \text{ for some } \delta \in (0,1),$$
$$\phi(1+\lambda) = 0 \text{ for some } \lambda > 0.$$

The variable, h_t, must satisfy $0 \le h_t \le 1$. This implies that the household cannot set k_{t+1}/k_t greater than $1+\lambda$ or less than $1-\delta$.

A household's effective labor input into production is the product of its time and human capital: $h_t k_t$. Total output is related to effective labor input by

$$f(h_t k_t) = (h_t k_t)^\alpha, \ \alpha \quad (0, 1).$$

The resource constraint for this economy is

$$c_t \quad f(h_t k_t),$$

and the initial level of human capital, k_0, is given. The utility value of a given sequence of consumption, c_t, is given by

$$\sum_{t=0} \beta^t u(c_t), \text{ where } u(c_t) = c_t^\sigma / \sigma, \ \sigma < 0.$$

(a) Express the planning problem for this economy as a sequence problem (SP). Write out the associated functional equation (FE).

(b) Show that $v(k) = Ak^{\sigma\alpha}$ is a solution to FE, and that the maximum is attained by the policy function, $g(k) = \theta k$ for some $(1 \quad \delta) < \theta < (1 + \lambda)$. (You may assume, without proof, that the maximum of the FE problem occurs in the interior of the relevant constraint set. Explain why this assumption is useful.)

(c) Suppose there are two separate economies, which differ only in how patient households are. In the more patient economy the discount rate is $\tilde{\beta}$ and in the less patient economy, the discount rate is $\beta < \tilde{\beta}$. Show that in the economy with more patient households, the growth rate of human capital is greater.

3. (15) An Alternative Transversality Condition. State and prove a version of the Theorem 4.15 (Sufficiency of Euler Equations and Transversality Condition) where the standard TC, is replaced by

$$\lim_{t \ >} \beta^t F_x(x_t, x_{t+1}) = 0$$

and X is a **bounded** subset of R^n (not R^n_+).

4. (20) Consider a household which solves the following problem:

$$v(k, r, w) = \max_{c,l \in B(k,r,w)} u(c, l),$$

where $u : \mathbb{R}^2_+ \to \mathbb{R}$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, c, and leisure, l. The constraints the household must obey in selecting c, l are summarized by B :

$$B(k, r, w) = \{c, l : 0 \leq c \leq rk + w(1 - l),\ 0 \leq l \leq 1\}.$$

Here, $r > 0$ is the market rental rate on capital and $w > 0$ is the market wage rate, neither of which the household can control. Also, $k > 0$ is the household's stock of capital. Prove that the derivative of v with respect to k exists, and display a formula for it. If you make use of a theorem to help prove your result, be sure to state it clearly.