

1. Introduction

Economists have for a long time been interested in the following statistic:

$$S_t = \frac{\text{GNP}_t}{K_t^{\alpha_k} n_t^{\alpha_n}},$$

where GNP_t is gross national product in period t , K_t is the total economy-wide stock of physical capital in period t , and n_t is economy-wide employment in period t . There are two interrelated reasons for this interest. One is empirical. The other, theoretical.

The empirical motivation for this interest is that S_t is observed to move in a systematic way with output. For example, in the business cycle frequencies, S_t moves up in a boom and down in a recession. In addition, during major economic episodes that extend over a period of one or two decades, one also finds S_t moving in a systematic way. For example, during the US Great Depression, output fell by about 1/3 between 1929 and 1933, and did not recover fully until about 10 years later. The variable, S_t , also fell over this period, by about one-half the fall in output. Similarly, the growth rate of S_t weakened in the 1990s in Japan, when it experienced its economic growth slowdown. Finally, S_t grows systematically at the very low frequencies, the ones that growth theorists think about. Economies that grow fast over long periods of time typically have fast growth in S_t .

On the theoretical side, S_t is a central object of study in some standard growth and business cycle theories. In Solow's model of exogenous disembodied technical change an exogenous increase in S_t is the engine of growth for society. That is why this variable is called the *Solow residual*. The reason is that it is a *residual* is that it is the part of GNP_t that can't be explained after measured factors of production are taken into account. In the Solow theory, S_t is also called total factor productivity, TFP. In real business cycle theory, S_t also plays a central role. There, like in the Solow growth model, S_t is viewed as an exogenous process. Business cycles are thought of as being a consequence of exogenous, stochastic variation in S_t .¹

¹In practice, what is called the Solow residual is usually, $\log(S_t/S_{t-1})$. In these notes, I refer to S_t as the Solow residual, or TFP.

The notion that S_t is *exogenous* - as in the Solow model or RBC models - is now less appealing. Still, people remain interested in S_t . Now, when people discuss S_t , they usually think of it as an endogenous variable. In the context of growth theory, for example, we talked about human capital, and how S_t could be a function of the stock of human capital. In the business cycle context, we discussed other possibilities. For example, S_t could be a function of capital utilization, or labor hoarding.² Alternatively, S_t could itself be a function of K_t and n_t in a world with external increasing returns. Economic theories that explain movements in S_t as arising endogenously, are sometimes referred to as a ‘theory of TFP’.

In these notes, we present the theory of TFP developed by Levon Barseghyan in his dissertation.³ In that theory, S_t is a function of entrepreneurial talent, which moves in a systematic way with movements in market prices.

2. The Model

Final goods are produced by a representative, competitive firm using the following constant returns production function:

$$Y = \int_J^1 Y_j dj.$$

Here, Y_j is total output of the j^{th} intermediate good, $j \in [J, 1]$. The range of intermediate goods that is available, $[J, 1]$, is taken to be exogenous by the final goods producer. The price of the j^{th} good is p . Here we impose the property of equilibrium, that the price of all intermediate goods is the same. The reason for this is that, with each intermediate good being a perfect substitute for the other, the elasticity of demand for any one good is infinite.

Intermediate goods are produced by entrepreneurs. There is free entry into being an entrepreneur. However, anyone who wants to be an entrepreneur must pay a fixed cost, ψ , to do so. After paying the fixed cost, the entrepreneur randomly draws a productivity level, A_j , $j \in [0, 1]$. The entrepreneur with productivity level,

²For a further discussion of labor hoarding, see Burnside, Eichenbaum and Rebelo, JPE, and Burnside and Eichenbaum, AER.

³Barseghyan’s theory builds on the work of Atkeson and Kehoe. Another interesting theory of TFP is presented in Nir Jaimovich’s Northwestern Phd dissertation, which is still in process. Yet one more theory appeared in Jeff Campbell’s Northwestern Phd dissertation, which can be found in Campbell, 1997, *Entry, Exit, Embodied Technology, and Business Cycles*, National Bureau of Economic Research Working paper number 5955.

A_j , has the option to produce or not. If such an entrepreneur chooses to produce, he can make use of the following production function:

$$y_j = A_j^{1-\gamma_n-\gamma_k} n_j^{\gamma_n} k_j^{\gamma_k},$$

where $\gamma_n, \gamma_k \geq 0$ and $\gamma_n + \gamma_k < 1$. Also, n_j and k_j denote the amount of labor and capital hired by the entrepreneur. The entrepreneur takes the output price, p , as well as the rental price of capital, r^k , and the wage rate, w , as exogenous. In addition, to produce and sell any positive amount of y_j , the entrepreneur must rent a fixed amount of capital, ϕ . So, the entrepreneur's problem is:

$$\max_{k_j, n_j} A_j^{1-\gamma_n-\gamma_k} n_j^{\gamma_n} k_j^{\gamma_k} - r^k k_j - w n_j - r^k \phi.$$

The j^{th} entrepreneur chooses not to produce if the optimized value of the above expression is negative.

Optimization by an entrepreneur who produces a positive amount leads to the following first order conditions:

$$\begin{aligned} r^k &= \gamma_k \left(\frac{A_j}{k_j} \right)^{1-\gamma_k} \left(\frac{A_j}{n_j} \right)^{-\gamma_n} \\ w &= \gamma_n \left(\frac{A_j}{n_j} \right)^{1-\gamma_n} \left(\frac{A_j}{k_j} \right)^{-\gamma_k} \end{aligned}$$

These conditions imply that the ratios,

$$\frac{A_j}{k_j}, \frac{A_j}{n_j},$$

are the same for all j .

We suppose that m people enter into being an entrepreneur. As noted above, these m are randomly distributed among $j \in (0, 1)$. Only those with $j \in (J, 1)$ actually produce, however. If n and k denote the aggregate supply of labor and capital, then,

$$\begin{aligned} m \int_J^1 n_j dj &= n \\ m \int_J^1 k_j dj &= k. \end{aligned}$$

Since all active entrepreneurs adopt the same A_j/n_j and A_j/k_j ratios, it follows:

$$\begin{aligned}\frac{A_j}{k_j} &= \frac{A}{k} \\ \frac{A_j}{n_j} &= \frac{A}{n}, \quad j \in (J, 1),\end{aligned}$$

where

$$A = m \int_J^1 A_j dj.$$

We can now develop an expression for Y :

$$\begin{aligned}Y &= \int_J^1 Y_j dj = m \int_J^1 y_j dj = m \int_J^1 A_j^{1-\gamma_n-\gamma_k} n_j^{\gamma_n} k_j^{\gamma_k} dj \\ &= m \int_J^1 A_j \left(\frac{A_j}{n_j}\right)^{-\gamma_n} \left(\frac{A_j}{k_j}\right)^{-\gamma_k} dj \\ &= \left(\frac{A}{n}\right)^{-\gamma_n} \left(\frac{A}{k}\right)^{-\gamma_k} m \int_J^1 A_j dj \\ &= A \left(\frac{A}{n}\right)^{-\gamma_n} \left(\frac{A}{k}\right)^{-\gamma_k},\end{aligned}$$

so that

$$Y = A^{1-\gamma_n-\gamma_k} n^{\gamma_n} k^{\gamma_k}.$$

Also,

$$\begin{aligned}r^k &= \gamma_k \left(\frac{A}{k}\right)^{1-\gamma_k} \left(\frac{A}{n}\right)^{-\gamma_n} = \gamma_k Y/k \\ w &= \gamma_n \left(\frac{A}{n}\right)^{1-\gamma_n} \left(\frac{A}{k}\right)^{-\gamma_k} = \gamma_n Y/n.\end{aligned}$$

The maximized level of profits are increasing in j , so that the cutoff, J , is defined by the condition:

$$\max A_J^{1-\gamma_n-\gamma_k} n_J^{\gamma_n} k_J^{\gamma_k} - r^k k_J - w n_J - r^k \phi = 0.$$

The best an entrepreneur with $j < J$ could do by producing a non-zero amount is generate negative profits. Such an entrepreneur would obviously prefer not to

produce at all. The entrepreneurs with $j > J$ would be producing positive profits. After substituting out the optimized values for the inputs, we obtain:

$$y_J (1 - \gamma_k - \gamma_n) - r^k \phi = 0,$$

or,

$$y_J = \frac{r^k \phi}{1 - \gamma_k - \gamma_n}.$$

As noted above, an entrepreneur must make the entry decision before his productivity is realized. We suppose that the entrepreneur makes the entry decision so that expected profits equal the fixed cost:

$$\psi = \int_J^1 [(1 - \gamma_k - \gamma_n) y_j - r^k \phi] dj.$$

Note that each profit level here is weighted by the corresponding density. Also, we impose that profits are zero for $j \in (0, J)$, and that the density of j is 1 for each $j \in (0, 1)$. Then,

$$\begin{aligned} \psi &= \int_J^1 [(1 - \gamma_k - \gamma_n) y_j - y_J (1 - \gamma_k - \gamma_n)] dj \\ &= y_J (1 - \gamma_k - \gamma_n) \int_J^1 \left[\frac{y_j}{y_J} - 1 \right] dj \\ &= r^k \phi \int_J^1 \left[\frac{A_j}{A_J} - 1 \right] dj \\ &= r^k \phi \left[\frac{A}{mA_J} - (1 - J) \right], \end{aligned}$$

so that

$$\psi = \gamma_k \frac{Y}{k} \phi \left[\frac{A}{mA_J} - (1 - J) \right].$$

A related observation is:

$$\frac{y_J}{Y} = \frac{A_J}{A},$$

so that

$$Y \frac{A_J}{A} = y_J = \frac{r^k \phi}{1 - \gamma_k - \gamma_n},$$

or,

$$\frac{A_J}{A} = \frac{\gamma_k \phi / k}{1 - \gamma_k - \gamma_n},$$

Aggregate labor is supplied by households, who are assumed to have a zero-income-effect-on-leisure utility function, so that:

$$\gamma n^n = w = \gamma_n \frac{Y}{n}.$$

The aggregate demand for capital is the sum of the capital demanded for production purposes by intermediate good firms, k , and for the fixed costs of entrepreneurs. This must equal total capital supplied, K :

$$K = k + m(1 - J)\phi.$$

We suppose that $K = 1$.

The resource constraint implies that total use of goods, $c + \psi m$, must equal supply:

$$c + \psi m = Y.$$

Total final output is c , or $Y - \psi m$. This must be equal to total income. To get the breakdown into the various income categories, note the following identity:

$$\begin{aligned} & (1 - \gamma_k - \gamma_n)Y - m\psi - r^k(1 - J)\phi m \\ & + \gamma_n Y \\ & + \gamma_k Y \\ = & Y - \psi m - r^k(1 - J)\phi m. \end{aligned}$$

Note that the first term is zero by the zero-profit condition on entrepreneurs. Taking this into account and rewriting:

$$\gamma_n Y + \gamma_k Y + r^k(1 - J)\phi m = Y - \psi m.$$

The first term on the left corresponds to total income paid to labor. The second and third terms correspond to total income paid to capital. The first of these is the income paid to the k units of capital employed in straight production and the second, $r^k(1 - J)\phi m$, is the amount of income paid to capital used in fixed costs. These second two terms are:

$$r^k k + r^k(1 - k) = r^k.$$

So, the share of income paid to labor, α_n , and the share of income paid to capital, α_k , are, respectively:

$$\alpha_n = \frac{\gamma_n Y}{Y - \psi m}, \quad \alpha_k = \frac{r^k}{Y - \psi m}, \quad \alpha_n + \alpha_k = 1.$$

The Solow residual for this economy is:

$$S = \frac{Y - \psi m}{n^{\alpha_n}}.$$

3. Computing the Equilibrium

We now collect the equations of the model. The aggregate production technology is:

$$Y = A^{1-\gamma_n-\gamma_k} n^{\gamma_n} k^{\gamma_k}. \quad (3.1)$$

The zero profit condition is:

$$\psi = \gamma_k \frac{Y}{k} \phi \left[\frac{A}{mA_J} - (1 - J) \right]. \quad (3.2)$$

The goods resource constraint is:

$$c + \psi m = Y. \quad (3.3)$$

Zero profits for the $j = J$ producer:

$$\frac{A_J}{A} = \frac{\gamma_k \phi / k}{1 - \gamma_k - \gamma_n}. \quad (3.4)$$

Labor supply:

$$\gamma_n n^\eta = \gamma_n \frac{Y}{n}. \quad (3.5)$$

Market clearing in capital services:

$$k = 1 - (1 - J)\phi m. \quad (3.6)$$

We adopt the following representation of A_j :

$$A_j = a + j^q,$$

so that:

$$A_J = a + J^q, \quad A = ma(1 - J) + \frac{m}{1 + q} (1 - J^{1+q}).$$

The six equations (3.1)-(3.6), determine the six unknowns, n , k , Y , m , J , c . One way to solve them works like this. Fix k . Solve (3.4), (3.6) for J , m (this can be

represented as a scalar, nonlinear, zero-finding problem in J). Solve (3.2) for Y . Solve (3.1) for n . Adjust k until (3.5) is satisfied. After this, it is straightforward to compute the Solow residual.

Because solving the model involves solving nonlinear equations, it can be difficult. We found that it is not easy to find parameter values for which there exists an equilibrium. An alternative way to go converts the variables, J and k , into exogenous variables, and converts two model parameters, ϕ and γ , into endogenous variables.

Here is the algorithm. Fix J, k , subject to:

$$\frac{1 - k}{1 - J} \frac{1}{\phi} > 0.$$

Compute m from (3.6). Compute γ_n from (3.4):

$$\begin{aligned} \phi &= (1 - \gamma_k) \frac{A_J k}{A \gamma_k} - \gamma_n \frac{A_J k}{A \gamma_k} \\ \gamma_n &= \frac{(1 - \gamma_k) \frac{A_J k}{A \gamma_k} - \phi}{\frac{A_J k}{A \gamma_k}} \end{aligned}$$

Compute Y from (3.2). Compute n from (3.1). Compute γ from (3.5). Equation (3.3) gives c , should this be of interest.

4. Numerical Example

Here is a numerical example. Let $\gamma_k = 0.25$, $\gamma_n = 0.2729$, $\gamma = 4.0601 \times 10^{-8}$, $q = 12$, $a = 0.001$, $\eta = 1$, $\psi = 0.1$, $\phi = 0.001$. For this economy,

$$\begin{aligned} J &= 0.6919, \quad k = 0.909, \quad m = 295.2823, \quad Y = 65.3117, \quad n = 20951, \\ c &= 35.7834, \quad w = 8.5062 \times 10^{-4}, \quad r^k = 17.9621. \end{aligned}$$

I also computed the equilibrium when the value of γ is reduced to 2.7067×10^{-8} . This corresponds to an increase in labor supply. In this case,

$$\begin{aligned} J &= 0.7000, \quad k = 0.900, \quad m = 333.3, \quad Y = 74.18, \quad n = 27346, \\ c &= 40.8, \quad w = 7.4019 \times 10^{-4}, \quad r^k = 20.6056. \end{aligned}$$

Thus, the increase in labor supply leads to a reduction in the wage, and to an increase in output. The increase in output in effect makes the acquisition of

an MBA cheaper, since the cost involves output goods. This leads to an increase in the supply of entrepreneurs, with m rising 12 percent, going from 295 to 333. Once the entrepreneurs have done the MBA, they find that rent for office space is now higher. Because capital complements labor in production, the increased quantity of labor used leads to an increase in the demand for capital services. Since these are in fixed supply, this leads to a 14 percent increase in the rental rate of capital, from $r^k = 18$ to $r^k = 21$. The rise in the cost of office space implies that only better entrepreneurs will actually end up producing. That is, J increases from 0.69 to 0.70.

One way to think about this is to look at the impact on A

$$A = m \int_J^1 A_j dj = m(1 - J) \times \frac{\int_J^1 A_j dj}{1 - J}.$$

The first term here, $m(1 - J)$, is the number of entrepreneurs who are actually producing. In the example, this rises by 9.9 percent, going from 90.9782 to 100.0000 with the rise in labor supply. The second term is the average quality of an entrepreneur. This increases by 2.6 percent, going from 0.2486 to 0.2549. These results in themselves are interesting, that the quantity and quality of entrepreneurs rises in an expansion.

Now, A is an input to TFP, but it is not TFP per se. For this, we need labor and capital's shares. In the example, α_n is 0.4980 initially, and goes to 0.4955. Also, α_k is 0.5020 initially and goes to 0.5045. In the example, Y increases by 14 percent. The rise in TFP is 2.54 percent. That is, for every one percent rise in output, 0.18 percent is 'due to' TFP.