

Economics 411-1  
Christiano  
Fall, 2002

## Notes on Bubbles in an OG Economy

The rise and subsequent collapse in the value of stock markets in the 1980s and 1990s has prompted renewed interest in macroeconomics in the possibility of ‘bubbles’. A bubble is the excess of the market price over its ‘fundamental value’. The latter is the present discounted value of the services generated by the asset. These notes describe a simple deterministic example using an overlapping generations model, which displays conditions under which a bubble can occur.

The idea is quite simple and can be summarized briefly. In the example, there is an asset with zero fundamental value. There is a unique fundamental equilibrium, but there may also be bubble equilibria. In a bubble equilibrium, arbitrage requires that the price of the asset rise at the same rate as the rate of return on physical capital. The requirement for the existence of a bubble is that the growth rate of income in the fundamental equilibrium exceed the rate of return on capital. If the growth rate of income were less than this, then the value of the asset would eventually outstrip income and households could then not afford to purchase it. If the growth rate of income is fast enough, then there is a continuum of bubble equilibria.

In what follows, we set the stage by describing the economy and studying the fundamental equilibrium. We then turn to the bubble equilibrium.

## 1 A Fundamental Equilibrium

Households born in period  $t$  solve the following problem:

$$\max_{c_t^t, c_{t+1}^t} \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \beta \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma},$$

subject to

$$\begin{aligned} c_t^t + k_{t+1} &\leq w_t, \\ c_t^{t+1} &\leq r_{t+1}k_{t+1} + (1-\delta)k_{t+1}, \end{aligned}$$

where  $w_t$  is the wage rate earned while in the first period of life, supplying one unit of labor inelastically. The households don't work in the second period of life. They finance their consumption in old age with rental income on capital and by selling the undepreciated part of their capital stock.

The first period of this economy is period  $t = 0$ , when there is an old generation alive who owns  $k_0$ , the initial stock of capital.

The first order condition associated with the saving decision while young is:

$$\frac{1}{\beta} \left( \frac{c_{t+1}^t}{c_t^t} \right)^\gamma = r_{t+1}^k, \quad t = 0, 1, 2, \dots$$

where  $r_{t+1}^k = r_{t+1} + 1 - \delta$ . Suppose, as in the Romer model, that

$$r_t = \alpha A, \quad w_t = (1 - \alpha) A k_t, \quad y_t = A k_t.$$

Then, household consumption growth is:

$$\frac{c_{t+1}^t}{c_t^t} = \lambda = [\beta (\alpha A + 1 - \delta)]^{\frac{1}{\gamma}}. \quad (1)$$

To determine consumption of the young, first determine the intertemporal budget constraint:

$$c_t^t + \frac{c_{t+1}^t}{\alpha A + 1 - \delta} \leq (1 - \alpha) A k_t, \quad t = 0, 1, 2, \dots \quad (2)$$

or, imposing the strict equality:

$$c_t^t = \frac{(1 - \alpha) A}{1 + \beta^{\frac{1}{\gamma}} (\alpha A + 1 - \delta)^{\frac{1}{\gamma} - 1}} k_t = \psi k_t,$$

say. So, a characterization of what the individual household does in equilibrium is based on  $\lambda$ ,  $\psi$  and  $k_t$ . From here on, we set  $\gamma = 1$ , so that

$$\psi = \frac{(1 - \alpha) A}{1 + \beta} \quad (3)$$

Thus, in its first period of life, the household consumes  $1/(1 + \beta)$  of its income (i.e.,  $w_t$ ) and saves the rest, so that:

$$k_{t+1} = \frac{\beta}{1 + \beta} w_t = \frac{\beta}{1 + \beta} (1 - \alpha) A k_t$$

Then, the growth rate of the aggregate stock of capital is:

$$\lambda^* \equiv \frac{k_{t+1}}{k_t} = \frac{\beta}{1 + \beta}(1 - \alpha)A = \beta\psi. \quad (4)$$

Note that the growth rate of the aggregate stock of capital need not be the same as the growth rate of household consumption.

The resource constraint for this economy is:

$$c_t^t + c_t^{t-1} + k_{t+1} = Ak_t + (1 - \delta)k_t, \quad t = 0, 1, 2, \dots \quad (5)$$

Let  $c_t = c_t^t + c_t^{t-1}$  denote aggregate consumption, so that

$$c_t = [A + (1 - \delta)] k_t - k_{t+1}.$$

From this, one can deduce the growth rate of aggregate consumption:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \frac{[A + (1 - \delta)] k_{t+1} - k_{t+2}}{[A + (1 - \delta)] k_t - k_{t+1}} \\ &= \frac{[A + (1 - \delta)] k_t - k_{t+1}}{[A + (1 - \delta)] k_t - k_{t+1}} \lambda^* \\ &= \lambda^*. \end{aligned}$$

The growth rate of aggregate consumption does correspond to the growth rate of the capital stock.

Suppose one period in this model corresponds to 30 years. Then,  $\beta = (1/1.03)^{30} = 0.40$ . Also,  $\delta = 1 - (1 - .1)^{30} = 0.96$ . And,  $\alpha = 0.36$ . Also, assuming consumption growth at the individual level of 3 percent per year,  $\lambda = 1.03^{30} = 2.43$ . In this case,  $\lambda^* = 3.0654$ ,  $r^k = 6.075$ .

## 2 Bubble Equilibria

Let's introduce a new asset, one that is a claim on an intrinsically useless object: one that does not generate utility and cannot be used to produce goods of any type. Individual households can purchase any amount,  $a_t \geq 0$ , of this asset. The asset is in permanently fixed supply,  $a > 0$ . It is owned by the initial old households, and let its price be denoted by  $P_t$ .

Suppose there is a market in which people can buy  $a_t$  units of the asset for  $P_t a_t$  when young and sell it when old for  $P_{t+1} a_t$ . Revising the household's budget constraint to reflect this new market:

$$\begin{aligned} c_t^t + k_{t+1} + P_t a_t &\leq w_t \\ c_{t+1}^t &\leq r_{t+1}^k k_{t+1} + P_{t+1} a_t. \end{aligned}$$

Implicitly, in the previous section we explored the 'fundamental' equilibrium in which  $P_t = 0$ , i.e., the price of the asset corresponds to its fundamental value. Below we show that if  $r^k > \lambda^*$ , then the fundamental equilibrium is the only equilibrium. However, when  $r^k < \lambda^*$  there exist ('bubble') equilibria with  $P_t > 0$ .

The intuition for this result is straightforward. Recall from the previous subsection that the growth rate of the aggregate capital stock is determined by the fraction of period  $t$  income,  $\beta/(1+\beta)$ , that households born in period  $t$  save. Since all their saving goes towards the accumulation of capital in the fundamental equilibrium we concluded that the growth rate of capital - and of period  $t$  income - is  $\lambda^* = \beta(1-\alpha)A/(1+\beta)$ . Now, in any equilibrium with  $P_t > 0$  arbitrage between  $a_t$  and  $k_t$  requires that  $P_{t+1}/P_t = r^k$ .<sup>1</sup> Relative to the fundamental equilibrium, in a bubble equilibrium a fraction (this is determined by the level of  $P_t$ ) of household income is diverted towards  $a$ . With the reduction in the fraction of income used to buy capital, the growth rate of the capital stock is reduced.

Suppose that the growth rate in the fundamental equilibrium,  $\lambda^*$ , is less than  $r^k$ . With the growth rate of capital in a candidate bubble equilibrium lower than  $\lambda^*$ , this means that eventually the price of  $a_t$  will outstrip the ability of young households to purchase it. A bubble cannot be part of an equilibrium in this case. Suppose now that  $\lambda^* > r^k$ . Then, as long as  $P_t$  is not too large, it is possible for there to exist a bubble equilibrium in which young households can always afford to purchase  $a$ . It is easy to see that there is a precise level of  $P_t$  where the growth rate of capital is  $r^k$  and the equilibrium of the economy is constant. We explore these possibilities carefully below.

---

<sup>1</sup>Suppose not. If  $P_{t+1}/P_t < r^k$ , then  $a_t = 0 < a$  and market clearing fails. If  $P_{t+1}/P_t > r^k$ , then demand by young households for  $k_t$  is zero in periods 0, 1, 2, ... . Market clearing in period 0 when period 0 old households supply  $k_0 > 0$  fails.

From the arbitrage relationship mentioned above, we conclude:

$$\frac{P_{t+1}}{P_t} = \alpha A + 1 - \delta.$$

Using this relationship, and substituting out for  $k_{t+1}$  in the household's budget constraint, we obtain the same intertemporal budget constraint we had before, (2). This is not surprising, since the inclusion of  $a_t$  in the household's budget constraint does not change its intertemporal consumption opportunity set. Then,  $c_t^t = \psi k_t$ , where  $\psi$  is defined in (3). The aggregate resource constraint is still (5). The substitutions done before are less convenient now. Divide both sides of (5) by  $k_t$ , impose  $c_t^t = \psi k_t$  for all  $t$ , and define  $\lambda_t^* = k_{t+1}/k_t$ , as before:

$$\begin{aligned} \frac{\lambda\psi}{\lambda_{t-1}^*} + \lambda_t^* &= A + 1 - \delta - \psi \\ &= A + 1 - \delta - \frac{(1 - \alpha)A}{1 + \beta} \\ &= A \frac{\beta + \alpha}{1 + \beta} + 1 - \delta \\ &= \phi, \end{aligned}$$

for  $t = 1, 2, 3, \dots$ , where

$$\phi = A \frac{\beta + \alpha}{1 + \beta} + 1 - \delta.$$

It is easy to verify that  $\lambda^*$  in (4) and  $r^k$  are the fixed points of the above

difference equation.<sup>2</sup> Write this as follows

$$\lambda_t^* = g(\lambda_{t-1}^*) = \phi - \frac{\lambda\psi}{\lambda_{t-1}^*}.$$

Since  $g' > 0$  and  $g'' < 0$ , this is a strictly increasing and strictly concave function. Also,  $g' \rightarrow -\infty$  as  $\lambda_{t-1}^* \rightarrow 0$  and  $g' \rightarrow 0$  as  $\lambda_{t-1}^* \rightarrow \infty$ . Thus, if  $g$  crosses the 45 degree line then it first crosses from below and then crosses from above. Obviously, when it crosses from below it has slope greater than unity, and when it crosses from above it has slope less than unity. That is,  $\lambda^* > r^k$  implies that the slope of  $g$  at  $\lambda_{t-1}^* = \lambda^*$  is less than unity while  $\lambda^* < r^k$  implies that the slope of  $g$  at  $\lambda_{t-1}^* = \lambda^*$  is greater than unity. These observations are confirmed by direct differentiation:

$$g'(\lambda^*) = \frac{\lambda\psi}{(\lambda^*)^2} = \frac{\lambda\psi}{(\beta\psi)^2} = \frac{\lambda}{\beta^2\psi} = \frac{\lambda}{\beta\lambda^*} = \frac{r^k}{\lambda^*}.$$

In the case of the parameter values discussed above,  $g'(\lambda^*) = 1.98$ .

---

<sup>2</sup>Substitute  $\lambda_t^* = \lambda_{t-1}^* = \lambda^* = \beta\psi$  into:

$$\frac{\lambda\psi}{\lambda_{t-1}^*} + \lambda_t^* + \psi$$

one obtains

$$\frac{\lambda\psi}{\beta\psi} + (1 + \beta)\psi = \alpha A + 1 - \delta + (1 - \alpha)A,$$

so that  $\lambda^*$  is indeed a fixed point of the difference equation. We can also try and find the other fixed point,  $\bar{\lambda}^*$ , by noting

$$(x - \lambda^*)(x - \bar{\lambda}^*) = x^2 - \phi x + \lambda\psi,$$

so that

$$\begin{aligned} \lambda^* + \bar{\lambda}^* &= \phi \\ \lambda^* \bar{\lambda}^* &= \lambda\psi. \end{aligned}$$

The latter result says that

$$\bar{\lambda}^* = \frac{\lambda\psi}{\lambda^*} = \frac{\lambda}{\beta} = r^k.$$

So, the fixed points are  $r^k$  and  $\lambda^*$ .

The difference equation determines  $\lambda_t^*$  for  $t = 1, 2, \dots$  given  $\lambda_0^*$ . But,  $\lambda_0^*$  is still to be determined. For this, use the period 0 resource constraint:

$$\psi k_0 + c_0^{-1} + k_1 = (A + 1 - \delta) k_0, \quad (6)$$

where  $\psi k_0$  is  $c_0^0$ , the first period consumption of the people born in period 0. The budget constraint in period 0 for the period 0 old is:

$$c_0^{-1} = [\alpha A + 1 - \delta] k_0 + P_0 a,$$

so the resource constraint becomes:

$$\psi k_0 + [\alpha A + 1 - \delta] k_0 + P_0 a + k_1 = (A + 1 - \delta) k_0.$$

Divide this by  $k_0$  :

$$\psi + [\alpha A + 1 - \delta] + \frac{P_0 a}{k_0} + \lambda_0^* = (A + 1 - \delta),$$

or

$$\begin{aligned} \lambda_0^* &= (1 - \alpha)A - \frac{(1 - \alpha)A}{1 + \beta} - \frac{P_0 a}{k_0} \\ &= (1 - \alpha)A \frac{\beta}{1 + \beta} - \frac{P_0 a}{k_0} \\ &= \lambda^* - \frac{P_0 a}{k_0} \end{aligned}$$

From this, it is evident that we cannot have a bubble if  $r^k > \lambda^*$ . If this were the case, then  $g'$  would be greater than unity at  $\lambda_{t-1}^* = \lambda^*$ , and so if  $P_0 > 0$ , so that  $\lambda_0^* < \lambda^*$ , then  $\lambda_t^* \rightarrow -\infty$  which cannot be an equilibrium. If, on the other hand,  $r^k < \lambda^*$ , then  $P_0$  positive but not too large (we don't want  $\lambda_0^* < r^k$ ), there may be bubble equilibria. These would produce a temporary period of low  $\lambda_t^*$ , which would permanently reduce the size of the capital stock.

Consider the parameter values,  $\beta = 0.80$ ,  $A = 31.05$ ,  $\delta = 0.96$ ,  $\alpha = 0.20$ ,  $\lambda = 5$ . In this case,  $r^k = 6.25$ ,  $\lambda^* = 11.04$ ,  $\lambda = 5$ . With  $P_0 = 0$ , then the unique equilibrium is one in which the aggregate economy grows at the gross rate 11.04, while individual consumption grows at the gross rate, 5. But, there are also equilibria in which  $P_0 > 0$  and  $P_{t+1}/P_t = r^k$ . We now describe two examples.

## 2.1 Example 1: A Constant Growth Equilibrium

Suppose, as in the previous example, that  $r^k < \lambda^*$ . There is one stationary equilibrium in which the growth rate of the economy is  $r^k$  while that is the growth rate of  $P_t$  too. We find this by setting  $P_0$  so that

$$\lambda_0^* = \lambda^* - \frac{P_0 a}{k_0} = r^k,$$

so that

$$P_0 = \frac{(\lambda^* - r^k) k_0}{a}.$$

To verify that  $k_{t+1} = r^k k_t$ ,  $t = 0, 1, 2, 3, \dots$  corresponds to an equilibrium when  $P_0$  takes on this value, we need to verify that the other equilibrium objects can be found such that these, together with the given  $k_t$ 's satisfy all the equilibrium conditions.

The budget constraint of the old in period 0 implies:

$$c_0^{-1} = [\alpha A + 1 - \delta] k_0 + P_0 a = [\alpha A + 1 - \delta + \lambda^* - r^k] k_0.$$

Their consumption is high in a  $P_0 > 0$  equilibrium, because a value is assigned to their intrinsically worthless asset,  $a$ . Expenditures of the period 0 young on consumption goods and capital is:

$$\begin{aligned} c_0^0 + k_1 &= (1 - \alpha) A k_0 - P_0 a \\ &= [(1 - \alpha) A - (\lambda^* - r^k)] k_0. \end{aligned}$$

Note, their consumption of goods and capital is less than what it is when  $P_0 = 0$ , reflecting the diversion to  $a$ . That the period 0 resource constraint is satisfied is easily verified. Similarly, the resource constraint for each  $t$  is satisfied as long as each person's budget constraint is satisfied, consistent with Walras' law.

Let's see if household optimization is satisfied, i.e.,  $c_t^t = \psi k_t$  and  $c_{t+1}^t / c_t^t = \lambda$  for  $t = 0$ . Imposing the given growth rate of capital in young households' budget constraint:

$$c_0^0 + r^k k_0 = [(1 - \alpha) A - (\lambda^* - r^k)] k_0,$$

or

$$\begin{aligned}
\frac{c_0^0}{k_0} &= (1 - \alpha)A - \lambda^* \\
&= (1 - \alpha)A - \beta \frac{(1 - \alpha)A}{1 + \beta} \\
&= \frac{(1 - \alpha)A}{1 + \beta} = \psi.
\end{aligned}$$

An implication of this result is that the entire amount of the fall in consumption by the young in period 0 comes from a diversion of purchases on  $k_1$  to  $a$ . Their consumption, being  $\psi k_0$ , is the same, whether  $P_0$  is positive or zero. The consumption in old age of the generation born in period 0 is:

$$\begin{aligned}
c_1^0 &= [\alpha A + 1 - \delta] k_1 + P_1 a = r^k \{[\alpha A + 1 - \delta] k_0 + P_0 a\} \\
&= r^k k_0 \{[\alpha A + 1 - \delta] + (\lambda^* - r^k)\} \\
&= r^k k_0 \lambda^* \\
&= k_0 \lambda \psi \\
&= \lambda c_0^0.
\end{aligned}$$

This establishes efficiency for the generation born in period 0. Since the quantity that they consume over their lifetime is invariant to  $P_0$ , their utility is unaffected by the presence of a bubble. It is easy to show that the efficiency conditions also hold for later generations, though their utility levels are affected by the presence of a bubble.

## 2.2 Example 2: A Nonstationary Equilibrium

It is easy to see that for any  $P_0 > 0$  but smaller than the one just described, the economy's low growth rate is just temporary. Eventually it will return to 11.04. There isn't a problem with the household's budget constraint here because  $P_t a$  is growing at the rate  $r^k$  whereas aggregate variables are growing at the gross rate,  $\lambda^* > r^k$ . Thus, there exists a continuum of nonstationary equilibria.