

Homework #7
Economics 4-11
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Due November 21, 2003.

Following is a deterministic economy composed of one representative, competitive household, and a representative, competitive firm. Preferences of the household are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \quad u(c, n) = \log c + \sigma \log(1 - n),$$

where c_t denotes consumption and n_t denotes hours worked. The household budget equation is

$$c_t + I_t \leq w_t n_t + r_t k_t,$$

where w_t is the wage rate and r_t is the rental rate on capital, k_t . Here, I_t denotes investment, which the household applies to increasing the stock of capital, using the following technology:

$$k_{t+1} = (1 - \delta)k_t + I_t, \quad 0 < \delta < 1.$$

The representative firm has access to the following technology:

$$y = Y^\gamma k^\alpha n^{1-\alpha}, \quad \gamma = 1 - \alpha, \quad \alpha = 1/3,$$

where Y is economy-wide average output, and y, k, n are firm output, capital, and employment, respectively. Note that the firm has constant returns to scale in the variables that it controls directly. Note too, the ‘externality’ in this production function. If all other firms are producing a lot (i.e., Y is big), this raises the productivity of an individual firm. Firms maximize profits given r_t and w_t . In equilibrium, $Y = y$.

1. Define a sequence-of-markets equilibrium for this economy.
2. Show that the Euler equations for labor and capital are, respectively:

$$w_t = -\frac{u_{n,t}}{u_{c,t}}, \quad u_{c,t} = \beta u_{c,t+1} [r_{t+1} + 1 - \delta],$$

where $u_{x,t}$ is the partial derivative of u with respect to $x = c_t, n_t$. Describe the transversality condition, and sketch how you would prove the sufficiency for household optimization of the transversality conditions and the Euler equations. (Hint: the proof can involve a lot of messy algebra. You need not do this. But, do indicate in a precise way what strategy you would take for establishing the proof.)

3. Show that in an equilibrium, the first order conditions for firms and the resource constraint are (hint: impose $y = Y$):

$$r_t = \alpha n_t^2, \quad w_t = \gamma k_t n_t, \quad c_t + k_{t+1} - (1 - \delta)k_t = k_t n_t^2.$$

4. Show that by combining the household and firm Euler equations, one obtains:

$$\frac{n_t^2 + 1 - \delta - \frac{\gamma}{\sigma} n_t (1 - n_t)}{n_t (1 - n_t)} = \beta \frac{\alpha n_{t+1}^2 + 1 - \delta}{n_{t+1} (1 - n_{t+1})}, \quad t = 0, 1, 2, \dots \quad (1)$$

Let (1) be represented as $v(n_t, n_{t+1}) = 0$. Note that this implicitly defines a map from n_t to n_{t+1} . Show that this map is composed of two functions, $n_{t+1} = f_i(n_t)$, $i = 1, 2$, where $f_1 > f_2$ for all n_t . Display analytic expressions for these functions. (Hint: remember the formula for the roots of a second order polynomial.)

5. The following proposition is true. ‘Suppose a sequence, n_0, n_1, \dots is found which satisfies (1) and also has the property, $a \leq n_t \leq b$ for $a > 0$ and $b < 1$ for all t . Then that sequence corresponds to an equilibrium.’ Sketch a proof of this proposition.
6. Explain why there are two stationary (i.e., equilibria with n_t constant) equilibria for this economy.
7. Let n^* be the greater of the two stationary equilibria just identified. Show that there is an interval, D , about n^* such that if $n_0 \in D$ and n_t , $t = 1, 2, \dots$, solves (1), then the given sequence, n_t , $t \geq 0$ represents an equilibrium. (Hint: note that $|\partial f_2(n)/\partial n| < 1$ for $n = n^*$.) Note that, since D is an interval, we have identified a continuum of equilibria. Explain why there are likely to be a great many more equilibria than just these two.

8. Show that the efficient allocations involve a constant value of n_t , \bar{n} , say. It turns out that $n^* < \bar{n}$. Give an economic interpretation of this.