Homework #2
Economics 411, fall 2005
Due Wednesday, October 5
Christiano

1. (Boldrin-Montrucchio 1986). Consider the policy rule, $g : [0,1] \rightarrow [0,1]:$

$$g(x) = 4x(1-x).$$

Draw this function, along with the 45 degree line, by hand in the unit box. Find an economy, (F, Γ, β, X), for which the above function is the policy rule, where the economy satisfies all of our assumptions (i.e., assumptions A4.3-A4.9 in Stokey and Lucas). Here, x is the aggregate stock of capital at the beginning of the period and x' is its value at the end of the period.

Some hints: Recall, $\Gamma, \beta \in (0,1)$ and F satisfy

$$g(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y),$$

where

$$v(x) = F(x, g(x)) + \beta v(g(x)).$$

Recall that the definition of a function or a correspondence must include a specification of the domain and range. Recall too that a function, say $f(x,y)$, is strictly concave iff:

$$f_{xx}(x,y) < 0,\ f_{yy}(x,y) < 0,\ f_{xx}(x,y) f_{yy}(x,y) - f_{xy}(x,y)^2 > 0,$$

for all (x,y) in the domain of f. Also, it is easy to verify that

$$g(x) = \arg \max_{y \in \Gamma(x)} \Psi(x,y),$$

when $\Psi(x,y)$ is defined as follows:

$$\Psi(x,y) = -\frac{1}{2}y^2 + yg(x) - \frac{1}{2}Lx^2 + ax,$$

and L, a are known constants. Finally, note that $v(x) = \Psi(x, g(x))$, and use this to back out F. You can think of your task as having to identify values of a and L that ensure assumptions A4.3-A4.9 are satisfied. How many such values are there?

2. (This is a continuation of question 1.) This question identifies an economy underlying the F and Γ functions you identified in the previous question. Consider the following two-sector economy: consumption goods are produced using the following production technology:

$$c \leq f(k_c, n),$$

where n and k_c are the labor and capital used in the consumption good sector. New capital, k', is produced using the following Leontieff technology:

$$k' \leq \min\{k_k/d, 1-n\},$$

where d is a positive real constant, k_k is capital used in the capital-producing sector, and $1-n$ is labor allocated to the capital-producing sector. Utility is linear and is a function of consumption alone:

$$u(c) = c.$$

The aggregate constraint on capital is:

$$k_c + k_k \leq k.$$

Let

$$f(k_c, n) = F(k_c + d(1-n), 1-n),$$

where F was obtained in the previous question.

(a) Explain why the g function of the previous question is the policy rule for this economy.

(b) Derive an expression for $P_{k'}$, the marginal cost of an extra unit of capital, measured in units of consumption foregone. Find the economic intuition for the fact that for low values of k, g is increasing in k, whereas it is decreasing in k for larger values of k (hint: recall our discussion of this point in class).

3. Plot the sequence of efficient x's for the above economy, for $x_0 = 1$, 0.5, 0.75,0.75001, 0.60. Do you observe monotone convergence to a unique steady state?

4. According to Theorem 4.15, the Euler and transversality conditions (equations (2) and (3) on page 98 of S-L) are sufficient for an interior sequence of x_t's to constitute an optimum of the SP problem. Necessity of the Euler equation is fairly obvious (a standard variational argument establishes this). Here is a sketch of an argument that establishes necessity of the transversality condition. This argument requires, in addition to the usual assumptions, $x_t = 0 \in X$, and $0 \in \Gamma(x_t)$ for $x_t \in X$. I sketch this argument below. Convert this sketch into a rigorous proof. In your proof you may use, without proof, any results from the book that you wish. However, you must be absolutely clear always about which assumptions you are using.

   Let x_t^*, $t = 0, 1, 2, ...$, denote a sequence of x_t's that solve the SP. Then,

$$v(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v(x_{t+1}^*), \ t = 0, 1, 2, ...,$$

where, of course, $x_0^* = x_0$, the initial condition. Let x_t be any other sequence that satisfies $(x_t, x_{t+1}) \in A$ (recall, A is the domain of F) for all t (does this imply that x_t is in the domain of v?). Then,

$$v(x_t) - v(x_t^*) \leq F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*).$$

The proof follows in a straightforward (not *trivial*) way by replacing x_t with 0, for all t, by multiplying both sides of this expression by β^t, and driving $t \to \infty$.

5. Consider an economy with a large number of identical households, each having preferences, $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Suppose the resource constraint is $c_t + i_t \leq f(k_t)$, where $k_{t+1} = i_t + (1-\delta)k_t$, f is strictly increasing and concave, $f'(k) \to \infty$ as $k \to 0$, $f'(k) \to 0$ as $k \to \infty$, $0 < \delta < 1$. Assume investment, i_t, is irreversible, i.e., it must be that $i_t \geq 0$. In addition, suppose $c_t, k_t \geq 0$ and that $k_0 > 0$ is given. Consider the functional equation associated with this problem:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} u\left(f(k) + (1-\delta)k - k'\right) + \beta v(k') \\ \Gamma(k) &= \{k' : (1-\delta)k \leq k' \leq f(k)\}. \end{aligned}$$

   (a) State a set of assumptions on β and u that guarantee there is a unique, differentiable, concave v that solves the above functional equation. For each property of v, explain which assumptions are used to get it.

   (b) Show that monotonicity of $\Gamma(k)$, Assumption 4.6 in S-L, fails so that one of the conditions of Theorem 4.7 which guarantee strictly increasing v, is not satisfied.

   (c) Show that the feasible set for this economy satisfies the following 'quasi-monotonicity property': if $\tilde{k} \geq k$, then $\Gamma(k) + (1-\delta)(\tilde{k} - k) \subseteq \Gamma(\tilde{k})$. Here, the sum of a set, say X, and a number, say a, is a new set, $X + a$, where $X + a \equiv \{x + a : x \in X\}$.

   (d) Show: v is an increasing function in k. (Hint: (i) following the basic strategy of the proof of Theorem 4.7, it's enough to establish that the assumptions of Theorem 4.7 with the monotonicity assumption on Γ replaced by quasi-monotonicity guarantee Tw is increasing if w is; (ii) make use of the fact that if $k' \in \Gamma(k)$, then $\tilde{k}' = k' + (1-\delta)(\tilde{k} - k) \in \Gamma(\tilde{k})$, $\tilde{k}' > k'$, and $f(\tilde{k}) + (1-\delta)\tilde{k} - \tilde{k}' > f(k) + (1-\delta)k - k'$.) Can you provide intuition for the fact that v is increasing even though Γ fails to satisfy monotonicity?