

Homework #8
Economics 411
Due Thursday, November 17
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1. (Dynamic Inefficiency in OG Models). Consider the overlapping generations model in which the utility of the generation born at t is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate, w_t . They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \quad c_{t+1}^t \leq r_{t+1}k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are r_t and w_t , respectively. Their production technology is $y = k^\alpha n^{1-\alpha}$, $0 < \alpha < 1$.

- (a) Define a sequence of markets equilibrium. Provide expressions for w_t and r_t in terms of k_t .
- (b) Consider a steady state equilibrium in which the aggregate stock of capital is constant, k , the consumption of each period's young is constant, c^y , and the consumption of each period's old is a constant, c^o . Time starts up in period 0, with the initial old generation owning the (steady state) capital stock, which they sell to the period 0 young. Show that the equilibrium rate of return on capital is

$$r_{k,t} = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}, \text{ for all } t.$$

Interpret this expression. Why is the interest rate infinite if $\beta = 0$? Why is it zero if $\alpha = 0$?

- (c) Show that, for parameter values where $r_{k,t} < 1$, the competitive equilibrium is inefficient. That is, prove the following: it is possible to deviate from the equilibrium consumption allocations by reallocating consumption between each period's old and the same period's young in a way that is compatible with the resource constraint and which makes everyone (i.e., the first generation, the second generation, the third, etc.) better off. How might the result be affected if there were a last date in the economy?
- (d) Identify parameter values for which the growth rate of household consumption deviates from the (zero) growth rate of aggregate consumption that occurs in the steady state equilibrium.
2. Consider the model economy associated with Romer's model of growth through specialization. That is, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The technology for producing final goods is:

$$y_t = \int_0^{M_t} x_t(i)^\alpha di, \quad M_t > 0, \quad 0 < \alpha < 1. \quad (1)$$

To produce $x_t(i)$ units of the i^{th} intermediate good requires

$$\frac{1}{2}(1 + x_t(i)^2) \quad (2)$$

units of capital if $x_t(i) > 0$ and zero units of capital if $x_t(i) = 0$. The following constraint must be satisfied:

$$\int_0^{M_t} \frac{1}{2}(1 + x_t(i)^2) di = k_t, \quad (3)$$

where k_t is the beginning-of-period t aggregate stock of capital. The initial capital stock, $k_0 > 0$, is given. The resource constraint is:

$$c_t + I_t \leq y_t, \quad (4)$$

and the aggregate capital accumulation technology is given by:

$$k_{t+1} = (1 - \delta)k_t + I_t.$$

The efficient allocations for this economy solve the planning problem, maximize utility with respect to $\{M_t, k_{t+1}, y_t, c_t, x_t(i), i \in (0, M_t)\}_{t=0}^{\infty}$, subject to the various constraints.

- (a) Explain why economic efficiency dictates $x_t(i) = \bar{x}_t$ for $i \in (0, M_t)$. From here on, you may simply assume $x_t(i) = \bar{x}_t$ for all $i \in (0, M_t)$.
- (b) Show that the planning problem for the Romer economy coincides with the planning problem for the Ak model. In particular, show that the problem can be written,

$$\max_{\{k_{t+1} \in \Gamma(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

where

$$F(k, k') = \frac{[(A + 1 - \delta)k - k']^{1-\gamma}}{1 - \gamma} = \max_{\bar{x}_t, M_t} \frac{c_t^{1-\gamma}}{1 - \gamma}.$$

The last maximization is subject to (1)-(4), and the given values of k_t, k_{t+1} . Display an expression for the value of A in terms of model parameters. In addition to verifying the form of F , show what the constraint set, Γ , is.

- (c) Identify a set of parameter values under which positive growth is efficient, although the growth rate in the market decentralization analyzed in class is zero.
- (d) The problem with monopoly power is that it results in an inefficiently low level of activity (in the Romer model, the root of this inefficiency is (??), according to which a firm that is a monopolist in the product market and competitive in factor markets will pay a rental rate on capital that is less than its marginal product). In the Romer model we have just seen that this manifests itself in the form of inefficiently low growth. The pace at which new varieties of specialized inputs (e.g., specialized manufactured goods, specialized labor) are introduced is too slow in the market economy. Some sort of intervention in the market economy is desirable. One possibility is to subsidize the activities of monopolists. Accordingly, let $p(i)x(i)$ be the revenues of the i^{th} monopolist in the

absence of taxes or subsidies. A subsidy rate, τ_t , raises the revenues of the i^{th} monopolists to $p(i)x(i)(1 + \tau_t)$. The total cost, G_t , to the government of this subsidy scheme is

$$G_t = \int_0^{M_t} p(i)x(i)\tau_t di.$$

Suppose G_t is financed by a lump sum tax applied to households. That is, the household budget constraint is modified as follows:

$$c_t + k_{t+1} - (1 - \delta)k_t = r_t k_t + w_t n_t - T_t,$$

where T_t represents taxes paid by the representative household to the government. Suppose the government balances its budget period by period:

$$T_t = G_t.$$

Find the subsidy rate, τ_t , that causes the allocations in the market economy to coincide with the efficient allocations.

These results have to be interpreted with caution. You have identified an ideal form of government intervention, which makes the private market economy efficient. However, the intervention we investigated abstracts from any social inefficiencies induced by having to raise the revenues needed to finance the subsidy to monopolists. We abstracted from this by assuming that the tax on households is administered in lump-sum form. In practice, such taxes are not available. So, the problem of ‘fixing’ the inefficiency in the Romer model is actually more complicated than this question makes it out to be.

3. This question explores the restrictions on preferences and technology that the existence of balanced growth path requires. Consider the following growth model:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to the resource constraint:

$$\begin{aligned} c_t + k_{t+1} - (1 - \delta)k_t &\leq f(k_t, \gamma^t n_t), \quad 0 < \alpha < 1, \\ k_{t+1}, c_t &\geq 0, \quad 0 \leq n_t \leq 1, \quad \gamma > 1. \end{aligned}$$

Also,

$$u(c_t, 1 - n_t) = \{(1 - \sigma)c_t^{\frac{\rho-1}{\rho}} + \sigma(1 - n_t)^{\frac{\rho-1}{\rho}}\}^{\frac{\rho\psi}{\rho-1}}/\psi, \text{ for } \rho > 0, \neq 1,$$

and

$$u(c_t, 1 - n_t) = [c_t^{(1-\sigma)}(1 - n_t)^\sigma]^\psi/\psi, \text{ for } \rho = 1,$$

where the (constant) elasticity of substitution between c_t and $(1 - n_t)$ is ρ , and

$$f(k, n) = \begin{cases} \{(1 - \alpha)k^{\frac{\nu-1}{\nu}} + \alpha n^{\frac{\nu-1}{\nu}}\}^{\frac{\nu}{\nu-1}}, & \text{for } \nu > 0, \neq 1 \\ k^{(1-\alpha)}n^\alpha, & \text{for } \nu = 1, \end{cases}$$

where ν is the elasticity of substitution between n and k in production.

(a) Define a balanced growth path as an equilibrium in which

$$n_t = n, \frac{k_{t+1}}{k_t} = \frac{c_{t+1}}{c_t} = \gamma.$$

Show that $\rho = 1$ is required for there to be a balanced growth path for this economy.

(b) Show that when $\rho = 1$, there is a scaling of the variables for this problem, in which the problem boils down to the standard growth model with no steady state growth.