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Recursive Methods and Participation Constraints

In class, we have discussed the advantages of the recursive methods for studying the dynamic properties of the neoclassical growth model and variants on it. The recursive approach can be used to study other important problems in dynamic macroeconomics. The purpose of this handout is to provide a taste of this broader set of examples.

The example developed below concerns financial markets when individual agents experience income uncertainty. When there are complete markets, as in Arrow-Debreu equilibrium, then agents can trade claims that are contingent on the realization of uncertainty - both at the individual level as well as at the aggregate level. In principle, the possible set of financial instruments could be so large that agents can achieve complete pooling of their idiosyncratic risk. However, the necessary exhaustive set of contingent claims over all possible dates and realizations of uncertainty is not observed in reality.

To construct empirically realistic models of financial markets in the presence of uncertainty, economists have proceeded in two ways. One is to suppose that there is only a one-period bond which pays off an equal amount in all states of the world. A feature of this approach is that the restrictions on financial contracts exceed what is observed in practice. For example, equity pays off different amounts in different states of nature. Another approach is associated with Alvarez and Jermann (2000,2001), Kehoe and Levine (1993), Kehoe and Perri (2002), Kocherlakota (1996) and Lochner and Monge (2002). This approach models directly the factors that account for the absence of Arrow-Debreu complete securities in the real world. The resulting restrictions on the degree of state-contingency of financial contracts is less severe. The restrictions are also more interesting, because they are derived endogenously from more primitive assumptions about the economic environment.

To understand what this literature is about, note that there is a natural income-sharing motive when agents experience idiosyncratic shocks to their income. Under efficient arrangements, people with unexpectedly high income will transfer some of it to people with unexpectedly low income. Financial markets or other arrangements which produce this outcome will look attractive from an ex ante perspective. The problem is that ex post, the agents who have to give up their income might have an incentive to renege. If they do renege, the whole income-sharing scheme collapses. This ex post incentive to renege imposes a restriction on how much income sharing there can occur. In particular, allocations must satisfy a particular participation constraint: agents giving up income must to be no worse off than if they choose to take their own income and depart from the income-sharing scheme. But, determining whether an agent's participation constraint is satisfied is fundamentally a complicated problem. This is because the value of remaining in the scheme is a function of utility now and in the future, and this in turn is a function of everything that goes into determining the dynamic equilibrium through time. It turns out that a practical way to compute this value is to solve an appropriate functional equation for a value function.

Participation constraints such as the one just described occur in a much larger class of economic settings. For example in efficiency wage models, contractual arrangements between firms and workers must take into account that workers potentially have an incentive to exert lower effort than the contract requires. In efficiency wage models a worker's decision about how much effort to exert is determined by comparing the utility gained over time from sticking to the rules, with the utility gained from deviating from the rules. This comparison is complicated because it involves future outcomes that occur as part of the general equilibrium. Again, a practical approach to this problem involves solving an appropriately constructed functional equation for a value function.

In the following section there is a description of the basic model environment and the incentive constraints. After that we look at how far we can go in thinking about the problem from a sequence point of view. Then we study a recursive formulation. After that we describe a practical computational strategy for approximating the efficient allocation. Finally, there are homework questions.

1 The Model

Suppose there are two types of agents. In period t, type 1 agent receives endowment e_t and type 2 agent receives endowment $1 - e_t$. The endowment, e_t can take on a finite set of values. Let s_t denote the exogenous uncertainty realized in period t, so that $s_t = e_t$. Let s^t denote a history of the exogenous uncertainty from the first period, date 0, until period t :

$$s^t = (s_0, s_1, ..., s_t).$$

The probability of a particular history, s^t, is denoted by $\pi_t(s^t)$, for $t = 0, 1, 2, ...$
.

Let $c_i(s^t)$ denote the consumption of agent i in history s^t, $i = 1, 2$.[1] The period utility for each agent is $u_i(c_i(s^t))$, for $i = 1, 2$. A planner values consumption allocations according to:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \left[\theta_1 u_1(c_1(s^t)) + \theta_2 u_2(c_2(s^t))\right]. \qquad (1)$$

Here, θ_1 and θ_2 are the weights assigned by the planner to the different types and u_1 and u_2 are two (potentially different) strictly increasing, concave and differentiable utility functions.

[1] To be precise, $c_i(s^t)$, is a scalar number indicating the consumption of person i in period t when events $s_0, s_1, ..., s_{t-1}$ occured in the past and the current realization of uncertainty is s_t.

The planner in our setup chooses consumption allocations to maximize utility subject to the resource constraint:

$$c_1(s^t) + c_2(s^t) = 1, \tag{2}$$

and one other condition to be discussed below. Note that if the resource constraint were the *only* constraint, then the solution to the planner problem is trivial. Preferences, (1), are completely separable in consumption across different histories. In addition, the technology, (2), imposes no restrictions across histories. Thus, the problem is optimized by choosing $c_i(s^t)$, $i = 1, 2$, to maximize period utility, history by history. What is efficient in one history has nothing to do with what happened in any other history. In this sense time and histories do not play an essential role in the problem posed so far. What is done in one history does not influence the planner's opportunities in any future history. What the planner can do in any given history is unrelated to what was done in previous histories.

The necessary and sufficient conditions for optimality in the problem posed so far are that $c_i(s^t)$, $i = 1, 2$ satisfy (2) and the following first order condition:

$$u_1'(c_1(s^t)) = \frac{\theta_2}{\theta_1} u_2'(c_2(s^t)). \tag{3}$$

Note that if the planner 'likes' people of type 1, so that θ_1 is large relative to θ_2, then the planner allocates consumption in such a way that the marginal utility of type 1 people is relatively low, i.e., the planner allocates a relatively large amount of consumption to agents of type 1. An interesting benchmark case is symmetry, where $\theta_1 = \theta_2$ and the two utility functions are the same. In this case, the solution is $c_1(s^t) = c_2(s^t) = 1/2$ for all s^t. We refer to the problem posed so far as the *unconstrained efficient allocation problem.*

In general, the unconstrained efficient allocations cannot be achieved by the planner in our model, because we allow agents the choice of not participating with the planner. In particular, we suppose that in the beginning of a period, say t, s_t is realized and then agents can decide whether to go into autarky or to interact with the planner. If they choose autarky, then they must consume only their own endowment in the current history and in all continuation histories. That is, an agent who chooses autarky in the current period is separated from the planner forever. Consider, for example, the unconstrained efficient allocations under symmetry. Suppose further that agents either receive an endowment of 3/4 or 1/4. In this case, agents receiving the high endowment are called upon by the planner to give up 1/4 and agents receiving the low endowment are given 1/4 by the planner. Obviously, an agent receiving the high endowment may choose to go into autarky. The benefit is that such an agent increases current consumption by 1/4, that is, by 50 percent. The cost is that the agent is shut out from receiving 1/4 in continuation histories in which their endowment is low. If the agent is not too risk averse and β is not too high, then such an agent might well choose autarky. But, in this case the insurance scheme collapses. Obviously, a planner contemplating the optimal allocations in this setting need not even consider allocations in which one or another type of agent strictly prefers autarky.

The constraint that agents do not choose autarky is one that must be applied in each history. Any consumption allocations that the planner considers must have the property that in each s^t, the utility of those allocations is no smaller than the utility of the allocations in autarky:

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{s^{t+j}|s^t}\beta^j\pi_{t+j}(s^{t+j}|s^t)u_1(c_1(s^{t+j})) &\geq \sum_{j=0}^{\infty}\sum_{s^{t+j}|s^t}\beta^j\pi_{t+j}(s^{t+j}|s^t)u_1(e(s^{t+j}))\equiv V_1(s^t)\\
\sum_{j=0}^{\infty}\sum_{s^{t+j}|s^t}\beta^j\pi_{t+j}(s^{t+j}|s^t)u_2(c_2(s^{t+j})) &\geq \sum_{j=0}^{\infty}\sum_{s^{t+j}|s^t}\beta^j\pi_{t+j}(s^{t+j}|s^t)u_2(1-e(s^{t+j}))\equiv V_2(s^t),
\end{aligned}$$

where $e(s^{t+j})=e_{t+j}$ and $s^{t+j}|s^t$ denotes a history, s^{t+j} consistent with history s^t. Also, $\pi_{t+j}(s^{t+j}|s^t)$ denotes the probability of s^{t+j} conditional on s^t :

$$\pi_{t+j}(s^{t+j}|s^t)=\frac{\pi_{t+j}(s^{t+j})}{\pi_t(s^t)}.$$

We refer to the above constraints as the agent's participation constraints. We refer to this problem as the *constrained efficient allocation problem.*

Note that requiring that the planner only consider consumptions that satisfy the participation constraints in no way limits the allocations that the planner can achieve in our environment. For example, the worst case scenario is one in which autarky is the best that can be achieved. Imposing the participation constraints on the problem does not rule out this allocation, since our specification of the participation constraints permits a strict equality. The problem is basically unchanged when there are more agent types. In this case, simply include all their participation constraints. This does not rule out any allocations of interest to the planner. For example, if the best that can be achieved is an allocation in which a subset of agents is in autarky and another subset is not, then this is potentially consistent with the participation constraints.

2 Sequence Representation of the Planner Problem, in Lagrangian Form

Let $\lambda_i(s^t) \geq 0$ denote the multiplier on the i^{th} agent's participation constraint in history s^t. The planner's problem is:

$$\max_{(c_1(s^t),c_2(s^t))\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \{ \left[\theta_1 u_1(c_1(s^t)) + \theta_2 u_2(c_2(s^t)) \right]$$
$$+\lambda_1(s^t) \left[\sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_{t+j}(s^{t+j}|s^t) u_1(c_1(s^{t+j})) - V_1(s^t) \right]$$
$$+\lambda_2(s^t) \left[\sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_{t+j}(s^{t+j}|s^t) u_2(c_2(s^{t+j})) - V_2(s^t) \right] \},$$

subject to the resource constraint, (2). We follow the strategy outlined in Marcet and Marimon (1992, 1999) for simplifying this expression. First, rewrite it as follows:

$$\max_{(c_1(s^t),c_2(s^t))\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \{ \left[\theta_1 u_1(c_1(s^t)) + \theta_2 u_2(c_2(s^t)) \right] \tag{4}$$
$$+\lambda_1(s^t) \left[u_1(c_1(s^t)) - V_1(s^t) \right] + \lambda_2(s^t) \left[u_2(c_2(s^t)) - V_2(s^t) \right]$$
$$+\lambda_1(s^t) \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_{t+j}(s^{t+j}|s^t) u_1(c_1(s^{t+j}))$$
$$+\lambda_2(s^t) \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_{t+j}(s^{t+j}|s^t) u_2(c_2(s^{t+j})) \},$$

Consider:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \lambda_1(s^t) \left\{ \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_{t+j}(s^{t+j}|s^t) u_1(c_1(s^{t+j})) \right\}. \tag{5}$$

Consider a given history, s^T, which we denote by $\tilde{s}^T = (\tilde{s}_0, \tilde{s}_1, ..., \tilde{s}_T)$, where $\tilde{s}_t$ are the realizations of uncertainty in $\tilde{s}^T$, for $t = 0, 1, 2..., T$. Define $\tilde{s}^l$ for $l = 1, ..., T$ analogously. Notice that $u_1(c_1(\tilde{s}^T))$ appears once for each value of t, $t = 0, 1, ..., T$ in (5). For $t = 0$ in (5) $u_1(c_1(\tilde{s}^T))$ appears with a coefficient of

$$\beta^T \pi_T(\tilde{s}^T) \lambda_1(\tilde{s}^0).$$

For $t = 1$, $u_1(c_1(\tilde{s}^T))$ appears with a coefficient of:

$$\beta^T \pi_T(\tilde{s}^T) \lambda_1(\tilde{s}^1),$$

and so on. Finally, for $t = T-1$, $u_1(c_1(\tilde{s}^T))$ appears with a coefficient of:

$$\beta^T \pi_T(\tilde{s}^T)\lambda_1(\tilde{s}^{T-1}).$$

We conclude that the coefficient on $u_1(c_1(\tilde{s}^T))$ in (5) is:

$$\beta^T \pi_T(\tilde{s}^T)\left[\lambda_1(\tilde{s}^0) + \lambda_1(\tilde{s}^1) + ... + \lambda_1(\tilde{s}^{T-1})\right].$$

for $T = 1, 2, ...$. As a result, (5) can be rewritten as follows:

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t(s^t)\left[\lambda_1(s^0) + \lambda_1(s^1) + ... + \lambda_1(s^{t-1})\right] u_1(c_1(s^t)).$$

Substituting this into (4), and rearranging:

$$\max_{(c_1(s^t), c_2(s^t))\}} \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t(s^t)\{M_1(s^t)u_1(c_1(s^t)) + M_2(s^t)u_2(c_2(s^t)) \quad (6)$$
$$-\lambda_1(s^t)V_1(s^t) - \lambda_2(s^t)V_2(s^t)\},$$

subject to the resource constraint, (2). Here,

$$M_i(s^t) = \lambda_i(s^0) + \lambda_i(s^1) + ... + \lambda_i(s^t) + \theta_i, \ i = 1, 2,$$

for $t = 0, 1, 2, ...$. It is convenient to note,

$$M_i(s^t) = M_i(s^{t-1}) + \lambda_i(s^t), \ i = 1, 2. \quad (7)$$

In solving this Lagrangian problem, the planner takes $M_i(s^{t-1})$ and $\lambda_i(s^t)$, $i = 1, 2$ as given sequences. The first order necessary condition for optimization is,

$$M_i(s^t)u_i'(c_i(s^t)) = \mu(s^t),$$

for $i = 1, 2$. Here, $\mu(s^t) \geq 0$ is the multiplier on the resource constraint. Combining the last equation for $i = 1, 2$, we obtain:

$$u_1'(c_1(s^t)) = z(s^t)u_2'(c_2(s^t)), \quad (8)$$
$$z\left(s^t\right) = \frac{M_2(s^t)}{M_1(s^t)}, \quad (9)$$

for all s^t and $t = 0, 1, 2, ...$. For later purposes, it is convenient to notice that (7) implies:

$$z(s^t) = z\left(s^{t-1}\right)\frac{1 - v_1(s^t)}{1 - v_2(s^t)}, \ v_i\left(s^t\right) = \frac{\lambda_i\left(s^t\right)}{M_i\left(s^t\right)}, \ i = 1, 2. \quad (10)$$

It is interesting to compare the constrained efficient problem of maximizing (6) subject to the resource constraint and the unconstrained efficient problem of maximizing (1) subject to the resource constraint. Recall that the second problem is static in the sense that choices in one history are unrelated to the choices in any other history. Superficially, the constrained problem satisfies the same property, but at a deeper level it clearly does not. For the allocations that optimize (6) to coincide with the constrained efficient allocations, the multipliers must be a very particular set of constants. Since the multipliers in (6) are constants, optimizing (6) represents a problem in which what is done in one history is independent of what is done in another history. In this sense, time seems not to matter, not even in the constrained efficient allocation problem. But, time actually does matter because the multipliers are themselves constructed from the efficient allocations. In effect, the multipliers are like prices in a competitive equilibrium. They are taken as given by agents when contemplating alternative choices, but the market clearing equilibrium prices are themselves functions of agents' particular choices in equilibrium.

To pursue this further, consider (8) for $t = 0$:

$$u_1'(c_1(s^0)) = \frac{\lambda_2(s^0) + \theta_2}{\lambda_1(s^0) + \theta_1} u_2'(c_2(s^0)). \tag{11}$$

If there were no participation constraints, then the allocations would satisfy the (3):

$$u_1'(c_1(s^0)) = \frac{\theta_2}{\theta_1} u_2'(c_2(s^0)).$$

Suppose, however, that this allocation involves transferring substantial resources from the type 1 agent to the type 2 agent. In this case, it is possible that the type 1 agent might prefer autarky, while the type 2 agent is happy. In this case, $\lambda_1(s^0) > 0$ and $\lambda_2(s^0) = 0$. From (11) we can see that in this case the type 1 agent receives more consumption than what he would get under the unconstrained efficient allocation. Consider the particular symmetric example discussed in the previous section. Thus, suppose the unconstrained efficient allocations call for type 1 agent to give up 1/4 in period 0 and type 2 agent to receive 1/4. In case type 1 agents would prefer autarky over this outcome, then the planner 'improves' the way he treats type 1 agents, by asking for less than 1/4. In effect, the planner behaves as though he 'likes' type 1 agent more, by increasing his weight in the problem above θ_1.

It is interesting to examine the planner's first order condition in a later history, say s^t :

$$u_1'(c_1(s^t)) = \frac{\lambda_2(s^0) + \lambda_2(s^1) + ... + \lambda_2(s^t) + \theta_2}{\lambda_1(s^0) + \lambda_1(s^1) + ... + \lambda_1(s^t) + \theta_1} u_2'(c_2(s^t)).$$

Suppose history s^t is one in which type 1 agents happen to have always received endowments, 3/4, and type 2 agents always received endowments, 1/4. In such a history, we can expect that $\lambda_1(s^j) > 0$, $j = 0, 1, ..., t$, while $\lambda_2(s^j) = 0$ for $j = 0, ..., t$. Notice that in this case, the type 1 agent is treated extremely well. That is, type 1 agents are treated better in any history in which in the past, they have been called upon to do a lot of transfers. In this sense what happens in one history is very much a function of what happens in other histories. Clearly, this feature of the constrained efficient allocation problem is purely a function of the presence of the participation constraints.

While the sequence formulation of the problem obviously yields important insights into the nature of the constrained efficient allocations, from a practical standpoint it quickly leads to a dead end. To see this, consider by way of contrast the problem of solving the neoclassical growth model in sequence form. As in the problem here, in principle one needs to search over the set of infinite sequences to maximize utility. This strategy is impossibly difficult both here and in the neoclassical growth model.

However, in the neoclassical growth model we have Theorem 4.15 from Stokey-Lucas, which provides a practical strategy for solving the sequence problem. The set of admissible infinite sequences of capital is reduced to a one-dimensional continuum using the Euler equation.[2] Then, Theorem 4.15 instructs us that the element in this one-dimensional continuum that satisfies the transversality condition is the solution we seek.

It does not appear that there is an analog of Theorem 4.15 in the present setting. For example, if we attempted to imitate the strategy based on Theorem 4.15 by solving forward, we would immediately encounter a severe road block. To see this, recall the choice of consumption in period 0. A natural first conjecture is that consumption is divided equally among the two agents. But, the only way to know whether this is consistent with the participation constraint is to know what all the future consumptions are. Only then is it possible to determine whether the agents prefer the equal division of consumption in period 0 over autarky. But, how are we supposed to know all future consumptions, when we have just started contemplating period 0 consumption? Below, we describe a recursive setup of the problem, which appears to lend itself more easily to solution.

3 Parameterizing the Model

It is useful at this point to describe our parameterization of the endowment process and of preferences. We specify that utility has the following constant elasticity form:

$$u_1(x) = \frac{x^{1+\gamma_1}}{1+\gamma_1},\ u_2(x) = \frac{x^{1+\gamma_2}}{1+\gamma_2}.$$

Turning to e_t, we suppose that it can take on only two values, e^h and e^l such that:

$$e^h = \bar{e} + \sigma,\ e^l = \bar{e} - \sigma,\ \bar{e} = \frac{1}{2}.$$

[2]Recall, the Euler equation is a second order difference equation with a given initial condition for date 0 capital. Given a value of the period 1 capital stock, one can compute an infinite sequence of capital stocks that satisfy the Euler equation. The observation in the text is motivated by the fact that this sequence is conditioned on the value of the period 1 stock of capital. This is a one-dimensional variable.

We let α denote the probability that e_t remains unchanged:

$$\alpha = prob(e_{t+1} = e^h | e_t = e^h) = prob(e_{t+1} = e^l | e_t = e^l).$$

The probability of switching between the two values of e_t is $1 - \alpha$. It is easy to verify that:

$$Ee_t = \bar{e},\ Var(e_t) = \sigma^2,\ \rho \equiv \frac{Cov(e_t, e_{t-1})}{Var(e_t)} = 2\alpha - 1.$$

Also, the one-step ahead forecast error variance in the linear Wold representation of e_t is $Var(\varepsilon_t)$, where

$$Var(\varepsilon_t) = \left(1 - \rho^2\right) Var(e_t).$$

To parameterize this, we can use the evidence on the time series properties of individual household income reported in, for example, Heaton and Lucas (1996).

With this specification of the exogenous uncertainty, computing the value of autarky is straightforward. Let the value of autarky for agent 1 when state $s_t = e_t^s$ be denoted V_1^s, for $s = h, l$. Then,

$$\begin{aligned} \begin{bmatrix} V_1^h \\ V_1^l \end{bmatrix} &= \begin{bmatrix} u_1\left(e^h\right) \\ u_1\left(e^l\right) \end{bmatrix} + \beta\pi \begin{bmatrix} u_1\left(e^h\right) \\ u_1\left(e^l\right) \end{bmatrix} + (\beta\pi)^2 \begin{bmatrix} u_1\left(e^h\right) \\ u_1\left(e^l\right) \end{bmatrix} + ... \quad (12) \\ &= [I - \beta\pi]^{-1} \begin{bmatrix} u_1\left(e^h\right) \\ u_1\left(e^l\right) \end{bmatrix}. \end{aligned}$$

For agent 2 the value of autarky is:

$$\begin{bmatrix} V_2^h \\ V_2^h \end{bmatrix} = [I - \beta\pi]^{-1} \begin{bmatrix} u_2\left(1 - e^h\right) \\ u_2\left(1 - e^l\right) \end{bmatrix}. \quad (13)$$

It is also useful to have the discounted utility associated with the allocations that are efficient when the participation constraints can be ignored. The associated consumptions are found by solving

$$u_1'(c_1) = \frac{\theta_2}{\theta_1} u_2'(1 - c_1)$$

for c_1 and then setting $c_2 = 1 - c_1$. Then, the discounted utility of type i agents is

$$\frac{u_i(c_i)}{1 - \beta}, \quad (14)$$

for $i = 1, 2$.

4 Recursive Formulation of the Problem

Here, we pursue the idea that the solution to the problem has a simple recursive structure. Let W_i denote the present discounted value of utility of agent i, under the constrained efficient allocations. Let z' take the place of $z(s^t)$ in (9) and let z take the place of $z\left(s^{t-1}\right)$ in (10). We specify the state of the economy to be composed of z and the current realization of the exogenous uncertainty, s. Suppose that V_i, v_i, W_i, z' and c_i are functions of this state. Then $W_i(z,s)$ has the following representation:

$$W_i(z,s) = u_i(c_i(z,s)) + \beta \sum_{s'|s} \pi(s'|s) W_i(z'(z,s),s'),\ i = 1,2, \tag{15}$$

where

$$\begin{aligned} u_1'(c_1(z,s)) &= z'(z,s)u_2'(c_2(z,s)) & (16)\\ c_1(z,s)+c_2(z,s) &= 1 & (17)\\ z'(z,s) &= z\frac{1-v_1}{1-v_2}, & (18)\\ W_i(z,s) &\geq V_i^s, & (19)\\ v_i\left[W_i(z,s)-V_i^s\right] &= 0,\ v_i \geq 0 & (20) \end{aligned}$$

for all i, s and z.

It is useful to use the above restrictions to define an operator that maps $w(z,s) = (w_1(z,s), w_2(z,s))$ into $w'(z,s) = (w_1'(z,s), w_2'(z,s))$. We consider functions $w_1(z,s)$ and $w_2(z,s)$ that satisfy a particular monotonicity assumption: $w_1(z,s)$ is continuous and weakly decreasing in z and $w_2(z,s)$ is continuous and weakly increasing in z. Let

$$\begin{aligned} w'(z,s) &= T[w](z,s) \qquad (21)\\ &\equiv [u_1(c_1(z,s)) + \beta \sum_{s'|s} \pi(s'|s) w_1(z'(z,s),s'),\\ &\quad u_2(c_2(z,s)) + \beta \sum_{s'|s} \pi(s'|s) w_2(z'(z,s),s')], \end{aligned}$$

where $c_1(z,s)$, $c_1(z,s)$ and $z'(z,s)$ must satisfy (16)-(20). We assume that the solution to these equations exists. To define $T[w](z,s)$, consider a particular point z, s and two particular functions, $w_i(z,s)$, $i = 1,2$. Consider a candidate solution to (16)-(20), in which $v_1 = v_2 = 0$. In this case, $z' = z$ (see (18)) and solve for $c_1(z,s)$, $c_2(z,s)$ using (16) and (17):

$$\begin{aligned} u_1'(c_1(z,s)) &= zu_2'(1-c_1(z,s)) \qquad (22)\\ c_2(z,s) &= 1-c_1(z,s). \end{aligned}$$

Denote the value of $T[w](z,s)$ under the candidate solution to (16)-(20) by $\tilde{w}(z,s) = [\tilde{w}_1(z,s), \tilde{w}_2(z,s)]$:

$$\begin{aligned} \tilde{w}(z,s) &= [u_1(c_1(z,s)) + \beta \sum_{s'|s} \pi(s'|s) w_1(z'(z,s),s'), \qquad (23)\\ &\quad u_2(c_2(z,s)) + \beta \sum_{s'|s} \pi(s'|s) w_2(z'(z,s),s')], \end{aligned}$$

where $z'(z,s) = z$ and $c_1(z,s)$, $c_1(z,s)$ are the solution to (22). For the candidate solution to (16)-(20) to be valid we must first verify that (19) is satisfied. If it is, then we set:

$$T[w](z,s) = \tilde{w}(z,s). \tag{24}$$

Now, suppose that (19) fails. In principle, this can happen for three reasons, either $\tilde{w}_1(z,s) < V_1^s$ or $\tilde{w}_2(z,s) < V_2^s$, or both. Consider the first case. In this case, we choose $c_1(z,s)$ to satisfy the participation constraint as a strict equality for agent of type 1 :

$$u_1(c_1(z,s)) + \beta \sum_{s'|s} \pi(s'|s) w_1(z'(z,s), s') = V_1^s, \tag{25}$$

where

$$\begin{aligned} z'(z,s) &= \frac{u_1'(c_1(z,s))}{u_2'(1-c_1(z,s))} \\ c_2(z,s) &= 1 - c_1(z,s). \end{aligned}$$

The value of $c_1(z,s)$ that solves this problem is bigger than the one that satisfies (22). To see this, recall that the value of $c_1(z,s)$ which satisfies (22) results in the left side of (25) being smaller than the right. Given our monotonicity assumption and the above functional form of $z'(z,s)$, only an increase in $c_1(z,s)$ will produce equality. At the same time, the increase in $c_1(z,s)$ produces a decrease in $c_2(z,s)$, and therefore (using the monotonicity assumption again) a decrease in:

$$u_2(c_2(z,s)) + \beta \sum_{s'|s} \pi(s'|s) w_2(z'(z,s), s').$$

If the values of $c_1(z,s)$, $c_2(z,s)$ that solve (25) cause the above expression to be smaller than V_2^s, then there is no set of values of $z'(z,s)$, $c_1(z,s)$, $c_2(z,s)$ that solves (16)-(20). As noted above, we assume this is not the case. Then we set $T[w](z,s)$ as in (24), where $\tilde{w}(z,s)$ is defined in (23).

Another possibility is that the $c_1(z,s)$, $c_2(z,s)$, $z'(z,s)$, which solve (22) imply $\tilde{w}_2(z,s) < V_2^s$ and $\tilde{w}_1(z,s) \geq V_1^s$. In this case, we proceed as in the previous case, with the obvious modifications.

The third and final possibility is that the $c_1(z,s)$, $c_2(z,s)$, $z'(z,s)$, which solve (22) imply $\tilde{w}_2(z,s) < V_2^s$ and $\tilde{w}_1(z,s) < V_1^s$. For a rigorous discussion of the fact that this cannot happen for $w = T[w]$, see Alvarez and Jermann (2000,2001). The intuition is simple. In general, at a point (z,s), one agent receives more than their initial endowment and the other, less. The one that receives more clearly benefits in the current period from receiving the transfer. But, the continuation utility of that person must satisfy (19), which means that staying out of autarky beginning the next period does not produce a fall in utility. For the person with utility strictly greater than autarky in the present and no worse than autarky in the future, the participation constraint is obviously not binding. Of course, it does not follow that our third possibility cannot happen for any possible candidate continuation function. In practice, however, we have not encountered this case.

5 Computing the Solution

The computational strategy focuses on finding a finite parameter approximation to $W_i(z,s)$, which 'approximately' solves (15). We can think of $W_i(z,s)$ as four different functions in z, one for each of the two possible values of s and the two possible values of i. Each of these functions is approximated by a function that is continuous and piecewise linear. That is, each function is composed of a series of straight lines that are joined end-to-end at a set of grid points, $z = Z_1, ...,$ $z = Z_N$. The interval, $z \in [Z_1, Z_N]$, is the domain of the functions we work with. To proceed, we must select values for Z_1, Z_N and N.[3] Whether our choice of grid and functional form for $W_i(z,s)$ is a good one can be assessed at the end of the calculations. More on this later.

Formally, our piecewise linear approximation to $W_i(z,s)$ is defined as follows. Let

$$W_i(z,s) = \begin{cases} W_i^h(z;a), & \text{for } s = e^h \\ W_i^l(z;a), & \text{for } s = e^l \end{cases}, \quad i = 1,2.$$

The parameter vector, $a \in R^{4N}$, records the value of the functions, $W_i^s(z;a)$, at each of the points, $z = Z_1, ..., z = Z_N$, for $i = 1,2$ and $s = h,l$. The value of the function for z lying between Z_i and Z_{i+1} is as follows:

$$\begin{aligned} W_i^s(z;a) &= \alpha W_i^s(Z_j;a) + (1-\alpha) W_i^s(Z_{j+1};a), \\ \alpha &= \frac{z - Z_{j+1}}{Z_j - Z_{j+1}}, \; j = 1, ..., N-1, \; s = h,l, \; i = 1,2. \end{aligned}$$

From this point of view, $a \in R^{4N}$ indexes a function, $[Z_1, Z_N] \to R$. The algorithm maps this space of functions into itself, and seeks a fixed point. The mapping begins with an initial guess for $a \in R^{4N}$. The associated function is inserted into the summation operator on the right side of (15) and (15) together with the equilibrium conditions (i.e., (2), (9) and (10)) are used to compute a new function. This corresponds to an $a' \in R^{4N}$. Call this mapping:

$$a' = T(a).$$

We seek an $a^* \in R^{4N}$ such that $a^* = T(a^*)$. Let a_0 be the initial guess for a^*. Then, $a_{r+1} = T(a_r)$ for $r = 1,2,...$. Our a^* is the vector to which this sequence converges, if it does converge.[4]

For our initial guess, a_0, we simply set our $W_i(z,s)$ functions to the value of the efficient allocations when the participation constraints are ignored, (14).

We now discuss how $a' = T(a)$ is evaluated for a given $a \in R^{4N}$. Set $z = Z_1$ and $s = e^h$. Conjecture that the participation constraints are not binding, so

[3] The grid points, $Z_2, Z_3, ..., Z_{N-1}$, could be selected in various ways. One way would be to choose them to be equally spaced. An alternative way is to write $z = (1-\theta)/\theta$, and choose an equally spaced grid of N θ's on the unit interval. These induce an unequally spaced grid on z. Obviously, other ways of selecting a grid for z can also be chosen.

[4] Another strategy treats $f(a)$ as a system of equations, $f(a) = T(a) - a$. A standard nonlinear equation solver can be used to find a^* such that $f(a^*) = 0$.

that $v_i(z,s)=0$ for $i=1,2$. In this case, $z'(z,s)=z$ and solve the following for $c_1(z,s)$, $c_2(z,s)$:

$$1=z\frac{[c_2(z,s)]^{\gamma_2}}{[1-c_2(z,s)]^{\gamma 1}},\ c_1(z,s)=1-c_2(z,s).$$

This is of course trivial when $\gamma_1=\gamma_2$. Verify that the participation constraints are in fact not binding, *i.e.*, that

$$\frac{[c_1(z,s)]^{1+\gamma_1}}{1+\gamma_1}+\beta\left[\alpha W_1^h(z';a)+(1-\alpha)\,W_1^l(z';a)\right] \geq V_1^h \tag{26}$$

$$\frac{[c_2(z,s)]^{1+\gamma_2}}{1+\gamma_2}+\beta\left[\alpha W_2^h(z';a)+(1-\alpha)\,W_2^h(z';a)\right] \geq V_2^h. \tag{27}$$

If both these conditions are satisfied, set

$$W_1^h(z;a') = \frac{[c_1(z,s)]^{1+\gamma_1}}{1+\gamma_1}+\beta\left[\alpha W_1^h(z';a)+(1-\alpha)\,W_1^l(z';a)\right] \tag{28}$$

$$W_2^h(z;a') = \frac{[c_2(z,s)]^{1+\gamma_2}}{1+\gamma_2}+\beta\left[\alpha W_2^h(z';a)+(1-\alpha)\,W_2^h(z';a)\right].$$

Note that since W_i^h, $i=1,2$ is being evaluated on our grid for z, finding the a' that satisfies the above equality is trivial. It simply requires setting the appropriate element of a' to the term on the right side of the equality.

These calculations are being described for the case, $s=e^h$. This leads us to consider the possibility that (26) is in fact violated, i.e., (26) is a strict inequality in the 'wrong' direction. Suppose it is. (When $s=e^l$, then we consider the possibility that it is (27) that is violated, and the following calculations are adjusted in the obvious way.) From (9) and our specification of z', as well as the resource constraint:

$$z'=\frac{(1-c_2(z,s))^{\gamma_1}}{(c_2(z,s))^{\gamma_2}}. \tag{29}$$

Choose $c_2(z,s)$ so that agent 1's participation constraint is satisfied as a strict equality. That is, solve the following nonlinear equation for $c_2(z,s)$:

$$\frac{[1-c_2(z,s)]^{1+\gamma_1}}{1+\gamma_1}+\beta\left[\alpha W_1^h(z';a)+(1-\alpha)\,W_1^l(z';a)\right] = V_1^h$$

$$c_1(z,s) = 1-c_2(z,s),$$

taking into account the link between z' and $c_2(z,s)$ in (29). Now, solve for $W_1^h(z;a')$ and $W_2^h(z;a')$ using (28).

Note that we do not pay attention to the possibility that both participation constraints are violated. Indeed, it seems possible that in the middle of the computations, this can occur. Nevertheless, the algorithm does not pay attention to this possibility. When an a^* is found at the end, it is necessary to confirm that this situation has in fact not occurred.

6 Homework Questions

1. Show $w = T[w]$, when w is the value function under autarky and T is as defined in section 4.

2. Show that depending on the value of β and γ, full risk sharing may be consistent with the participation constraint.

3. Suppose $w(z, s)$ satisfies the monotonicity assumption stated just before equation (21). Does $T[w](z, s)$ also satisfy that monotonicity property?

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