Handout for "Chaos, Sunspots, and Automatic Stabilizers in a Business Cycle Model" Christiano/Harrison

Outline

1. Model: household
   firm
   equilibrium

2. Characterizing Equilibrium

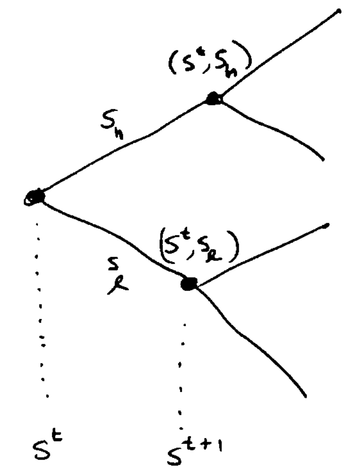
3. Nonstochastic Equilibria

4. Stochastic Equilibria

5. Tax policy

6. Optimal Allocations

Household



At each s^t, $t \geq 0$,

$$\max \sum_{j=t}^{\infty} \sum_{s^j | s^t} \beta^{j-t} \mu(s^j)\, u[c(s^j), n(s^j)]$$

s.t.

$$c(s^t) + k(s^t) - (1-\delta) k(s^{t-1}) \leq r(s^t) k(s^{t-1}) + w(s^t) n(s^t)$$

$$k(s^t), c(s^t) \geq 0, \quad 0 \leq n(s^t) \leq 1.$$

$$u(c,n) = \log c + \sigma \log(1-n)$$

- Let $\{r(s^t), w(s^t)\}$, all s^t, $t \geq 0$, and

  $k(s^{-1}) = k_0$ be given.

- Let $\pi = \{c(s^t), n(s^t), k(s^t)\}$ denote a "plan"

  that is feasible (i.e., satisfies budget constraint).

- Following are Euler equations (E):

(inter) $$u_c(s^j) = \beta \sum_{s_{j+1}} \mu(s^{j+1} | s^j) u_c(s^{j+1}) [r(s^{j+1}) + 1 - \delta]$$

(intra) $$\frac{-u_n(s^j)}{u_c(s^j)} = w(s^j)$$

all s^j, $j \geq 0$.

- Following is a transversality condition (TVC):

$$\lim_{T \to \infty} \beta^T \sum_{s^T} \mu(s^T) u_c(s^T) k(s^T) = 0$$

Result :

If π^* is <u>interior</u> and satisfies E, TVC, and π is any other feasible plan, then

$$U(\pi^*) \geq U(\pi).$$

Proof: Mechanical application of strategy in S-L for proving Th. 4.15.

Firm

Technology: $y = Y^{\gamma} k^{\alpha} n^{(1-\alpha)} = f(Y, k, n)$, $\gamma = 1-\alpha$

Objective: $\max \; y - rk - wn$

$$\Rightarrow \quad f_n = w, \; f_k = r.$$

$$f_n = (1-\alpha) N^{\frac{2\gamma-1}{1-\gamma}} K, \quad f_k = \alpha N^{\frac{\gamma}{1-\gamma}}$$

$$f = K N^{\frac{\gamma}{1-\gamma}}$$

With $\gamma = \frac{2}{3}$:

$$\boxed{f_n = (1-\alpha) N K, \quad f_k = \alpha N^2 \quad f = K N^2}$$

Sequence-of-Markets Equilibrium

Prices and probabilities: $\{r(s^t), w(s^t), \mu(s^t);\ \text{all } s^t, t \geq 0\}$

Quantities: $\{c(s^t), k(s^t), n(s^t);\ \text{all } s^t, t \geq 0\}$

such that:

(i) given prices and given $Y(s^t) = k(s^{t-1})\, n(s^t)^2$, quantities solve firm problem, each s^t, $t \geq 0$

(ii) given prices and probabilities, quantities solve household problem

Characterizing Space of Equilibria

If μ, and a plan, $\{n(s^t), c(s^t), k(s^t)\}$, satisfies

E, TVC, RC:

$$(E)\quad \begin{cases} \dfrac{1}{c(s^t)} = \beta \displaystyle\sum_{s_{t+1}} \mu(s^{t+1}|s^t) \dfrac{1}{c(s^{t+1})} \left[\alpha\, n(s^{t+1})^2 + 1 - \delta\right] \\[2em] \dfrac{\sigma\, c(s^t)}{1 - n(s^t)} = (1-\alpha)\, n(s^t)\, k(s^t) \end{cases}$$

$$(TVC)\quad \lim_{T\to\infty} \beta^T \sum_{s^T} \mu(s^T) \frac{k(s^T)}{c(s^T)} = 0$$

$$(R.C.)\quad c(s^t) + k(s^t) - (1-\delta)k(s^{t-1}) = k(s^{t-1})\, n(s^t)^2$$

Then that plan corresponds to an equilibrium.

Change of Variables:

$$\tilde{c}(s^t) = \frac{c(s^t)}{k(s^{t-1})}, \quad \lambda(s^t) = \frac{k(s^t)}{k(s^{t-1})}.$$

Then:

$$(E) \begin{cases} \dfrac{\lambda(s^t)}{\tilde{c}(s^t)} = \beta \sum_{s_{t+1}} \mu(s^{t+1} \mid s^t) \dfrac{1}{\tilde{c}(s^{t+1})} \left[\alpha\, n(s^{t+1})^2 + 1 - \delta\right] \\[2ex] \dfrac{\tilde{c}(s^t)}{1 - n(s^t)} = \dfrac{\gamma}{\sigma}\, n(s^t) \end{cases}$$

$$(TVC) \qquad \lim_{T \to \infty} \beta^T \sum_{s^T} \mu(s^T) \frac{\lambda(s^T)}{\tilde{c}(s^T)} = 0$$

$$(RC) \qquad \tilde{c}(s^t) + \lambda(s^t) = n(s^t)^2 + 1 - \delta$$

By intratemporal (E):

$$\tilde{c}(s^t) = \frac{\gamma}{\sigma} n(s^t)(1-n(s^t))$$

By (R.C.):

$$\lambda(s^t) = n(s^t)^2 + 1-\delta - \tilde{c}(s^t)$$

$$= n(s^t)^2 + 1-\delta - \frac{\gamma}{\sigma} n(s^t)(1-n(s^t))$$

Substitute into TVC:

$$\frac{n(s^t)^2 + 1-\delta - \frac{\gamma}{\sigma} n(s^t)(1-n(s^t))}{\frac{\gamma}{\sigma} n(s^t)(1-n(s^t))} - \beta \sum_{s_{t+1}} \mu(s^{t+1}|s^t) \left[\frac{\alpha\, n(s^{t+1})^2 + 1-\delta}{\frac{\gamma}{\sigma} n(s^{t+1})(1-n(s^{t+1}))}\right]$$

Ω,

$$\sum_{s_{t+1}} \mu(s^{t+1}|s^t)\, v(n(s^t), n(s^{t+1})) = 0$$

Where

$$v(n, n') = \frac{n^2 + 1 - \delta - \frac{\gamma}{\sigma} n(1-n)}{\frac{\gamma}{\sigma} n(1-n)} - \beta \frac{\alpha (n')^2 + 1 - \delta}{\frac{\gamma}{\sigma} n'(1-n')}$$

Result:

If $\{n(s^t)\}, \{\mu(s^t)\}$, $s^t, t \geq 0$

with property:

$$0 < n(s^t) \leq 1 \quad \text{all } s^t, \; t \geq 0,$$

then $\{n(s^t)\}$ corresponds to an equilibrium.

Proof

Compute remaining objects in equilibrium (i.e., prices and quantities) and verify that they satisfy TVC, E, RC.

Construction of equilibrium:

Step 1

$$\tilde{c}(s^t) = \frac{\gamma}{\sigma} n(s^t)(1 - n(s^t))$$

$$\lambda(s^t) = n(s^t)^2 + 1 - \delta - \frac{\gamma}{\sigma} n(s^t)(1 - n(s^t))$$

Step 2

$$k(s^t) = \lambda(s^t)\, k(s^{t-1}), \quad k(s^{-1}) = k_0 .$$

$$c(s^t) = \tilde{c}(s^t)\, k(s^{t-1})$$

$$r(s^t) = \alpha\, n(s^t)^2$$

$$w(s^t) = (1-\alpha)\, n(s^t)\, k(s^{t-1})$$

Easily verified: E & RC.

To verify TVC; Note:

$$\beta^T \mu(s^T) \frac{k(s^T)}{c(s^T)} = \beta^T \mu(s^T) \frac{\lambda(s^T)}{\tilde{c}(s^T)} = \beta^T \mu(s^T) \frac{n(s^T)^2 + 1 - \delta - \frac{\gamma}{\sigma} n(s^T)(1-n(s^T))}{\frac{\gamma}{\sigma} n(s^T)(1 - n(s^T))}$$

$$\xrightarrow[T \to \infty]{} 0$$

Q.E.D.

The v function:

A saddle in 3-D

two branches in 2-D

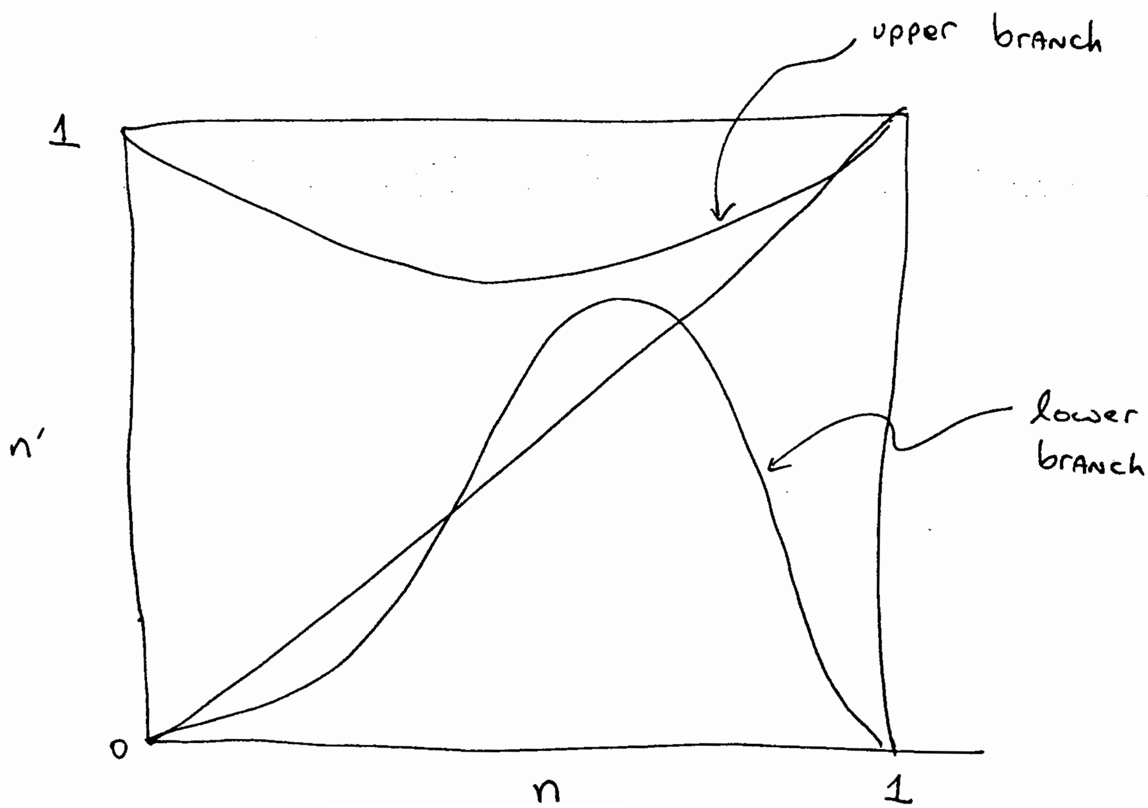


Figure 1a: The v(n,n') function

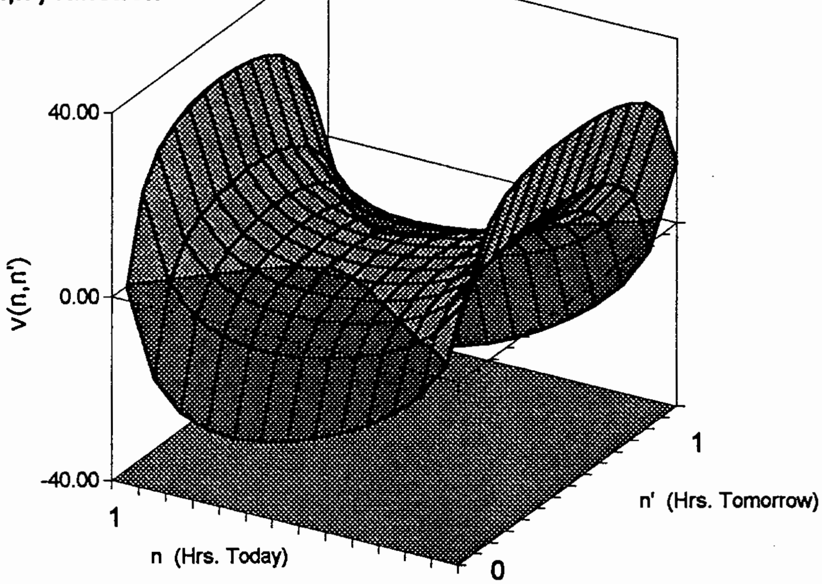


Figure 1b: Contour: v(n,n')=0

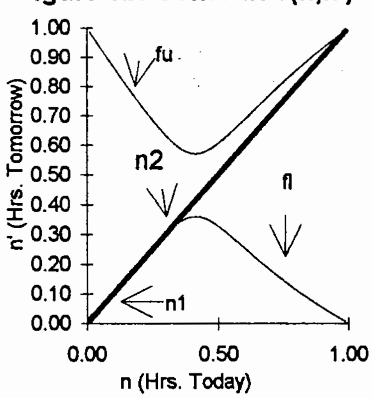
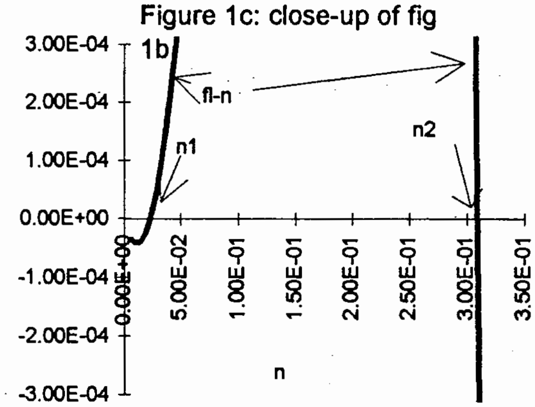


Figure 1c: close-up of fig 1b



Note: Figure 1a is a three dimensional view of the function v in equation (2.23), computed using the standard parameter values. The dark and light regions identify the parts of v that are less and greater than zero, respectively. Figure 1b shows the values of n' that set v to zero, given n. Here, fl and fu denote the lower and upper branch functions defined in (2.25), respectively. Also, n1 and n2 denote the points where fl crosses the 45 dregree line. Figure 1c is the difference between the lower branch function and the 45 degree line in Figure 1b, in a neighborhood of the origin. It shows that fl first cuts the 45 degree line from below, at n1, and then again from above, at n2.

Nonstochastic Equilibria

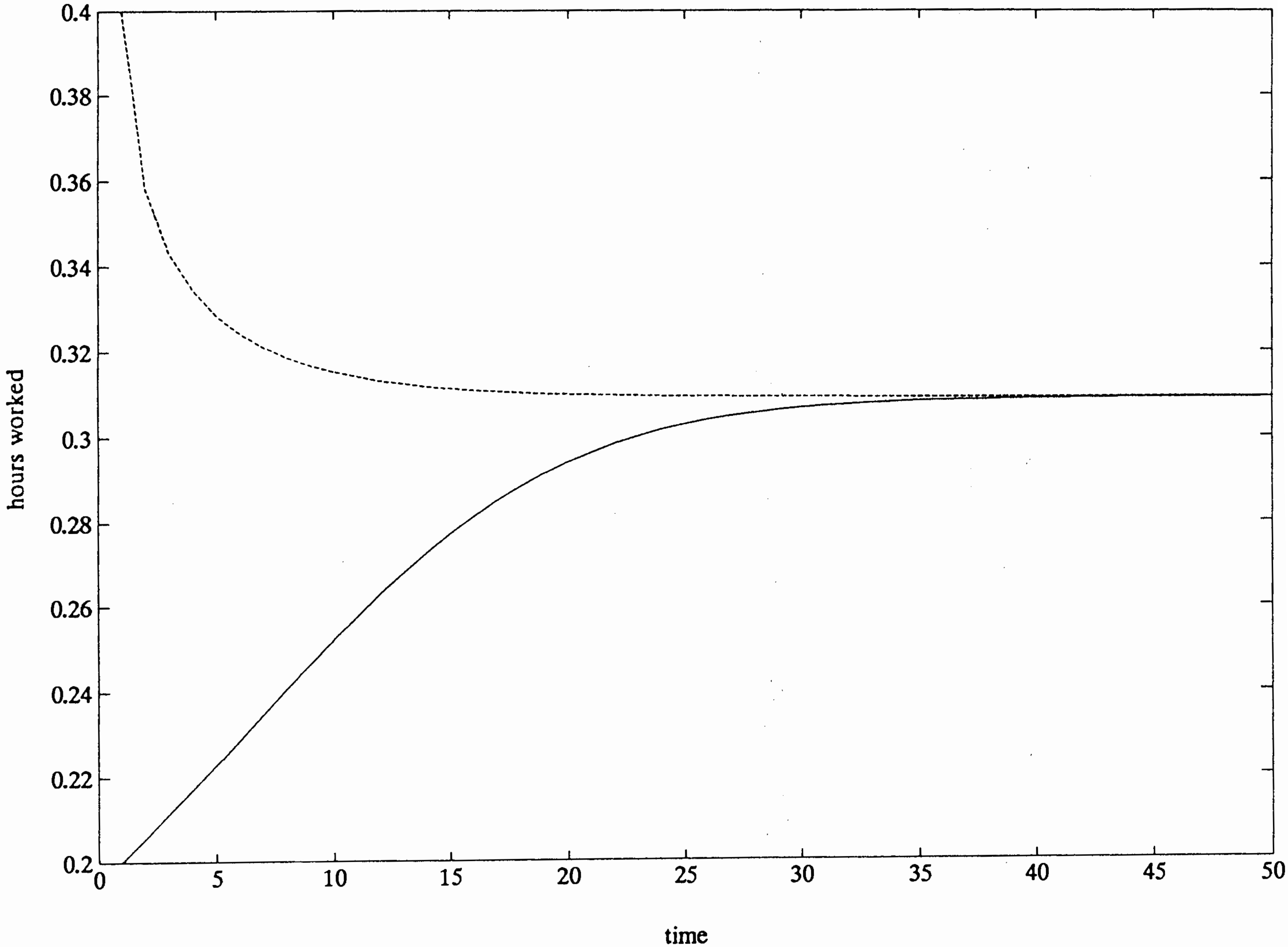
- Constantn equilibria
- Boring, monotone convergence equilibria
- Regime switching equilibria

  Nonstationary Regime Switching

  Stationary Regime Switching (Chaos?)

Examples of Boring Equilibria

Figure 2: Two Equilibria on the Lower Branch



Review of Dynamics

Let $f : J \to J$ be a map.

definition: Orbit of $x \in J$: $x, f(x), f^2(x), \ldots$

definition: Periodic point of f: $x \in J$ s.t. $\exists k > 0$ with $x = f^k(x)$

definition:
topological transitivity for any pairs of open sets, $U, V \in J$, $\exists k > 0$ s.t. $f^k(U) \cap V \neq \emptyset$

"for almost all initial conditions, orbit of map f goes through every open interval"

definition: Sensitive dependence on Initial Conditions:

$f : J \to J$ has sensitive dependence if there exists $\delta > 0$ s.t. for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ s.t.

$$|f^n(x) - f^n(y)| > \delta.$$

" Consider the orbit of $x \in J$: $x, f(x), f^2(x), \ldots$
there is a $\delta > 0$ that is a function of f, not x,
such that even for y extremely close to x,
the orbit of y is δ away from the orbit of x "

Definition $f: J \to J$ is chaotic on J

if:

1. f has sensitive dependence
2. f is topologically transitive
3. periodic points are dense in J.

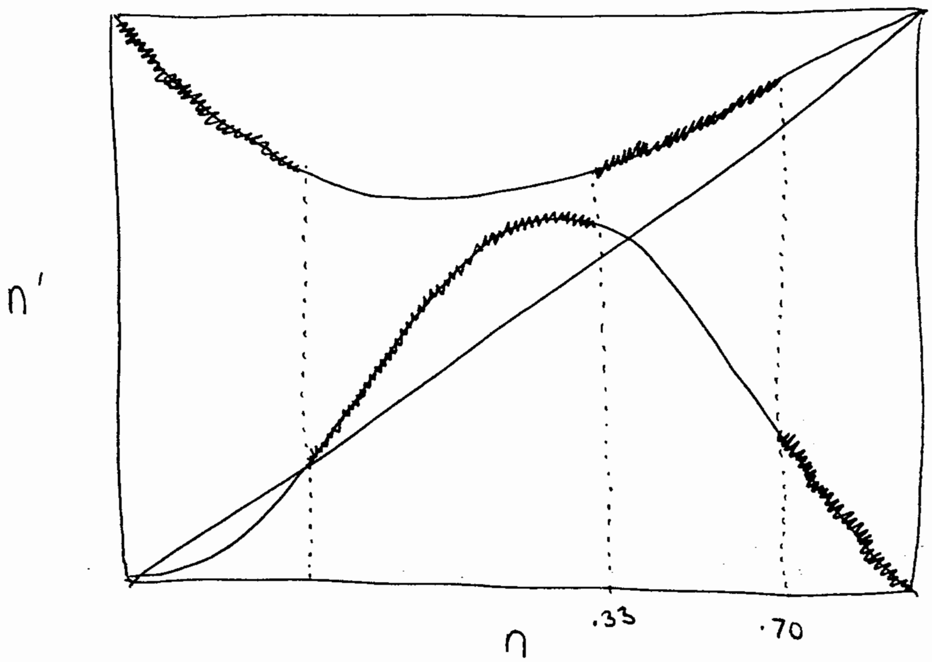
Classic example of chaotic map:

$$f(x) = 4x(1-x), \quad J = [0,1].$$

Periodic points: $x = 0, \frac{3}{4}$

eventually periodic point: $x = \frac{1}{4}$

A chaos-Like Equilibrium Map



f :

Regime-switching equilibrium.

(18)

Orbits of two initial conditions: $n(0) = .454, .455$

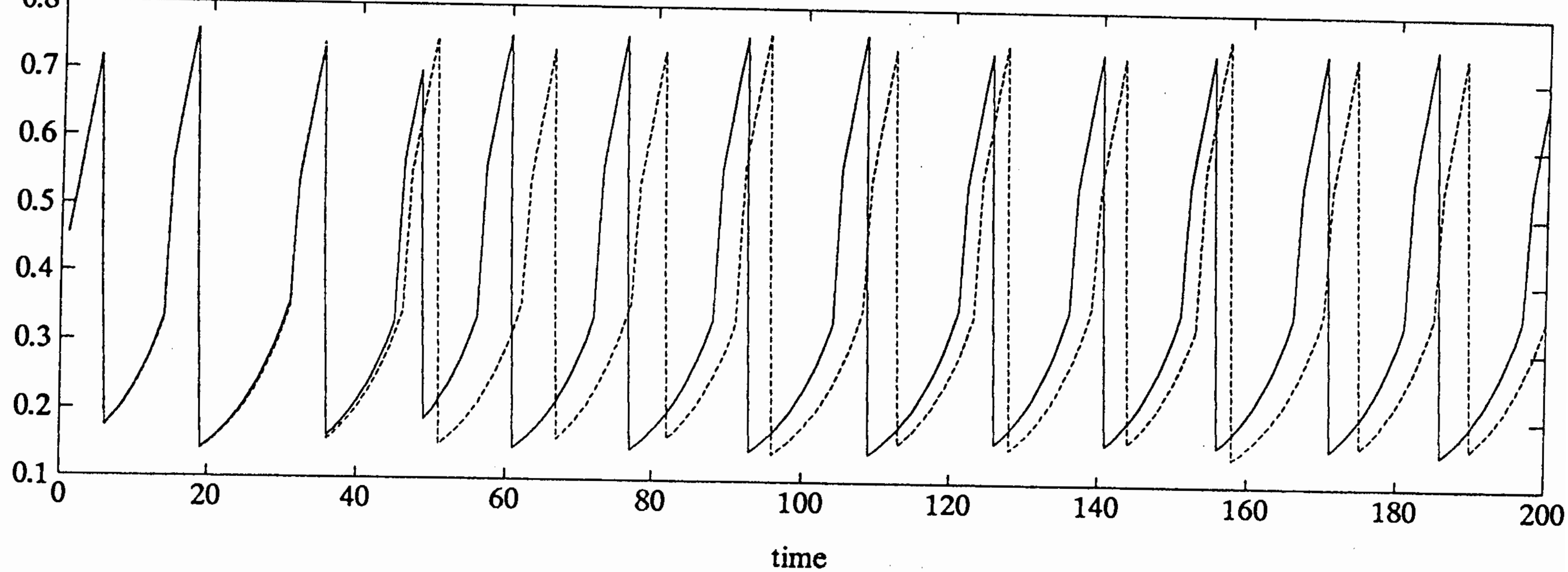


Fig4a: fraction of available time worked

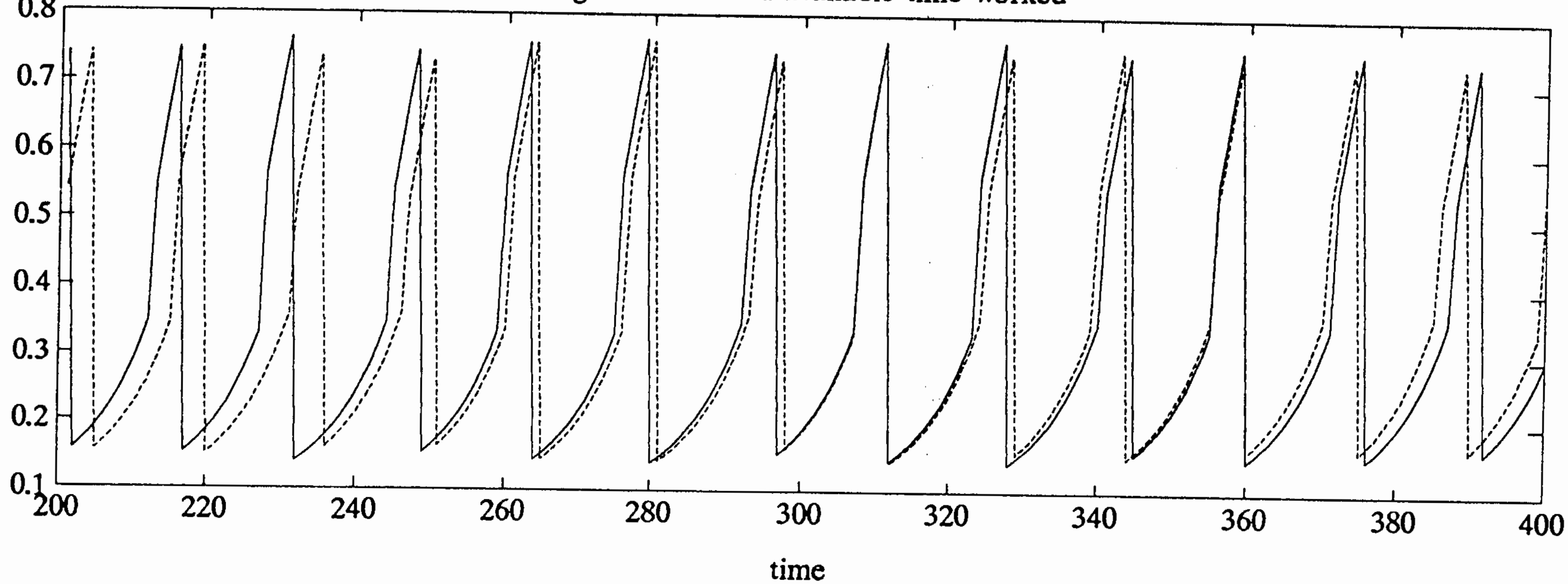
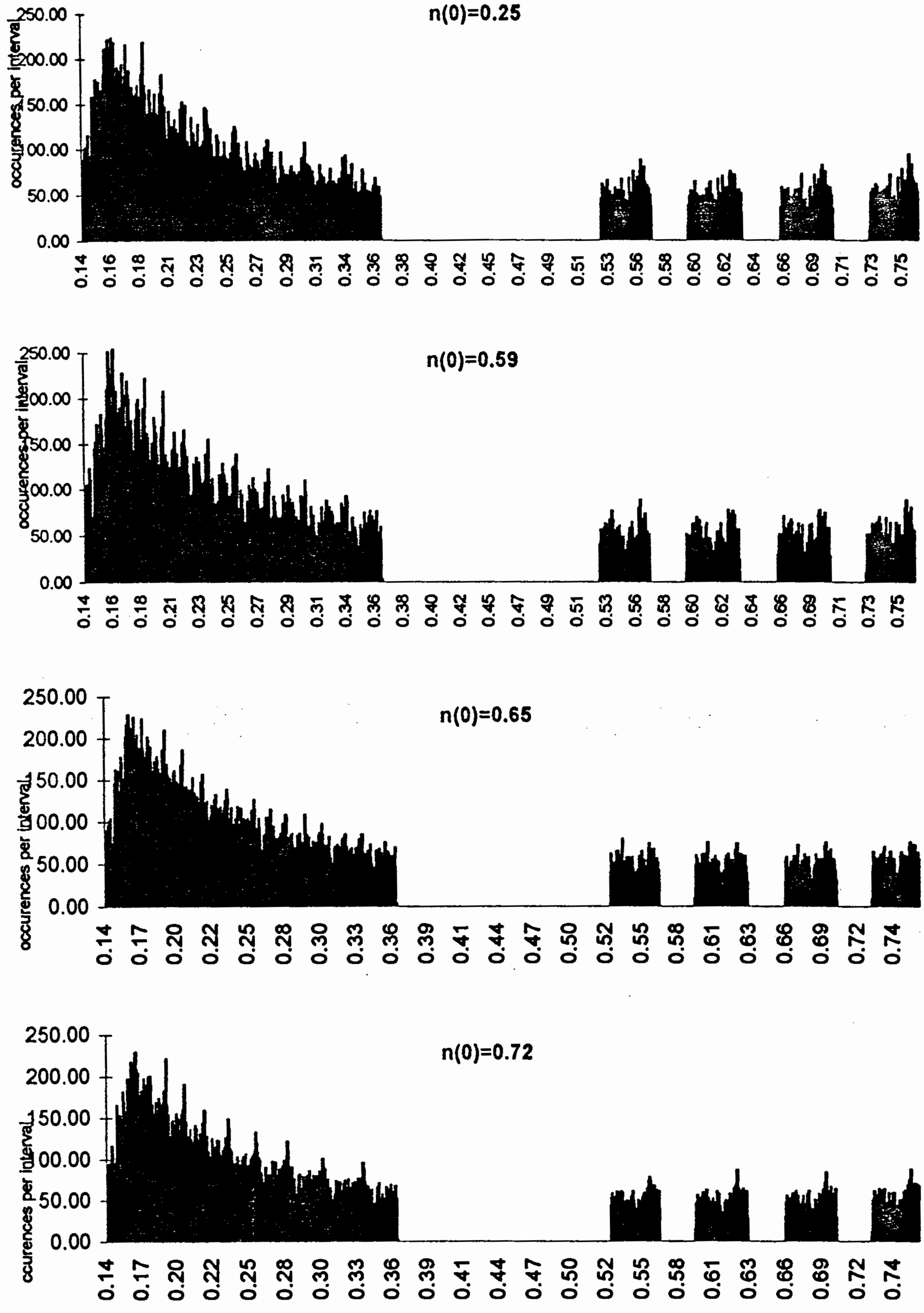


Fig4b: fraction of available time worked

Evidence on Topological Transitivity

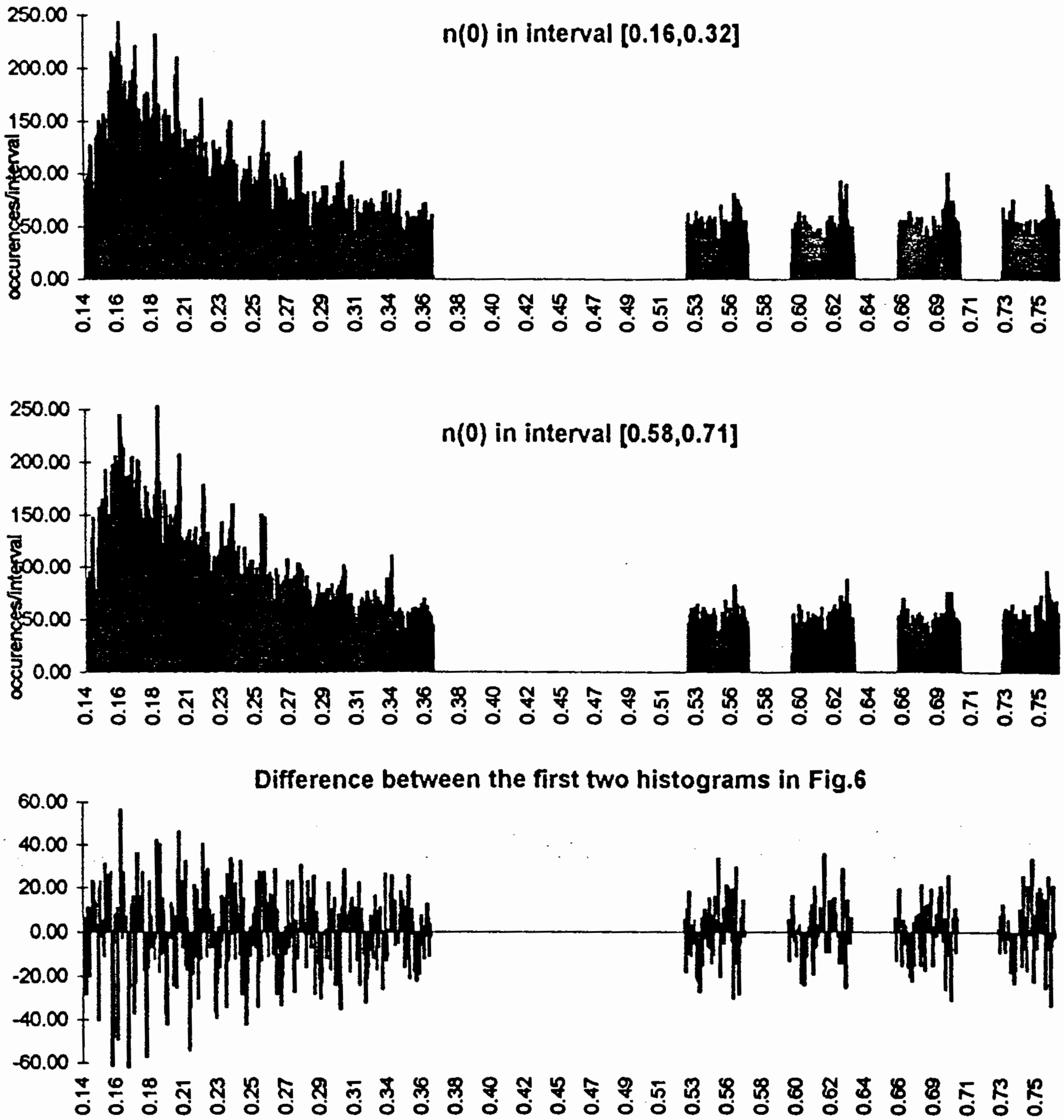
Figure 5: Histograms of Four Orbits



Note: Four histograms of 50,000 iterates on f, defined in (3.1), using standard parameter values, except σ=1.25. The iterates are differentiated according to the initial condition on n, as indicated.

Evidence on Mixing

Figure 6: Density Functions Induced by f



Note: Top two graphs - density of iterate number 3,000 on the map f defined in (3.1) with initial condition drawn uniformly from the indicated interval.

Stochastic Equilibria

There are two types:

conventional sunspots

Markov Switching sunspots

$\Rightarrow$ $s \in \mathbb{R}^2$

$$s = \begin{bmatrix} s(1) \\ s(2) \end{bmatrix}$$

$s(1) \sim$ realization of 2-state Markov chain, selecting branch of υ.

$s(2) \sim$ realization of "Euler error" shock.

Conventional Sunspots

$$v(n, n') = \omega \,, \quad \omega \in [-.06, .06],$$

each with probability $\frac{1}{2}$.

Equil. Stochastic process on n induced by ω as follows:

Step 1: pick n_0 arbitrarily

draw $\omega_1 \in [-.06, .06]$

Step 2: choose n_1 s.t.

$$v(n_0, n_1) = \omega_1$$

draw $\omega_2 \in [-.06, .06]$

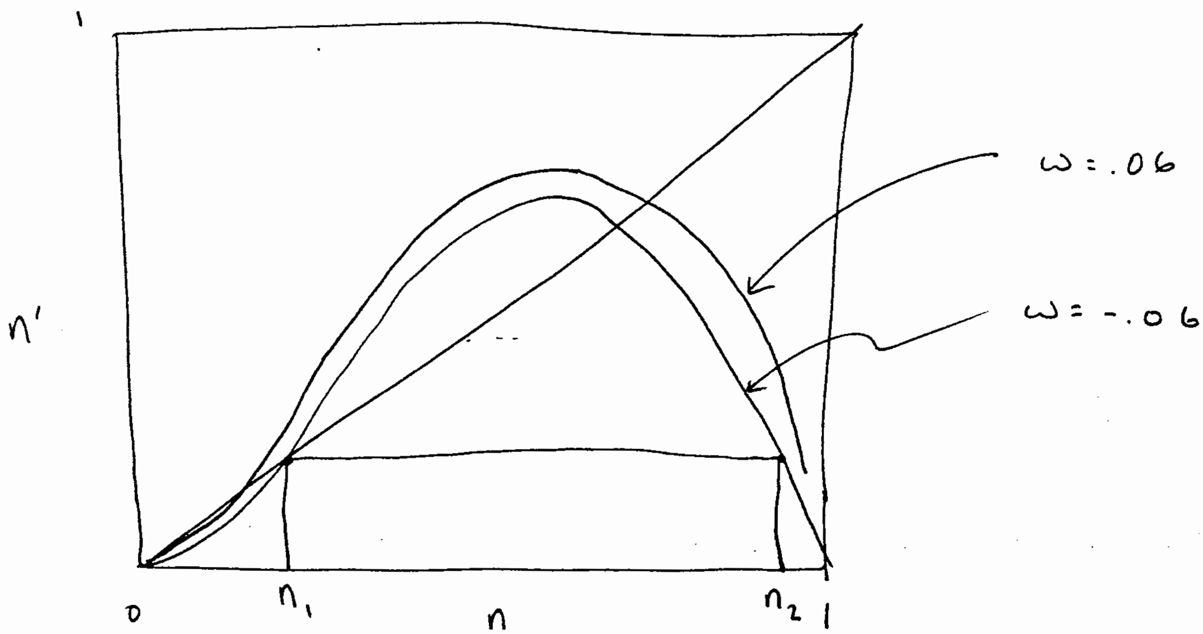
Step 3: choose n_2 s.t.

$$v(n_1, n_2) = \omega_2$$

draw $\omega_3 \in [-.06, .06]$

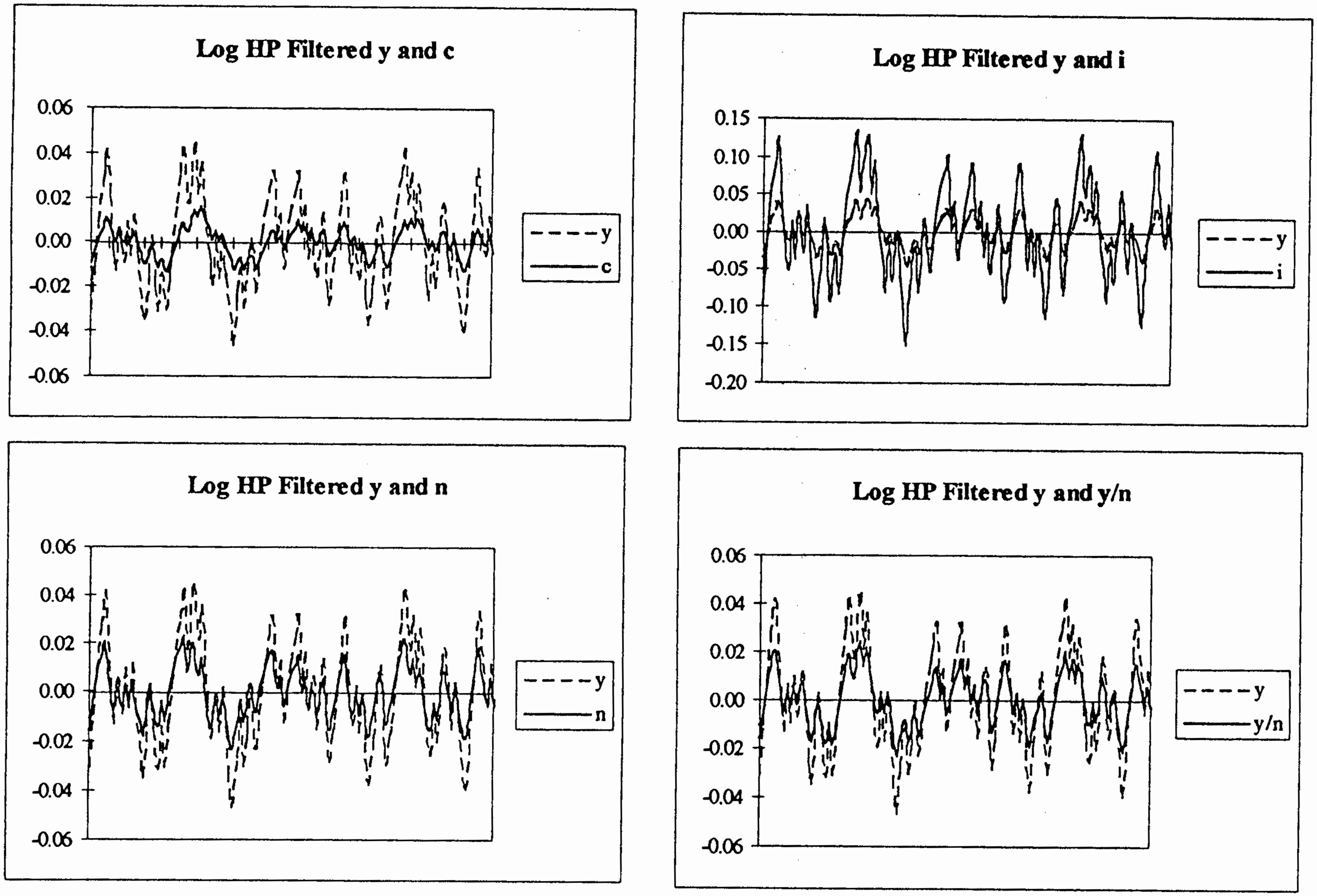
etc etc

Easy to verify that this stochastic process is an equilibrium



Conventional Sunspot

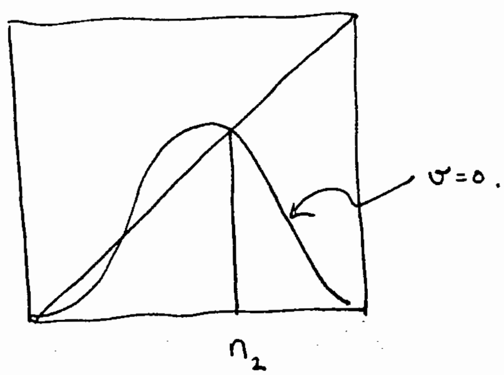
Figure 7: Logged and HP Filtered Data from the Lower Branch Only Equilibrium



Note: ~~Graphs are means of 100 simulations of length 114.~~

Welfare Analysis

Comparison with constant $(n_2, n_2, n_2, \ldots\ldots)$ equilibrium



Conventional sunspot equilibrium: .9% better than constant equilibrium

Markov Switching sunspot equilibrium: 289% worse than constant equilibrium

Policy Analysis

$$\tau_t = 1 - \frac{\bar{n}}{N_t}, \quad \bar{n} = n_2$$

Aggregate employment

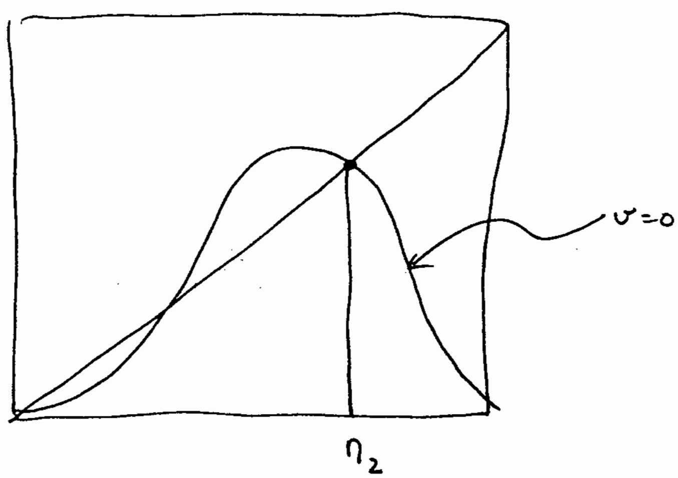
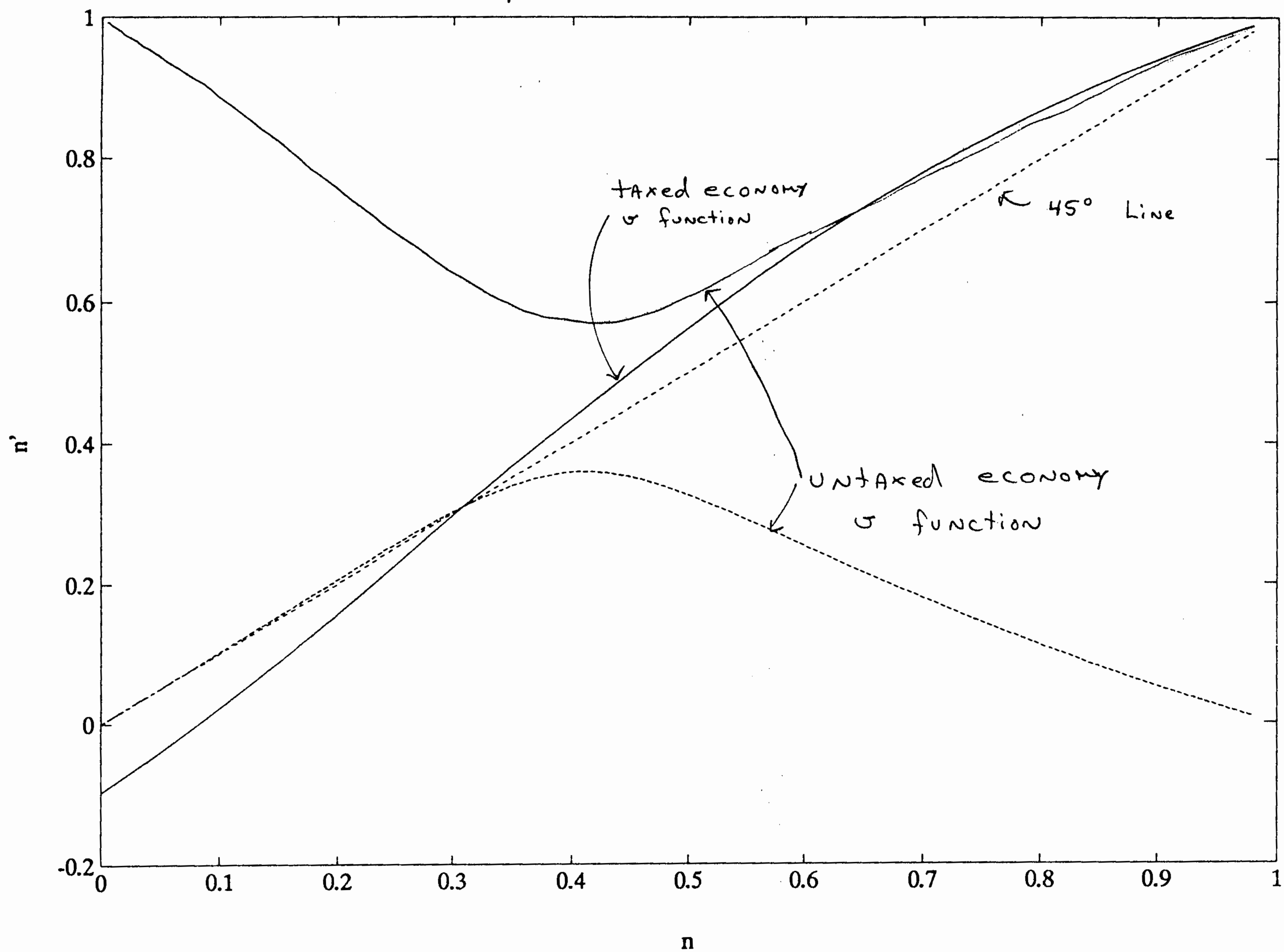


Figure ~~8~~ 9: v(n,n') for taxed and untaxed economy



Optimal Policy

Problem:

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \mu(s^t) \{ \log c(s^t) + \sigma(1-n(s^t)) \}$$

s.t.

$$c(s^t) + k(s^t) - (1-\delta)k(s^{t-1}) \leq k(s^{t-1})\{n(s^t)\}^2$$

Rescale:

$$\max_{n,\lambda} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \mu(s^t)\; f(n(s^t), \lambda(s^t))$$

$$+ \frac{1}{1-\beta} \log k_0$$

$$f(n(s^t), \lambda(s^t)) = \log(n^2 + 1 - \delta - \lambda) + \frac{\beta}{1-\beta} \log \lambda$$
$$+ \sigma \log(1-n)$$

s.t.

$$0 \leq \lambda \leq n^2 + 1 - \delta$$

$$0 \leq n \leq 1$$

Figure ~~X~~ 10: Planner's Objective Function