Christiano
411, Winter 2005

FINAL EXAM

Answer three of the following equally-weighted four questions. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 55 minutes. Good luck!

1. Consider the Krusell-Rios Rull model, in which there are 4 possible strategies available to each agent at the beginning of her 3-period life (young, middle age, old): (1) be unskilled in each period, (2) be unskilled in the same vintage for the first 2 periods and then be skilled in that technology in the 3^{rd} period, (3) be unskilled in the first period, a fast learner in some vintage in middle age and skilled in that vintage when old, (4) be an innovator in the first 2 periods and then skilled in the new technology in the last period. The availability of strategy (4) depends on whether innovation is, or is not, allowed. Whether innovation is allowed is determined by majority rule. There is a mass of 1 for each generation, so that the total mass of agents at a given date t is 3. Agents discount the future at the rate, $0 < \beta < 1$, and maximize their discounted income. The aggregate production function for technology τ is Cobb-Douglas:

$$f_\tau\left(\varphi_{s,\tau}, \varphi_{u,\tau}\right) = A_\tau \varphi_{s,\tau}^{\alpha} \varphi_{u,\tau}^{1-\alpha},$$

where $\varphi_{s,\tau}$ and $\varphi_{u,\tau}$ are the number of skilled and unskilled workers, respectively, in vintage τ. Productivity improves with vintage:

$$\frac{A_\tau}{A_{\tau+1}} = \gamma.$$

Consider the following modification to the structure of the economy seen in class. Assume that innovators receive a transfer, μ_t, from the government in each of the 2 periods of their life. The transfer is financed through labor taxes. If $\omega_t(\varphi_{s,\tau}, \varphi_{u,\tau})$ is the gross wage paid to the unskilled worker at time t in vintage τ, then the net wage is $(1-\delta)\,\omega_t(\varphi_{s,\tau}, \varphi_{u,\tau})$ where δ is an exogenous constant and $\delta \in [0, 1]$.

(a) For given $\varphi_{s,\tau}$, $\varphi_{u,\tau}$, derive the gross wage paid to the unskilled workers working in the best technology in a given period of time, t. Derive the net profits of a skilled worker in vintage τ at time t.

(b) Derive the discounted lifetime income for the 4 possible strategies.

(c) Conjecture a stationary equilibrium where in any period t a constant fraction, $\bar{\varphi}$, of the young choose to innovate and develop new technologies and the remainder, $1-\bar{\varphi}$, choose to remain unskilled.

Derive the ratio of skilled to unskilled workers at each period of time t as a function of $\bar{\varphi}$. Derive the wage of unskilled workers and the profits of skilled workers as a function of this ratio. Derive the budget constraint the government must satisfy in each period t.

(d) Derive the equilibrium value of $\bar{\varphi}$ as a function $(\alpha, \gamma, \beta, \delta)$. Discuss how $\bar{\varphi}$ changes as we increase δ. Provide economic intuition.

(e) Derive conditions under which the conjectured equilibrium is actually an equilibrium, i.e., make sure that no one would deviate and choose strategy (2) or (3).

(f) Suppose the economy is in a no-innovation equilibrium at time t, and a vote is held on whether to allow innovation. (50% of votes in favor of innovation are needed to make innovation possible.) If innovation is allowed, innovators get a subsidy as above. Discuss the incentives each different agent in the economy has to vote in favor or against innovation as a function of her current and future income. In particular, how is the subsidy to innovation going to affect this decision? Explain carefully.

2. Suppose there are 2 types of agents. In period t, type 1 agent receives endowment e_t, and type 2 agent receives endowment $1-e_t$. Suppose e_t can take on only two values: 3/4 and 1/4. Let s_t denote the exogenous uncertainty in period t, with $s_t = e_t$. Let s^t denote the history of exogenous uncertainty from the first period, date 0, until period t :

$$s^t = (s_0, s_1, ..., s_t).$$

The probability of a particular history, s^t, is $\pi_t(s^t)$.

A planner chooses a sequence of consumption allocations across the 2 types by maximizing the following objective function:

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left[\theta_1 u_1\left(c_1\left(s^t\right)\right)+\theta_2 u_2\left(c_2\left(s^t\right)\right)\right],$$

subject to the resource constraint:

$$c_1\left(s^t\right)+c_2\left(s^t\right)\leq 1,\ \text{for all } s^t,$$

and the participation constraints:

$$\sum_{j=0}^{\infty}\sum_{s^{t+j}|s^t}\beta^t\pi_{t+j}\left(s^{t+j}|s^t\right)u_1\left(c_1\left(s^{t+j}\right)\right)\geq\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_{t+j}\left(s^{t+j}|s^t\right)u_1\left(e\left(s^{t+j}\right)\right)\equiv V_1\left(s^t\right)$$

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_{t+j}\left(s^{t+j}|s^t\right)u_2\left(c_2\left(s^{t+j}\right)\right)\geq\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_{t+j}\left(s^{t+j}|s^t\right)u_2\left(1-e\left(s^{t+j}\right)\right)\equiv V_2\left(s^t\right),$$

for each s^t. The utility functions, u_1 and u_2, are strictly concave and increasing.

(a) Let $\lambda_i\left(s^t\right)\geq 0$ be the Lagrange multiplier on the i^{th} agent's participation constraint and $\mu\left(s^t\right)$ be the Lagrange multiplier on the resource constraint in history s^t. Write down the Lagrangian version of the planner problem.

Define

$$M_i\left(s^t\right)=\theta_i+\lambda_i\left(s^0\right)+\lambda_i\left(s^1\right)+...+\lambda_i\left(s^t\right),\ i=1,2.$$

Rewrite the planner's lagrangian problem in terms of $M_i\left(s^t\right)$. Use this formulation to argue that $M_i\left(s^t\right)$ can be interpreted as the weight the planner assigns to agent i.

(b) Derive the first order conditions to the planner problem.

Define

$$\begin{aligned}z\left(s^t\right)&=\frac{M_2\left(s^t\right)}{M_1\left(s^t\right)},\\ \nu_i\left(s^t\right)&=\frac{\lambda_i\left(s^t\right)}{M_i\left(s^t\right)},\ i=1,2.\end{aligned}$$

Derive a law of motion for $z(s^t)$ as a function of $z(s^{t-1})$, $\nu_1(s^t)$ and $\nu_2(s^t)$. Interpret these results and in particular discuss how the optimal allocations, $c_1(s^t)$ and $c_2(s^t)$, change as $z(s^t)$ changes. What does $z(s^t)$ measure? Provide economic intuition on the relationship between $\lambda_i(s^t)$ and $z(s^t)$.

(c) Write down the recursive formulation of the problem and define carefully the state variables. Carefully analyze the role of each of them.

From now on, consider a particular (low probability!) history with the property:

$$e_t = 3/4, \text{ for all } t = 0, 1, 2, ..., +\infty.$$

(d) Suppose that, given the above realization of e_t from $t = 0$ to $t = t^*$, the social planner has chosen allocations up to time t^*, such that $c_1\left(s^{t^*}\right) < 3/4$ and $c_2\left(s^{t^*}\right) > 1/4$. Prove that the participation constraint of one of the two agents will not be binding at t^* given s^{t^*}. What about the participation constraint of the other agent? Will it bind? Provide economic intuition.

(e) Given your results in (b)-(d), and given the realization of e_t and $c_1\left(s^{t^*}\right)$, $c_2\left(s^{t^*}\right)$ specified above, would $c_1\left(s^{t^*+1}\right)$ and $c_2\left(s^{t^*+1}\right)$ be greater or smaller than, respectively, $c_1\left(s^{t^*}\right)$ and $c_2\left(s^{t^*}\right)$? Provide a proof and economic intuition for your answer. (Hint: study how $z(s^t)$ changes.)

3. Consider the following 2 period economy. The representative household maximizes

$$u(c_1 + c_2, l)$$

subject to the following two budget constraints:

$$\begin{aligned} c_1 + k &\leq \omega \\ c_2 &\leq (1-\delta) Rk + (1-\tau)\,\omega l. \end{aligned}$$

Here, ω is the wage rate (which corresponds to the marginal product of labor), R denotes the rental rate of capital (its marginal product), and c_i denotes consumption in period i, $i = 1, 2$. One unit of labor is supplied inelastically in period 1 and l units of labor are supplied in period 2. Finally, k denotes saving in period 0, while δ and τ denote the tax rates, respectively, on capital income and labor income. We suppose that preferences have the following parametric form:

$$u(c_1 + c_2, l) = c_1 + c_2 - \frac{1}{2} l^2.$$

When $(1 - \delta) R = 1$, the household is assumed to choose $c_1 = 0$, $k = \omega$. The government must finance an exogenously given level of consumption, G, in period 2, subject to the following government budget constraint:

$$G \leq \delta R k + \tau \omega l.$$

The government chooses values for δ and τ so that its budget constraint is satisfied and utility of the representative household is as large as possible.

For the problem to be interesting, G must exceed the maximum revenues from capital taxes alone. In addition, it must not be so large as to exceed the maximum possible revenues from capital and labor taxes. We suppose G satisfies these conditions.

(a) Find the competitive equilibrium allocations for a given set of δ and τ that satisfy the government budget constraint.

(b) Derive the best ex-ante (beginning of period 1) equilibrium relative to the given social welfare function. Call the policy in the best equilibrium the Ramsey policy.

(c) Prove rigorously that in the absence of commitment, the government would deviate from the Ramsey policies in period 2. Explain.

(d) Suppose everyone understands that the government will choose taxes according to its incentives as of period 2. Find the sustainable equilibrium and compare the value of the social welfare function in this equilibrium with the one you obtain if the government could commit in period 1 to the Ramsey policy. Explain.

4. Consider an economy in which the representative agent has the following preferences:

$$E\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\frac{(1-n_t)^{1+\psi}}{1+\psi},\ \psi>0,\ \sigma\neq 1,$$

where σ and ψ are such that the utility function is concave, with positive marginal utility to consumption, c_t, and negative marginal utility to labor effort, n_t. Consumption is restricted to be non-negative, and $0 \leq n_t \leq 1$. Output, y_t, is produced as follows:

$$y_t = s_t A k_t n_t^{1-\alpha},$$

where k_t is the beginning-of-period t stock of capital. Also, s_t is an exogenous shock to productivity, with

$$\begin{aligned} s_t &= 1+\varepsilon \text{ with probability } 1/2 \\ &= 1-\varepsilon \text{ with probability } 1/2. \end{aligned}$$

The shock is independently distributed over time. Capital depreciates completely in one period, so that the aggregate resource constraint is:

$$c_t + k_{t+1} \leq y_t.$$

The efficient allocations are sequences of consumption, labor and capital that maximize utility subject to the various technology constraints. When $\sigma < 1$, we suppose that:

$$\beta A^{1-\sigma} < 1.$$

When $\sigma > 1$, we suppose that $k_{t+1} = k_t$ is feasible.

a. Let $\varepsilon = 0$, so that there is no uncertainty. Let $v(k_0)$ denote the maximized value of discounted utility under the efficient allocations. Explain clearly why the following are true:

  i. $v(k_0) = k_0^{(1-\sigma)} w$, where $-\infty < w < \infty$. (Hint: scale the economy using the capital stock.)

  ii. $k_{t+1} = \lambda k_t$, and $n_t = n$, constant.

b. Let $\varepsilon > 0$. Write out the Euler equations for n_t and k_{t+1}. Verify that these are satisfied for $k_{t+1} = \phi y_t$ and $n_t = n$ for $t = 0, 1, \ldots$.

c. Derive a simple expression for y_{t+1}/y_t in terms of s_{t+1} and ϕ.

d. Let $\sigma > 1$ and suppose there is an increase in ε. Use the Euler equations to predict what happens to n, ϕ, and the average growth rate of the economy with this increase in uncertainty. What is the intuition behind the result?