Homework #4
Economics 411, Winter 2005
Due Wednesday, February 2
Christiano

1. Suppose a planner chooses to maximize, by choice of $c_0, c_1, c_2, \ldots,$ the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], u(c_t) = \log(c_t) \tag{1}$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, \;\; k_0 \text{ given,}$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

   (a) Let $k_{t+1} = g_t(k_t)$ denote the policy rule that solves this problem, $t = 0, 1, \ldots.$. From the perspective of period 0, the part of the problem from $t = 1$ and on looks exactly like the problem with $\delta = 1$. As a result, you know that the optimized value of $u(c_1) + \beta^2 u(c_2) + \ldots$ has the form, $v(k_1)$, and you know how to compute $v(k_1)$ because it has a simple log-linear form. Use this to show that the optimal choice of k_1 has the form:
$$k_1 = g k_0^\alpha,$$
where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ. How does the saving rate from period $t = 1$ and on compare with the date 0 saving rate?

   (b) Is there a unique k^* with the property $k_t \to k^*$ as $t \to \infty$ for all k_0? Display a formula relating k^* to the parameters of the model.

   (c) Suppose $\beta = 1/1.03$, $\alpha = 1/3$, $\delta = 0.85$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.

   (d) Now suppose that when date 1 happens, the planner decides to reoptimize with respect to $k_2, k_3, \ldots$. The initial condition for this problem is k_1, the decision implemented by the planner last period. From the perspective of $t = 1$, the planner's preferences over c_t, $t \geq 1$ are as follows:
$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \ldots]$$
and the resource constraint is as before. (Note how different the problem for $t \geq 1$ looks from the point of view of period 1 than it does from the point of view of period 0.) What values will the planner choose for k_1, k_2, k_3, k_4, k_5? If the planner chooses to reoptimize in this way every period, to what value will k_t actually tend?

(e) Why are the values for k_2, k_3, k_4, k_5 chosen by the planner in date 1 different from the values planned for these variables as of date 0? Because of this difference, the problem is said to be *time inconsistent.* If δ had been set to one, we would not have had this problem. Why not?

Basically, the attitude of the planner is 'I'm very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I'll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I'll consume a lot today and save a lot tomorrow.' Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that 'tomorrow I'll quit smoking, but tonight I'll just have one or two more'. It also has been used to explain the low US saving rate. The notion is that many people say, 'today I'll spend, and tomorrow I'll save', day after day. (See the papers of David Laibson, of Harvard.) Does the solution that we have used in (d) make any sense? Would a rational person really make decisions in the time-inconsistent way described there?

2. Consider a two-sector economy with the following preferences:

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right),$$

where

$$u\left(c, l\right) = \begin{cases} \frac{\left[c(1-l)^{\eta}\right]^{1-\gamma}}{1-\gamma}, \gamma \neq 1,\ \gamma > 0 \\ \log c + \eta \log\left(1-l\right),\ \gamma = 1 \end{cases}.$$

Consumption goods are produced using the following technology:

$$c_t \leq A k_{ct}^{\alpha} l_t^{(1-\alpha)}$$

and investment goods are produced using the following technology:

$$I_t = b k_{I,t},$$

where

$$\begin{aligned} k_{t+1} &= \left(1-\delta\right) k_t + I_t \\ k_t &= k_{c,t} + k_{I,t},\ k_{c,t}, k_{I,t} \geq 0, \end{aligned}$$

where $\delta \in (0,1)$ and $b > \delta$. Also, the initial stock of capital, $k_0 > 0$, is given. The following condition will later be useful to guarantee boundedness of utility:

$$\beta\left(1-\delta+b\right)^{\alpha(1-\gamma)} < 1.$$

(a) Show that the efficient allocations solve the following problem:

$$V(k_0) = \max_{k_{t+1}, l_t \in \Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}, l_t).$$

Display the function, F, and the correspondence, Γ.

(b) Show that V has the following form:

$$V(k_0) = k_0^{\alpha(1-\gamma)} w,$$

where w is finite. Establish the finiteness of w for $\gamma > 1$ and $\gamma < 1$.

(c) Show that w is the fixed point of a particular functional equation. Does the functional equation satisfy Blackwell's conditions to be a contraction? Display formulas which can be used to solve for w as well as the optimal level of employment and the optimal growth rate of capital.

(d) Compute the price of capital, $P_{k',t}$, in this model. Show that along a growth path, $P_{k',t} \to 0$. Show that the marginal product of capital also converges to zero. Show that the rate of return on capital is constant.

(e) Does the economy satisfy the convergence property?

3. Consider an economy with capital of different vintages. At time t, the amount of capital of vintage τ, $k_{t,\tau}$, $\tau = 1, 2, 3, ...$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1-\delta)^{\tau-1} i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t - \tau$ consumption units, applied in period $t - \tau$. Capital which has vintage τ in period t has vintage $\tau + 1$ in period $t + 1$. Investment expenditures at time t, i_t, must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period t+1 installed capital goods. Consider a given amount of investment, i. Note that this investment applied in period $t + 1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1} i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^{\alpha} n_{t,\tau}^{1-\alpha}, \ 0 < \alpha < 1, \ \tau = 1, 2, 3, ... \ .$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t, since capital is measured in common efficiency units. Similarly, the wage rate is w_t.

(a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^{\alpha} n_t^{1-\alpha},\ (1-\alpha)\left(\frac{k_t}{n_t}\right)^{\alpha} = w_t,\ \alpha\left(\frac{k_t}{n_t}\right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau},\ k_t = \sum_{\tau=1}^{\infty} k_{t,\tau},\ n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(Hint: refer to your class notes on the indeterminacy of firm size under constant returns to scale.)

(b) Show that 'aggregate capital', k_t, evolves according to:

$$k_{t+1} = (1-\delta)k_t + \gamma^t i_t.$$