

Homework #5  
Economics 411, winter 2005  
Due Wednesday, February 9  
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1. Monopoly power in a variant of the neoclassical growth model. In the neoclassical growth model, there is only one good, so monopoly power would entail assuming that there is only one firm. To avoid this silly implication, we introduce a Dixit-Stiglitz technology which allows for the presence of monopoly power, without the implication that there is only one firm. This part of the homework is basically a review of simple price theory. This is useful for understanding the Romer model (more on this in question 2), and more generally.

In this discussion, we first discuss the firm sector. Because this problem is static, it is convenient from the point of notation, to temporarily drop the time subscript,  $t$ . In discussing the firm sector, we take the aggregate supply of capital,  $k$ , and labor,  $n$ , as given. These cannot be determined until the households have been brought into the picture.

At first, the firm sector may seem algebra-intensive. However, at the very end the whole setup collapses into three simple equations, equations that are very similar to three analogous equations that characterize the firm sector in our decentralization of the neoclassical growth model.

Final goods,  $y$ , are produced by a representative, competitive firm, using a continuum of intermediate goods,  $x(i)$ ,  $i \in (0, 1)$  :

$$y = \left[ \int_0^1 x(i)^\lambda di \right]^{\frac{1}{\lambda}}, \quad 0 < \lambda < 1. \quad (1)$$

The price of the  $i^{th}$  intermediate good is  $p(i)$ , which the final good producer takes as given. The profits of the final good producer are:

$$\pi = y - \int_0^1 p(i)x(i)di,$$

where the price of the final good has been normalized at unity. The problem of the final good producer is to choose  $x(i)$ ,  $i \in (0, 1)$  to maximize profits subject to the budget constraint. The first order necessary condition for profit maximization is:

$$p(i) = y^{1-\lambda}x(i)^{\lambda-1}, \quad i \in (0, 1).$$

There is a single monopolist who produces  $x(i)$ . The monopolist sets its price,  $p(i)$ , and output,  $x(i)$ , treating the first order condition of the final good producer as its demand curve,  $p(i) = P(x(i), y)$ , where

$$P(x(i), y) = y^{1-\lambda}x(i)^{\lambda-1}. \quad (2)$$

The monopolist knows the value of  $y$  and correctly understands that its choice of  $x(i)$  has no impact on  $y$ . The monopolist uses capital and labor to produce  $x(i)$  using the following technology:

$$x(i) = k(i)^\alpha n(i)^{1-\alpha} = f(k(i), n(i)), \quad 0 < \alpha < 1, \quad (3)$$

where  $k(i)$  and  $n(i)$  are the capital and labor hired by the monopolist in the capital rental and labor markets, respectively. The monopolist is small in the factor markets, and so it takes the rental rate on capital,  $r$ , and the wage rate,  $w$ , as given. The monopolist's problem is to choose  $x(i)$  to maximize profits, subject to (2), the technology, (3), and the given  $r, w$ . It is convenient to derive an expression for the monopolist's cost function,  $C(x(i), w, r)$  :

$$C(x(i), w, r) = \min_{k(i), n(i)} rk(i) + wn(i),$$

subject to (3). In Lagrangian form,

$$C(x(i), w, r) = \min_{k(i), n(i)} rk(i) + wn(i) + \mu [x(i) - f(k(i), n(i))],$$

where  $\mu \geq 0$  is the multiplier. The first order conditions for this problem are:

$$r = \mu f_k \quad (4)$$

$$w = \mu f_n \quad (5)$$

$$x = f.$$

Here,  $\mu, k$  and  $n$  are to be determined as functions of  $x, r, w$ . To obtain an expression for  $\mu$ , solve (4) for  $k/n$ :

$$\frac{k}{n} = f_k^{-1} \left( \frac{r}{\mu} \right).$$

Substitute this into (5) and obtain:

$$w = \mu f_n \left[ f_k^{-1} \left( \frac{r}{\mu} \right) \right].$$

Solving this for  $\mu$  and taking into account our functional form assumption, we find:

$$\mu = \mu(r, w) = \left( \frac{1}{1-\alpha} \right)^{1-\alpha} \left( \frac{1}{\alpha} \right)^\alpha (r)^\alpha (w)^{1-\alpha}.$$

To interpret  $\mu$ , note that it is the derivative of  $C$  with respect to  $x(i)$  in the Lagrangian expression of the firm minimization problem. Thus,  $\mu(r, w)$  is the marginal cost of producing  $x(i)$ . Another intuitive way to see this interpretation of  $\mu$  is to consider (4), which can be rewritten,  $\mu = r/f_k$ . Note that  $r$  is the change in cost with respect to a unit change in capital, while  $f_k$  is a change in output with respect to a unit change in capital,

so that the ratio is the change in cost with respect to a change in output. That is,

$$\mu = \frac{r}{f_k} = \frac{\frac{dCost}{dk}}{\frac{dx}{dk}} = \frac{dCost}{dx} \times \frac{dk}{dk} = \frac{dCost}{dx}.$$

This establishes that the marginal cost of producing  $x(i)$  is independent of the value of  $x(i)$ . (The linear homogeneity of the production function is what guarantees this.) From this observation, together with the fact that  $C(0, r, w) = 0$ , we conclude that the cost function of the  $i^{th}$  intermediate good firm has the following representation:

$$C(x(i), r, w) = \mu(r, w) \times x(i).$$

Taking into account that the intermediate good firm must respect its demand curve, the profit maximization problem is:

$$\pi(i) = \max_{x(i)} P(x(i), y)x(i) - \mu(r, w)x(i).$$

The first order condition equates marginal revenue,  $P'(x(i), y)x(i) + P(x(i), y)$ , to marginal cost,  $\mu(r, w)$ . Taking into account our functional form assumptions,

$$\lambda \left( \frac{y}{x(i)} \right)^{1-\lambda} = \mu(r, w).$$

Note that the value of  $x(i)$  that solves this is independent of  $i$ . Similarly, the choice of  $p(i)$  is independent of  $i$ . These observations are not surprising in view of the symmetry of the intermediate good firm problems. Denote the optimal values of  $x(i)$  and  $p(i)$  by  $x$  and  $p$ , respectively. From the fact that (1) must be satisfied in equilibrium, we have that in equilibrium,  $x = y$ , so that, using the previous expression and the demand curve:

$$\begin{aligned} \lambda &= \mu(r, w) \\ p &= 1. \end{aligned}$$

That is, in equilibrium the marginal cost is equated to  $\lambda$ , the parameter in the final good production function. In addition, the price of the intermediate input is unity. It is interesting to see the value of profits in equilibrium:

$$\begin{aligned} \pi &= P(x, y)x - \mu(r, w)x \\ &= x[1 - \lambda]. \end{aligned}$$

Note that when  $\lambda = 1$ , then profits are zero. This reflects that, in this case, the intermediate inputs are perfect substitutes so that the 'monopolist' does not have any exploitable monopoly power. In this case, marginal revenue is just the price of the intermediate good and  $P' = 0$ . For lower values of  $\lambda$ , each intermediate good firm becomes less substitutable with

the others, and the monopolist has more monopoly power. As a result, profits per unit of sales,  $\pi/x$ , rises. A crucial thing to note is that, according to (4)-(5), the monopolist pays factors of production less than their marginal products:

$$\begin{aligned} r &= \lambda f_k < f_k \\ w &= \lambda f_n < f_n. \end{aligned} \tag{6}$$

The equations that summarize the firm sector are these, together with (1) which reduces to

$$y = k^\alpha n^{1-\alpha},$$

given that  $y = x(i)$  for all  $i$ . These equations are conditional on the values of  $k$  and  $n$ , which we have taken as given in our discussion of the firm sector. Notice that these three equations look exactly like the equations corresponding to the firm sector in our competitive decentralization of the neoclassical model, with the exception of the appearance of  $\lambda$  in the firm first order conditions.

To determine  $k$ , and  $n$ , we have to bring in the households. Their problem is to solve, at each date,  $t$  :

$$\max \sum_{j=0}^{\infty} \beta^j u(c_{t+j}),$$

subject to

$$c_{t+j} + k_{t+1+j} - (1 - \delta)k_{t+j} \leq r_{t+j}k_{t+j} + w_{t+j}n_{t+j} + \pi_{t+j} + \int_0^1 \pi_{t+j}(i)di,$$

and  $n \leq 1$ . Note that households are treated as receiving profits from all firms, final and intermediate. The household optimally chooses  $n = 1$  and its first order condition for capital is:

$$u'(c_{t+j}) = \beta u'(c_{t+1+j}) [r_{t+1+j} + 1 - \delta], \quad j = 0, 1, 2, \dots$$

Substituting out for the equilibrium value of the rental rate of capital,

$$u'(c_{t+j}) = \beta u'(c_{t+1+j}) [\lambda f_{k,t+j+1} + 1 - \delta], \quad j = 0, 1, 2, \dots$$

Notice that this first order condition resembles the first order condition of the efficient allocations in the neoclassical model, with a crucial exception. Instead of  $f_k$  appearing here, it is something less. A consequence of this is that when households make their decisions about saving, they treat the payoff from capital as  $r + 1 - \delta$ , which is less than the actual payoff,  $f_k + 1 - \delta$ . Because, in effect, the equilibrium offers households insufficient incentive to save, it will produce a socially inefficiently low level of capital.

A sequence of markets equilibrium for this economy is a sequence of prices,  $\{p_t(i), r_t, w_t\}_{t=0}^{\infty}$ , and quantities,  $\{n_t(i), k_t(i), n_t, k_{t+1}, c_t\}_{t=0}^{\infty}$ , such

that the household and firm problems are satisfied given the prices and the quantities. In addition, we require market clearing,  $\int n_t(i)di = n_t$ ,  $\int k_t(i)di = k_t$ , for each  $t$ .

- (a) Use the linear homogeneity of the production function, and expressions for profits to show that, in equilibrium,

$$\begin{aligned} r_t k_t + w_t n_t + \pi_t + \int_0^1 \pi_t(i)di \\ = f(k_t, n_t), \end{aligned}$$

so that the resource constraint is satisfied.

- (b) Show that, in a steady state,

$$k = \left[ \frac{\lambda \alpha}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\alpha}}.$$

- (c) Show that the efficient allocations of the economy with monopolists solve:

$$\max_{k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} - (1 - \delta)k_t \leq f(k_t, n_t).$$

(Hint: note that I dropped the whole thing about (1). In effect, you need to show that this not a binding restriction on the problem of determining the efficient allocations.) Thus, the efficient allocations in the monopoly economy coincide with those of the neoclassical economy. However, because of the presence of monopoly power, the equilibrium allocations are not efficient. This is consistent with the result for the steady state capital stock in (b) above, which establishes that when  $\lambda < 1$ , the equilibrium steady state capital stock in the economy is less than the efficient level of capital, which we derived in class.

- (d) Modify the household budget constraint in the following way:

$$c_{t+j} + k_{t+1+j} - (1 - \delta)k_{t+j} \leq (1 + \tau_{t+j})r_{t+j}k_{t+j} + w_{t+j}n_{t+j} + \pi_{t+j} + \int_0^1 \pi_{t+j}(i)di - T_{t+j}$$

Here,  $\tau_{t+j}$  is a tax subsidy on the household's capital income. In addition,  $T_{t+j}$  is a lump sum transfer from the household to the government. By 'lump sum' I mean that the magnitude of the transfer is independent of any choice by the household. We require that the government balance its books in each period:

$$\tau_t r_t k_t = T_t,$$

where  $k_t$  is the economy-wide average stock of capital, and *not* the quantity of capital chosen by any particular household (otherwise, we could not maintain our assumption that from the perspective of the individual household,  $T_t$  is lump sum). Show that there is a value of  $\tau_t$ ,  $t = 0, 1, 2, \dots$  such that the allocations in a sequence of market equilibrium are efficient. Note that this value is positive. This reflects that, to steer households towards the efficient level of investment, they need an additional incentive beyond what the market gives them. The market systematically gives them too little incentive because of the presence of monopoly power. In thinking about whether a tax policy can be found which selects the efficient allocations, be sure to also think about the allocations in date 0.

2. Consider the model economy associated with Romer's model of growth through specialization. That is, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The technology for producing final goods is:

$$y_t = \int_0^{M_t} x_t(i)^\alpha di, \quad M_t > 0, \quad 0 < \alpha < 1. \quad (7)$$

To produce  $x_t(i)$  units of the  $i^{\text{th}}$  intermediate good requires

$$\frac{1}{2}(1 + x_t(i)^2) \quad (8)$$

units of capital if  $x_t(i) > 0$  and zero units of capital if  $x_t(i) = 0$ . The following constraint must be satisfied:

$$\int_0^{M_t} \frac{1}{2}(1 + x_t(i)^2) di = k_t, \quad (9)$$

where  $k_t$  is the beginning-of-period  $t$  aggregate stock of capital. The initial capital stock,  $k_0 > 0$ , is given. The resource constraint is:

$$c_t + I_t \leq y_t, \quad (10)$$

and the aggregate capital accumulation technology is given by:

$$k_{t+1} = (1 - \delta)k_t + I_t.$$

The efficient allocations for this economy solve the planning problem, maximize utility with respect to  $\{M_t, k_{t+1}, y_t, c_t, x_t(i), i \in (0, M_t)\}_{t=0}^{\infty}$ , subject to the various constraints.

- (a) Explain why economic efficiency dictates  $x_t(i) = \bar{x}_t$  for  $i \in (0, M_t)$ . From here on, you may simply assume  $x_t(i) = \bar{x}_t$  for all  $i \in (0, M_t)$ .

- (b) Show that the planning problem for the Romer economy coincides with the planning problem for the  $Ak$  model. In particular, show that the problem can be written,

$$\max_{\{k_{t+1} \in \Gamma(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

where

$$F(k, k') = \frac{[(A + 1 - \delta)k - k']^{1-\gamma}}{1 - \gamma} = \max_{\bar{x}_t, M_t} \frac{c_t^{1-\gamma}}{1 - \gamma}.$$

The last maximization is subject to (7)-(10), and the given values of  $k_t, k_{t+1}$ . Display an expression for the value of  $A$  in terms of model parameters. In addition to verifying the form of  $F$ , show what the constraint set,  $\Gamma$ , is.

- (c) Identify a set of parameter values under which positive growth is efficient, although the growth rate in the market decentralization analyzed in class is zero.
- (d) The problem with monopoly power is that it results in an inefficiently low level of activity (in the Romer model, the root of this inefficiency is (6), according to which a firm that is a monopolist in the product market and competitive in factor markets will pay a rental rate on capital that is less than its marginal product). In the Romer model we have just seen that this manifests itself in the form of inefficiently low growth. The pace at which new varieties of specialized inputs (e.g., specialized manufactured goods, specialized labor) are introduced is too slow in the market economy. Some sort of intervention in the market economy is desirable. One possibility is to subsidize the activities of monopolists. Accordingly, let  $p(i)x(i)$  be the revenues of the  $i^{th}$  monopolist in the absence of taxes or subsidies. A subsidy rate,  $\tau_t$ , raises the revenues of the  $i^{th}$  monopolists to  $p(i)x(i)(1 + \tau_t)$ . The total cost,  $G_t$ , to the government of this subsidy scheme is

$$G_t = \int_0^{M_t} p(i)x(i)\tau_t di.$$

Suppose  $G_t$  is financed by a lump sum tax applied to households. That is, the household budget constraint is modified as follows:

$$c_t + k_{t+1} - (1 - \delta)k_t = r_t k_t + w_t n_t - T_t,$$

where  $T_t$  represents taxes paid by the representative household to the government. Suppose the government balances its budget period by period:

$$T_t = G_t.$$

Find the subsidy rate,  $\tau_t$ , that causes the allocations in the market economy to coincide with the efficient allocations.

These results have to be interpreted with caution. You have identified an ideal form of government intervention, which makes the private market economy efficient. However, the intervention we investigated abstracts from any social inefficiencies induced by having to raise the revenues needed to finance the subsidy to monopolists. We abstracted from this by assuming that the tax on households is administered in lump-sum form. In practice, such taxes are not available. So, the problem of ‘fixing’ the inefficiency in the Romer model is actually more complicated than this question makes it out to be.

3. We have not yet discussed the overlapping generation model. However, when we discuss Jones-Manuelli later on, we will use the OG model. This exercise is to help you start thinking about this model. You are asked to derive a classic result about the possible inefficiency of competitive equilibrium.

Consider the overlapping generations model in which the utility of the generation born at  $t$  is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate,  $w_t$ . They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \quad c_{t+1}^t \leq r_{t+1}k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are  $r_t$  and  $w_t$ , respectively. Their production technology is  $y = k^\alpha n^{1-\alpha}$ ,  $0 < \alpha < 1$ .

- (a) Define a sequence of markets equilibrium. Provide expressions for  $w_t$  and  $r_t$  in terms of  $k_t$ .
- (b) Consider a steady state equilibrium in which the aggregate stock of capital of capital is constant,  $k$ , the consumption of each period’s young is constant,  $c^y$ , and the consumption of each period’s old is a constant,  $c^o$ . Time starts up in period 0, with the initial old generation owning the (steady state) capital stock, which they sell to the period 0 young. Show that the equilibrium rate of return on capital is

$$r_{k,t} = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}, \text{ for all } t.$$

Interpret this expression. Why is the interest rate infinite if  $\beta = 0$ ? Why is it zero if  $\alpha = 0$ ?

- (c) Show that, for parameter values where  $r_{k,t} < 1$ , the competitive equilibrium is inefficient. That is, prove the following: it is possible to deviate from the equilibrium consumption allocations by reallocating consumption between each period's old and the same period's young in a way that is compatible with the resource constraint and which makes everyone (i.e., the first generation, the second generation, the third, etc.) better off. How might the result be affected if there were a last date in the economy?
- (d) Identify parameter values for which the growth rate of household consumption deviates from the (zero) growth rate of aggregate consumption that occurs in the steady state equilibrium.