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D11, Winter 2005

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (25) Consider a household which solves the following problem:

$$v(k,r,w) = \max_{c,l \in B(k,r,w)} u(c,l),$$

where $u : \Re^2_+ \to \Re$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, c, and leisure, l. The constraints the household must obey in selecting c, l are summarized by B:

$$B(k,r,w) = \{c, l : 0 \leq c \leq rk + w(1-l),\ 0 \leq l \leq 1\}.$$

Here, k, r, $w > 0$ are numbers over which the household has no control. Prove that v is concave in k and that the derivative of v with respect to k exists for 'interior k'. An interior k means $k > 0$ and that the optimal choice of l satisfies $0 < l < 1$. Also, display a formula for the derivative of v. If you make use of a theorem to help prove your result, be sure to state it clearly.

2. (35) Consider the following sequence problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1,$$

subject to:

$$\begin{gathered} c_t + k_{t+1} - (1-\delta)k_t + g \leq f(k_t), \\ c_t, k_{t+1} \geq 0,\ u, f \text{ strictly concave, increasing, } g > 0, \\ u(c),\ u'(c) \text{ continuous for all } c \geq 0 \\ f'(k) \to 0 \text{ as } k \to \infty. \end{gathered}$$

Note that this resource constraint differs from the type of constraint considered in class because of the presence of $g > 0$. We can think of this as a fixed level of government spending.

Consider the following Stokey-Lucas canonical form:

$$\max_{k_{t+1}\in\Gamma(k_t)} \sum_{t=0} \beta^t F(k_t, k_{t+1}),$$

where

$$\begin{aligned}
\Gamma &: K \to K,\ K \subseteq R^1,\ K \text{ convex} \\
A &= \{x, y : y \in \Gamma(x),\ x \in K\} \\
&\quad\ \Gamma \text{ non-empty, compact, continuous, monotone, convex} \\
F &: A \to R \text{ bounded, continuous}
\end{aligned}$$

In responding to the following questions, you may occasionally need to make the assumption that g is sufficiently small (*). Also, feel free to use graphs to illustrate your argument.

(a) Consider the existence of Γ, K, F, A that satisfy the properties of the Stokey-Lucas canonical form.

  i. Show that there is some $\underline{k} > 0$ such that the feasible set of c, k' is empty for all $0 \leq k < \underline{k}$.

  ii. Show that there is some $k^* > \underline{k}$ such that for all $k < k^*$, $c_{t+j} \geq 0$ all $j \geq 0$ is technologically infeasible.

  iii. Show that there is some $\overline{k} > k^*$ such that if $k_t \leq \overline{k}$, then technological feasibility requires $k_{t+j} \leq \overline{k}$ for all $j \geq 0$.

  iv. Explain why (*) is necessary to establish (i)-(iii).

  v. Use results (i)-(iii) to establish that Γ, K, F, A consistent with the properties of the Stokey-Lucas canonical form can be constructed. Display expressions for Γ, K, F, A.

  vi. Does Γ satisfy monotonicity? (Hint: Γ is monotone if $\Gamma(x) \subseteq \Gamma(y)$ whenever $x \leq y$.)

(b) Define a steady state stock of capital as a value of the initial capital stock, $k_0 > 0$, such that the efficient allocations are, $k_{t+1} = k_0$

for $t = 0, 1, 2, ...$ Establish, as rigorously as you can, that there exist two such values of k_0. (You may invoke theorems without proof, but be sure to state the theorem and assumptions carefully.) Describe formulas that can be solved to compute the two steady states.

3. (20) Suppose households are all identical, and have preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

and the resource constraint, expressed in per capita terms, is:

$$c_t + k_{t+1} - (1-\delta)k_t \leq f(k_t, n_t),$$

where $c_t, k_{t+1} \geq 0$ denote consumption and capital, respectively, and $\beta, \delta \in (0,1)$. Also, n_t denotes hours worked, which are constrained by $0 \leq n_t \leq 1$. Utility is strictly increasing, concave and f is linearly homogeneous, strictly increasing. As you answer this question, make up whatever additional assumptions about u, and f you feel you need.

(a) Define an Arrow-Debreu date 0 equilibrium for this economy, by giving households a budget constraint and firms the production technology. Set up markets and prices.

(b) Establish the following properties of the Arrow-Debreu equilibrium: (i) if a system of prices and profits are part of an equilibrium, then any positive scalar multiple of these prices and profits are also an equilibrium; (ii) all prices are strictly positive; (iii) firm profits are zero; (v) a particular relationship must hold between the return on capital and the wage rate.

(c) Define the *efficient* allocations as the allocations which maximize the representative household's utility subject to the resource constraint. Prove that the allocations in an Arrow-Debreu equilibrium are efficient.

(d) Set up a sequence of markets equilibrium. Show that the allocations in a sequence of markets equilibrium and in an Arrow-Debreu equilibrium satisfy the intertemporal Euler conditions satisfied by the efficient allocations.

4. (20) Suppose the planner maximizes, by choice of c_t, k_{t+1}, and n_t,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} - (1-\delta)k_t &= y_t, \\ y_t &= k_t^{\alpha}[\exp(\mu t) n_t]^{(1-\alpha)}, \\ c_t &\geq 0, k_{t+1} \geq 0, 1 \leq n_t \leq 1 \end{aligned}$$

where $\mu > 0$ and δ, α, β are all positive and less than 1. Suppose u has the constant elasticity of substitution form:

$$u(c,n) = \begin{cases} \left(\{[c^{(1-1/\sigma)} + \gamma(1-n)^{(1-1/\sigma)}]^{1/(1-1/\sigma)}\}^{\psi} - 1 \right) / \psi & \sigma \neq 1 \\ \frac{\{c(1-n)^{\gamma}\}^{\psi} - 1}{\psi} & \sigma = 1 \end{cases}$$

The parameters are such that utility is strictly increasing and concave in consumption and leisure. Explain why it is that if $\sigma = 1$, then the variables converge to a path in which c_t/y_t, k_t/y_t and n_t are constant. However, when $\sigma \neq 1$, the economy is unlikely to have this property.